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ON A VARIANT OF VERTEX EDGE DOMINATION

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ABSTRACT. A new variant of vertex edge domination, namely semi total vertex edge domination has been introduced in the present paper. A subset S of the vertex set V of a graph G is said to be a semi total vertex edge dominating set(stved - set), if it is a vertex edge dominating set of G and each vertex in S is within a distance two of another vertex in S. An stved-set of G having minimum cardinality is said to be an $\gamma_{stve}(G)-set$ and its cardinality is denoted by $\gamma_{stve}(G)$. Bounds for $\gamma_{stve}(G)-set$ have been given in terms of various graph theoretic parameters and graphs attaining the bounds have been characterized. In particular, bounds for trees have been obtained and extremal trees have been characterized.

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1. Introduction & Preliminaries

A graph G consists of a finite non empty set V of p vertices together with a set E of q edges joining pairs of distinct vertices in V. By the open neighbourhood of a vertex v of G, we mean the set $N_G(v) = \{u \in V : uv \in E\}$. The closed neighbourhood of a vertex v of G, $N_G[v] = \{u \in V : uv \in E\} \cup \{v\}$. The degree $d_G(v)$ of a vertex v in a graph G is the number of edges of G incident with v. We denote by $\delta(G)$, $\Delta(G)$ the minimum, maximum degrees of the vertices of G, respectively. The distance between two vertices u and v in G, denoted by $d_G(u, v)$ is the length of a shortest $u - v$ path in G. The eccentricity of a vertex v in a graph G, denoted by $e(v)$, is the maximum of the distances from v to the remaining vertices of G. The radius of a graph G, denoted by $r(G)$, is the minimum of the eccentricities of the vertices in G. The diameter of a graph G, denoted by $diam(G)$, is the maximum of eccentricities of the vertices in G. A vertex of minimum eccentricity is called a central vertex of G and the set of all central vertices of G, denoted by $C(G)$, is called the center of G. The eccentricity of the center of G, denoted by $\hat{r}(G)$, is the maximum distance from

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the center to the vertices not in the center, where the distance from a vertex to a set is the smallest distance from the vertex to any of the vertices in the set. A cut - vertex in a graph G is a vertex whose deletion from G increases the number of components in the resultant graph. A leaf is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. Finally, support vertex is said to be strong if it is adjacent to at least two leaves, else it is said to be weak.

A set $S(\subseteq V)$ is called a dominating set for G, provided each vertex of $V-S$ is adjacent to a member of S. The domination number of G, denoted by $\gamma(G)$, is the cardinality of the smallest dominating set in G. For a comprehensive survey of domination in graphs, refer[8].

Suppose $S \subseteq V$. The subgraph weakly induced by S is the graph $< S >_w = (N[S], E \cap (S \times N[S]))$. A set S is a weakly connected dominating set of G, if S is a dominating set and $< S >_w$ is connected. The weakly - connected domination number of G, denoted by $\gamma_w(G)$, is the minimum cardinality of the weakly connected dominating set for G[4].

A subset S of the vertex set V is said to be a *vertex edge dominating set* of the graph G if for each edge uv in G there is a vertex w in S such that $w \in \{u, v\}$ or w dominates at least one of u, v. The *vertex edge domination number* $\gamma_{ve}(G)$ is the minimum cardinality of a vertex edge dominating set of G. Vertex edge domination in graphs was introduced in [6], and further studied in [7].

Wayne Goddard., et al[9], introduced a variant of vertex - vertex domination, namely semi total domination as follows. A set S of vertices in a graph G without isolated vertices is said to be semi total dominating set, if S is a dominating set and each vertex in S is within a distance two of another vertex in S. The semi total domination number of G, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi total dominating set of G[9].

In [5], a new type of graphs, called *semi complete graphs*, are introduced as follows. A connected graph G is said to be *semi complete*, if any two different vertices in G have a common neighbour.

Analogous to semi total domination in vertex - vertex domination, we introduce a new variant of vertex - edge domination, namely semi total vertex edge domination as follows. A vertex edge dominating set $S(\subseteq V)$ is said to be semi total vertex edge dominating set of G, if S is a vertex edge dominating set for G and each vertex in S is within a distance 2 from another vertex in S. The semi total vertex edge domination number of G, denoted by $\gamma_{stve}(G)$, is the minimum cardinality of a semi total vertex edge dominating set of G. A semi total vertex edge dominating set of minimum cardinality is said to be a $\gamma_{stve}(G) - set$.

In the present paper, we give necessary and sufficient conditions for a stved - set of G to be a minimal stved - set. Bounds for this variant are given in terms of various other graph theoretic parameters.

B. Krishna Kumari., et al. [2], established bounds for the vertex edge domination number of trees. For every tree of order $p \geq 3$ with l leaves and s support vertices, $\frac{p-s-l+3}{4} \leq \gamma_{ve}(G) \leq \frac{p}{3}$. They also characterized the extremal trees. We prove that for any tree of order $p \geq 6$, $\frac{p-s-l+3}{3} \leq \gamma_{stve}(G)$ and also characterize the extremal trees.

All graphs considered in this paper are simple, finite, undirected and connected with order $p \geq 2$. For all graph theoretic terminology not defined here, the reader is referred to [1] and [8].

2. Main results

Now, we give necessary and sufficient condition for an stved - set to be minimal.

Theorem 2.1. *A semi total vertex edge dominating set S of G is minimal if and only if for each $u \in S$, one of the following conditions hold:*

(1) *for some v in S, there is no $w \in S - \{u, v\}$ such that $d(v, w) \leq 2$.*
(2) *there is an edge v_1v_2 in G such that, $(N[v_1] \bigcup N[v_2]) \bigcap S = \{u\}$.*

Proof. The proof is straightforward. □

Note: For a graph G, we have $\gamma_{ve}(G) \leq \gamma_{stve}(G)$.

Now, we give the semi total vertex edge domination numbers of some standard graphs.

Proposition 2.2. (1) *For a path $P_n (n \geq 2)$,*

$$\gamma_{stve}(P_n) = \begin{cases} 2, & 2 \leq n < 7 \\ 2m, & n = 7m \\ 2m+1, & n = 7m+1, 7m+2 \\ 2m+2, & n = 7m+3, 7m+4, 7m+5, 7m+6 \end{cases}$$

(2) *For a cycle $C_n (n \geq 3)$,*

$$\gamma_{stve}(C_n) = \begin{cases} 2, & 3 < n \leq 6 \\ 2m, & n = 6m (m \geq 2) \\ 2m+1, & n = 6m+1, 6m+2 \\ 2m+2, & n = 6m+3, 6m+4, 6m+5 \end{cases}$$

(3) $\gamma_{stve}(S_n) = 2$.
(4) $\gamma_{stve}(K_{m,n}) = 2$.
(5) $\gamma_{stve}(K_n) = 2, n \geq 2$.

Proposition 2.3. *For a graph G with atleast two vertices, $2 \leq \gamma_{stve}(G) \leq p$. Also, $\gamma_{stve}(G) = p$ if and only if $G = K_2$.*

Theorem 2.4. $\gamma_{stve}(G) = p - 1$ *if and only if $G = K_3$ or $G = P_3$.*

Proof. Assume that $\gamma_{stve}(G) = p - 1$.
Suppose that $p \geq 4$. Let $V = \{v_1, v_2, ..., v_{p-1}, v_p\}$ be the vertex set of G. If the graph G has a pendant edge, say v_iv_j, then it can be easily verified that $V - \{v_i, v_j\}$ is an stved - set for G. If the graph does not have pendant edges,

then in this case $V - \{v_i, v_j\}$ is an stved - set for any pair of adjacent vertices $v_i and v_j$. In either case, G has an stved-set of cardinality atmost $p - 2$. This implies, $\gamma_{stve}(G) \leq p - 2$, a contradiction to our assumption. By the above proposition 2.3 $p \neq 2$. Hence, $G = K_3$ or $G = P_3$. The converse part is clear. □

Theorem 2.5. *For a graph G, $\gamma_{stve}(G) = p - 2$ if and only if $p = 4$.*

Proof. Assume that $\gamma_{stve}(G) = p - 2$.
Suppose that $p \geq 5$. Form a spanning tree $G^{'}$ from G.

Let V_1 and V_2 be the partite sets of $G^{'}$. If $|V_1| = 1$, then $V_1 \bigcup \{v\}_{v \in V_2}$ is an stved - set of $G^{'}$ of cardinality $2 = \gamma_{stve}(G^{'}) = \gamma_{stve}(G) < p - 2$ which is a contradiction to our assumption. Hence $\min\{|V_1|, |V_2|\} \geq 2$.

Observe that V_1 or V_2 is an stved-set of $G^{'}$. In other words, $V - V_1$ or $V - V_2$ is an stved-set of $G^{'}$. If one of $V - V_1$ or $V - V_2$ is of cardinality 2, then

$$\gamma_{stve}(G) = 2 < p - 2,$$

a contradiction. Thus,

$$min\{V - V_1, V - V_2\} \geq 3$$

This implies that $\gamma_{stve}(G) \leq p - 3 < p - 2$, again a contradiction. So, $p \leq 4$. By theorem 2.4, $p = 4$.

Assume that the converse is true.

Let G be a graph with four vertices and $G^{'}$ be a spanning tree of G. Since any pair of distinct vertices in $G^{'}$ forms an stved-set of $G^{'}$,

$$\gamma_{stve}(G^{'}) = \gamma_{stve}(G) = 2 = p - 2$$

.

Hence the proof.

□

Theorem 2.6. *If $\Delta(G) \geq p - 3$, then $\gamma_{stve}(G) = 2$.*

Proof. Suppose $\Delta(G) = p - 3$.
Let $d(v) = p - 3$ and $\{v_1, v_2, v_3, ..., v_{p-3}\}$ be the neighbours of v. Then, $V = \{v\} \bigcup \{v_1, v_2, v_3, ..., v_{p-3}\} \bigcup \{v_{p-2}, v_{p-1}\}$.
Case:1 Suppose v_{p-1}, v_{p-2} are adjacent. Without loss of generality. assume that $< v_{p-3}v_{p-2}v_{p-1} >$ is a path in G. Then, clearly $\{v, v_{p-3}\}$ is a stved - set for G.
Case:2 Suppose $\{v_{p-1}, v_{p-2}\}$ are not adjacent. Clearly, each $v_i, i \in \{p-1, p-2\}$ is adjacent to at least one vertex in G. Let $v_{p-1}v_i, v_{p-2}v_j$ are edges in G for some $1 \leq i, j \leq p - 3$. Without loss of generality assume that $i = p - 3$. Then, $\{v, v_{p-3}\}$ is a stved - set for G. The proof is trivial for $\Delta(G) \geq p - 2$. □

Theorem 2.7. *If $\delta(G) \geq 2, d(G) \leq 2$, then $\gamma_{stve}(G) \leq \delta(G)$.*

Proof. Suppose that $\delta(G) = d(v)$, for some v in G. By the hypothesis $N(v)$ forms an stved - set for G. Hence,

$$\gamma_{stve}(G) \leq \delta(G).$$

. □

Remark 2.1. Since for a *semi complete graph* G[5], $d(G) \leq 2, \delta(G) \geq 2$, by the above theorem, $\gamma_{stve}(G) \leq \delta(G)$.

Theorem 2.8. *For a graph G,*

$$\gamma_{stve}(G) \geq \left\lceil \frac{3q}{6(\Delta-1)^2+1} \right\rceil.$$

Proof. Let S be an stved - set for G. For edge uv in G, define $f : E \to [0,1]$ by

$$f(uv) = \frac{1}{l(u)+m(v)},$$

where

$$l(u) = |N[u] \cap S|, m(v) = |N[v] \cap S|$$

Let $v \in S$. For S is an stved - set of G, there is atleast one edge, av, in $< N[v] >$ that is dominated by a vertex of S different from v. So, $f(av) \leq \frac{1}{3}$. Then,

$$\begin{aligned}
\sum_{vx\in<N[v]>} f(vx) &= \sum_{vx\in<N[v]>-\{va\}} f(vx) + f(va) \\
&\leq \sum_{vx\in<N[v]>-\{va\}} d_G(vx) + \frac{1}{3} \\
&= \sum_{vx\in<N[v]>-\{va\}} [d_G(v) + d_G(x) - 2] + \frac{1}{3} \\
&\leq 2(\Delta-1)^2 + \frac{1}{3}.
\end{aligned}$$

Hence, each vertex in S dominates atmost $[6(\Delta-1)^2+1]/3$ edges in G. So, S dominates atmost $[6|S|(\Delta-1)^2+1]/3$ edges in G. This implies,

$$q \leq [6|S|(\Delta-1)^2+1]/3.$$

Hence the result. □

Note: The bound is sharp, as it is attained in the case of C_4.

Theorem 2.9. *For a graph G of order $p \geq 4$, $\gamma_{stve}(G) \leq \frac{p}{2}$.*

Proof. Assume that S is weakly connected dominating set for G. Since every edge in $< S >_w$ has atleast one end point is S and $< S >_w$ is the union of the closed stars at the vertices in S, any edge in G lies on a star centred on a vertex in S or has the neighbors of vertices(vertex) in S as its end points. This implies, S is a stved - set of G. Hence by Proposition 13.[4] the result holds. □

Note: The bound is sharp, as it is attained in the case of P_4.

Theorem 2.10. *If G is a graph with $p \geq 4$, then $\gamma_{stve}(G) = \frac{p}{2}$ if and only if G has a spanning subgraph isomorphic to P_4 or $K_{1,3}$.*

Proof. Assume that $\gamma_{stve}(G) = \frac{p}{2}$. Suppose $d(G) \geq 4$. Form a spanning tree $G^{'}$ of G. Let S be a weakly connected dominating set for $G^{'}$. $G^{''}$ be the resultant tree obtained by removing all the pendant vertices from $G^{'}$. If S is not a weakly connected dominating set for $G^{''}$, then atleast one of the pendant vertices is the unique vertex within a distance 2 to a vertex in S(say v). This implies, that there is an edge in $G^{''}$ which is not a member of $< S >_w$. So, S is not a weakly connected dominating set for $G^{'}$, a contradiction. Thus, S is a weakly connected dominating set for $G^{''}$. Clearly,

$$\gamma_w(G^{'}) \leq \frac{p-l}{2} < \frac{p}{2}$$

where l is the number of pendant vertices in $G^{'}$. Since $G^{'}$ is a spanning tree of G, $\gamma_{stve}(G) < \frac{p}{2}$, a contradiction to our assumption that $\gamma_{stve}(G) = \frac{p}{2}$. Hence, $d(G) \leq 3$.

If $d(G) = 3$ or $d(G) = 2$ and $p = 2n, n \geq 3$, by using the construction as above, we get a contradiction to our assumption. Also, if $d(G) = 3$ and $n = 2$, $G = P_4$.

Suppose $d(G) = 2, n = 2$. Then, from any vertex in G all the remaining vertices are at a distance atmost 2. Clearly, G has a spanning subgraph isomorphic to $K_{1,3}$.

Hence the result. □

Theorem 2.11. *For every connected graph G of diameter atleast 3, there is a $\gamma_{stve}(G) -$ set without leaves.*

Proof. The proof follows from the fact that, if any leaf is a member of a $\gamma_{stve}(G)-$ *set*, then it can be replaced by the vertex adjacent to it or by a non leaf vertex adjacent to its support vertex. □

Theorem 2.12. *For every connected graph G of diameter atleast 5, there is a $\gamma_{stve}(G) -$ set without leaves and support vertices.*

Proof. The proof follows from the fact that, if any leaf or support vertex is a member of a $\gamma_{stve}(G) -$ *set*, then it can be replaced by the non leaf vertex adjacent to the support vertex. □

In order to characterize the trees T for which $\gamma_{stve}(T) = (p-s-l+3)/3$(where p, s, l are the number of vertices, support vertices, leaves respectively), we define a family $\mathcal{T}$ of trees to consist of all trees T that can be obtained from a sequence $T_1, T_2, ..., T_k (k \geq 1)$ such that $T_1 =< v_1v_2v_3v_4v_5v_6v_7 >$. If $T = T_i (i \geq 1)$, then

T_{i+1} can be obtained recursively from T_i from one of the following operations. Let $A(T_1) = \{v_3, v_5\}$ and H be a path $P_6 =< v_8v_9v_{10}v_{11}v_{12}v_{13} >$ with $A(H) = \{v_9, v_{11}\}$.

- Operation $\mathcal{O}_1$: Attach a vertex by adding an edge to any support vertex of T_k. Let $A(T_{i+1}) = A(T_i)$.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining an edge from any vertex of P_2 to a vertex of T_k, which is not a leaf and is adjacent to a support vertex. Let $A(T_{i+1}) = A(T_i)$.
- Operation $\mathcal{O}_3$: Attach a copy of H by joining an edge from one of its leaves to a leaf of T_k adjacent to a weak support vertex. Let $A(T_{i+1}) = A(T_i) \cup A(H)$.
- Operation $\mathcal{O}_4$: Attach a path P_2 by joining an edge from a vertex in P_2 to a vertex of $A(H)$. Let $A(T_{i+1}) = A(T_i) \cup A(H)$.

Now, we prove that for every tree T of the family $\mathcal{T}$, we have $\gamma_{stve}(T) = (p - s - l + 3)/3$.

Theorem 2.13. *If $T \in \mathcal{T}$, then*

$$\gamma_{stve}(T) = \frac{p - s - l + 3}{3}.$$

Proof. We use the terminology mentioned in the construction of the family of trees, $\mathcal{T}$. To show that $\gamma_{stve}(T) = (p - s - l + 3)/3$ for $T \in \mathcal{T}$, we use induction on the number of operations k performed in the construction of T. The property is true for $T_1 = P_7$. Suppose that the property is true for all trees constructed with $k - 1$ operations. Let $T = T_k$ with $k \geq 2$. S be a $\gamma_{srve}(G) - set$ and $T^{'} = T_{k-1}$. Assume that $T^{'}$ has $p^{'}$ vertices, $l^{'}$ leaves and $s^{'}$ support vertices.

If T is obtained from $T^{'}$ by using the operation $\mathcal{O}_1$, then $\gamma_{stve}(T^{'}) = \gamma_{stve}(T)$, $p = p^{'} + 1, s = s^{'}, l = l^{'} + 1$. By induction on $T^{'}$, $A(T^{'}) = A(T)$ is a $\gamma_{stve}(T) - set$ of cardinality $(p - s - l + 3)/3$.

If T is obtained by using the operation $\mathcal{O}_2$, then $\gamma_{stve}(T^{'}) = \gamma_{stve}(T)$, $p = p^{'} + 2, s = s^{'} + 1, l = l^{'} + 1$. Again, by using induction on $T^{'}$, $A(T^{'}) = A(T)$ is a $\gamma_{stve}(T) - set$ of cardinality $(p - s - l + 3)/3$.

If T is obtained from $T^{'}$ by using the operation $\mathcal{O}_3$, then $\gamma_{stve}(T^{'}) = \gamma_{stve}(T)$, $p = p^{'} + 6, s = s^{'}, l = l^{'}$. Assume that a copy of H is added to a leaf adjacent to a weak support vertex in $T^{'}$. Since $A(T) = A(T^{'}) \cup \{v_9, v_{11}\}$ is a (unique) stved - set of G, we have $\gamma_{stve}(T) = \gamma_{stve}(T^{'}) + 2$. Also, it can be easily observed that $S^{'} = S - \{v_9, v_{11}\}$ is a (unique) stved - set of $T^{'}$. This implies, $\gamma_{stve}(T^{'}) \leq \gamma_{stve}(T) - 2$. It follows that $\gamma_{stve}(T) = \gamma_{stve}(T^{'}) + 2$ and $A(T)$ is a $\gamma_{stve}(T) - set$. By induction on $T^{'}$, it can be easily checked that cardinality of $A(T)$ is $(p - s - l)/3$.

Suppose that T is obtained from $T^{'}$ by using the operation $\mathcal{O}_4$. If T is obtained from $T^{'}$ by joining an edge from a vertex in P_2 to a vertex in A(H), then it is easy to check as in the case of T is obtained from $T^{'}$ by using the operation $\mathcal{O}_2$. $\square$

Theorem 2.14. *If T is a nontrivial tree of order p with l leaves and s support vertices, then*

$$\gamma_{stve}(T) \geq \frac{p-s-l+3}{3}$$

with equality if and only if $T \in \mathcal{T}$.

Proof. If $d(T) \leq 5$, then $(p-s-l+3)/3 < 2 = \gamma_{stve}(T)$. Assume that $d(T) \geq 6$. Thus the order p of the tree is atleast 7. Now, we obtain the result by using induction on the order p. Assume that the theorem is true for every tree of order $2 \leq p^{'} < p$ having $l^{'}$ pendant vertices and $s^{'}$ support vertices.

First assume that some support vertex of T, say x, is strong. Let y be the leaf adjacent to x. Let $T^{'} = T - \{y\}$. Then by Theorem 2.11., we have $\gamma_{stve}(T) = \gamma_{stve}(T^{'})$, $p^{'} = p-1, s^{'} = s, l^{'} = l-1$. By using inductive hypothesis on $T^{'}$, we have $\gamma_{stve}(T^{'}) \geq (p^{'} - s^{'} - l^{'} + 3)/3$. This implies, $\gamma_{stve}(T) = \gamma_{stve}(T^{'}) \geq (p^{'} - l^{'} - s^{'} + 3)/3 = (p-l-s+3)/3$. Further if $\gamma_{stve}(T) = (p-l-s+3)/3$, then obviously $\gamma_{stve}(T^{'}) = (p^{'} - l^{'} - s^{'} + 3)/3$ and $T^{'} \in \mathcal{T}$. The tree T can be obtained by adding a leaf to a support vertex of $T^{'}$. Thus T is obtained from $T^{'}$ by using operation $\mathcal{O}_1$. Henceforth, assume that every support vertex of T is weak.

Let us assume that $< v_1v_2v_3v_4v_5...v_{n-1}v_n >$ be a diammetral path in T. Since $d(T) \geq 6$, it follows that $n \geq 7$. Let us root the tree at v_1. By T_x, we mean the subtree induced by x and its descendents in the rooted tree T.

Assume that some child of v_3, say v, is a leaf. Let $T^{'} = T - \{v\}$. By Theorem 2.11., it follows that $\gamma_{stve}(T) = \gamma_{stve}(T^{'})$. Also, we have $p^{'} = p-1, l^{'} = l-1, s^{'} = s-1$. By using induction hypothesis on $T^{'}$, we have $\gamma_{stve}(T) = \gamma_{stve}(T^{'}) \geq (p^{'} - l^{'} - s^{'} + 3)/3 = (p-l-s+3)/3$. Furthermore, as in the above case observe that T is obtained from $T^{'}$ by using the operation $\mathcal{O}_1$.

Now assume that among the children of v_3, there is a support vertex, say v, other than v_2. Let $T^{'} = T - T_{v_2}$. We have $p^{'} = p-2, l^{'} = l-1, s^{'} = s-1$. We get $\gamma_{stve}(T) \geq \gamma_{stve}(T^{'}) \geq (p^{'} - l^{'} - s^{'} + 3)/3 = (p-s-l+3)/3$. If $\gamma_{stve}(T) = (p-s-l+3)/3$, then obviously $\gamma_{stve}(T^{'}) = (p^{'} - s^{'} - l^{'} + 3)/3$. By the induction hypothesis, $T^{'} \in \mathcal{T}$. The tree T can be obtained by adding a path P_2 to a vertex of $T^{'}$ adjacent to a support vertex of $T^{'}$. Thus T is obtained from $T^{'}$ by operation $\mathcal{O}_2$. If among the children of v_3, there are support vertices different from v, v_2, then also T can be obtained from $T^{'}$ by operation $\mathcal{O}_2$.

Now assume that $d_T(v_3) = 2$. Assume that $d_T(v_4) \geq 3$. Assume that some child of v_4, say v, is a leaf. Let $T^{'} = T - v$. Then, $p^{'} = p-1, s^{'} = s-1$ and $l^{'} = l-1$. This implies, $\gamma_{stve}(T) = \gamma_{stve}(T^{'}) \geq (p^{'} - l^{'} - s^{'} + 3)/3 = (p-l-s+3)/3$.

Now assume that no child of v_4 is a leaf. Observe that v_3 is a member of stved - set of the tree T.

case 1.: Suppose that v_4 is adjacent to paths P_2 and P_3, where P_2, P_3 are not paths on the diammetral path. Then, observe that v_4, u, where u is a vertex adjacent to v_4 on the path P_3, are members of stved -

set T. If S is an stved - set of T, then $S-\{v_3\}$ is an stved - set of $T^{'}=T-T_{v_3}$. Let $p^{'}=p-3, l^{'}=l-1, s^{'}=s-1$. This implies, $\gamma_{stve}(T)\geq\gamma_{stve}(T^{'})+1\geq(p^{'}-l^{'}-s^{'}+3)/3+1>(p-l-s+3)/3$.

case 2.: Suppose that v_4 is adjacent to k paths of length one, where $k\geq 1$ and not adjacent to any path of length 2. Observe that v_4 is a member of stved - set of T.

subcase 1.: Suppose v_3 is the unique vertex within a distance 2 from v_4. Observe that any stved - set consists v_3, v_4. Consider $T^{'}=T-T_{v_4}$. Clearly $p^{'}=p-2d_T(v_4), l^{'}=l-d_T(v_4)+2, s^{'}=s-d_T(v_4)+2$. Also, $S-\{v_3,v_4\}$ is an stved - set of $T^{'}$. We get $\gamma_{stve}(T)\geq\gamma_{stve}(T^{'})+2\geq(p^{'}-l^{'}-s^{'}+3)/3+2>(p-l-s+3)/3$.

subcase 2.: Suppose v_4 has a vertex within distance 2, different from v_3. Observe that any stved - set contains v_3. Consider $T^{'}=T-T_{v_3}$. Clearly $p^{'}=p-3, l^{'}=l-1, s^{'}=s-1$. Also, $S-\{v_3\}$ is an stved - set of $T^{'}$. We get $\gamma_{stve}(T)\geq\gamma_{stve}(T^{'})+1\geq(p^{'}-l^{'}-s^{'}+3)/3+2>(p-l-s+3)/3$.

case 3.: Suppose that v_4 is adjacent to k paths of length two, where $k\geq 1$ and not adjacent to any path of length two. Observe that v_3 is a member of stved - set of T.

subcase 1.: Suppose $k\geq 2$. Consider $T^{'}=T-T_{v_3}$. Clearly $p^{'}=p-3, l^{'}=l-1, s^{'}=s-1$. Also, $S-\{v_3\}$ is an stved - set of $T^{'}$. We get $\gamma_{stve}(T)\geq\gamma_{stve}(T^{'})+1\geq(p^{'}-l^{'}-s^{'}+3)/3+2>(p-l-s+3)/3$. Suppose v_4 is adjacent to exactly two paths of length 2, which are not vertex disjoint. Observe that v_3, u(where u is a vertex on the path of length 2, adjacent to v_4) are members an stved - set of T, say S. Suppose no other vertex in S is within a distance two from v_3 or u(where u is a vertex on the path of length 2, adjacent to v_4). Let $T^{'}=T-T_{v_4}$. Then $p^{'}=p-9, s^{'}=s-3, l^{'}=l-3$. As in the earlier case, we get, $\gamma_{stve}(T)\geq\gamma_{stve}(T^{'})+2\geq(p^{'}-l^{'}-s^{'}+3)/3+2>(p-l-s+3)/3$. If $p^{'}=0$, then observe that T is obtained from $T^{'}=P_7$ by joining a path P_2 to a non leaf vertex adjacent to a support vertex of P_7. This implies T is obtained from $T^{'}$ by the operation $\mathcal{O}_2$.

subcase 2.: Suppose $k=1$.

If v_4 is a member of an stved - set, say S, of T, then by considering $T^{'}=T-T_{v_3}$, we get, $\gamma_{stve}(T)\geq\gamma_{stve}(T^{'})+1\geq(p^{'}-l^{'}-s^{'}+3)/3+2>(p-l-s+3)/3$.

Suppose v_4 is not a member of an stved - set, say S, of T. If no other vertex in S is within a distance two from v_3 or u(where u is a vertex on the path of length 2, adjacent to v_4). Then by

> considering $T' = T - T_{v_4}$, we get, $\gamma_{stve}(T) \geq \gamma_{stve}(T') + 2 \geq (p' - l' - s' + 3)/3 + 2 > (p - l - s + 3)/3$.

Now assume that $d_T(v_4) = 2$. Assume that $d_T(v_5) \geq 3$. Assume that some child of v_5, say v, is a leaf. Let $T' = T - v$. Then, $p' = p - 1, s' = s - 1$ and $l' = l - 1$. This implies, $\gamma_{stve}(T) = \gamma_{stve}(T') \geq (p' - l' - s' + 3)/3 = (p - l - s + 3)/3$.

Now assume that no child of v_5 is a leaf. Observe that v_3, v_5 are members of any stved - set, say S, of the tree T. By considering various possibilities like in the earlier cases, we get, $\gamma_{stve}(T) \geq \gamma_{stve}(T') + 1 \geq (p' - l' - s' + 3)/3 + 2 > (p - l - s + 3)/3$ or $\gamma_{stve}(T) \geq \gamma_{stve}(T') + 2 \geq (p' - l' - s' + 3)/3 + 2 > (p - l - s + 3)/3$.

Continuing like this, assume that $d_T(v_6) = 2$ and $d_T(v_7) \geq 3$. Observe that any stved - set, say S, of the tree T contains the vertices v_3, v_5, v_7, also any path(outside the considered diammetral path) adjacent to v_7 will be of length atmost 5.

If v_7 is adjacent to path lengths of all possibilities, then consider $T' = T - T_{v_5}$. Clearly $p' = p - 5, l' = l - 1, s' = s - 1$. We get, $\gamma_{stve}(T) \geq \gamma_{stve}(T') + 2 \geq (p' - l' - s' + 3)/3 + 2 > (p - l - s + 3)/3$.

If v_7 is adjacent to paths of length one or three and not to adjacent paths of other possible lengths, then by considering $T' = T - T_{v_3}$, we get $\gamma_{stve}(T) \geq \gamma_{stve}(T') + 1 \geq (p' - m' - s' + 3)/3 + 2 > (p - m - s + 3)/3$.

Suppose v_7 is adjacent to a path of length 5, say $< uvwxyz >$. Let $T' = T - T_{v_5}$. Clearly $p' = p - 5, l' = l - 1, s' = s - 1$. If $p' = 8$, observe that v, x are also members of S and T is obtained from T' by adding a path P_6 to a leaf of T'. Hence T is obtained from T' by using the operation $\mathcal{O}_3$, also $T \in \mathcal{T}$.

Suppose $p' = 10$. Observe that v, x are also members of S. Consider the case in which there is a path of length one adjacent to x or v. Then, in this case T is obtained from T' by using the operation $\mathcal{O}_4$. □

Theorem 2.15. *Let S be a minimum stved - set of G. Then, there is a spanning tree T of G such that S is an stved - set of T.*

Proof. If G is a tree, then the result is trivial. Otherwise, consider a cycle C in G. Remove an edge from G as follows.

- **Case:1:** C has an edge uv, where $u, v \notin V - S$. Then, it can be easily observed that S is an $stved - set$ of $G - \{uv\}$.
- **Case:2:** C has an edge uv, where $u \in S$ and $v \notin S$. Suppose that u is a unique vertex, vertex edge dominating $vw(w \neq u)$. Then, there is an edge $vx(x \neq u)$ in C, such that $x \notin S$. Hence, Case:1 applies.
- **Case:3:** C has an edge uv, where $u, v \in S$. Then, it can be easily observed that S is an $stved - set$ of $G - \{uv\}$, for any edge uv in C.

Continue this process, until we obtain a spanning tree T of G. Observe that $\gamma_{stve}(T) \leq |S| = \gamma_{stve}(G)$. □

Theorem 2.16. *For a graph G with x cut - vertices and s support vertices,*

$$\gamma_{stve}(G) \geq \frac{x-s+3}{3}.$$

Moreover, this bound is sharp.

Proof. Let S be a $\gamma_{stve}(G) - set$. Form a spanning tree T of G, as in Theorem 2.15., so that S is an stved - set of T. Let $x(T)$ denote the number of cut - vertices of G. Since any cut - vertex of G is also a cut-vertex of T, we have $x(T) \geq x$. Now, by applying Theorem 2.14. to T, we have

$$\gamma_{stve}(G) \geq \gamma_{stve}(T) \geq \frac{p-l-s+3}{3} = \frac{x(T)-s+3}{3} \geq \frac{x-s+3}{3}.$$

□

Theorem 2.17. *For a connected graph G,* $\left\lceil \frac{diam(G)}{3} \right\rceil \leq \gamma_{stve}(G)$.

Proof. Let S be a $\gamma_{stve}(G) - set$. Any diammetral path in G includes at most two edges from $< N[v] >$, for each $v \in S$. Also, since S is a $\gamma_{stve}(G) - set$, the diammetral path in G includes at most $\gamma_{stve}(G) - 2$ edges joining the neighbourhoods of the vertices in S. Hence, $diam(G) \leq 2\gamma_{stve}(G) + \gamma_{stve}(G) - 2 + 2 = 3\gamma_{stve}(G)$. This completes the proof. □

Theorem 2.18. *For a connected graph G,*

$$\left\lceil \frac{2r(G)-1}{4} \right\rceil \leq \gamma_{stve}(G).$$

Proof. Let S be the minimum stved - set for G. Form a spanning tree T of G as in Theorem 2.15. Then, $\gamma_{stve}(T) \leq \gamma_{stve}(G)$. Since $r(G) \leq r(T)$ and $2r(T) - 1 \leq diam(T)$, by applying Theorem 2.16 to T, the proof follows. □

Theorem 2.19. *For a graph G, we have*

$$\frac{2}{3}\hat{r} \leq \gamma_{stve}(G).$$

Proof. By using the Theorem 2.15. and Theorem 2.17. , the proof follows. □

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