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A STUDY ON INVARIANT REGIONS, EXISTENCE AND UNIQUENESS OF THE GLOBAL SOLUTION FOR TRIDIAGONAL REACTION-DIFFUSION SYSTEMS

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Abstract. In this article, we are devoted to study the problem of the existence, uniqueness and positivity of the global solutions of the 3×3 reaction-diffusion systems with the total mass of the components with time. We also suppose that the nonlinear reaction term has a critical growth with respect to the gradient. The technique that we used to prove the global existence is the method of the compact semigroup.

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1. Introduction

In this article, we study the following mathematical model of the reaction-diffusion system using our techniques that are based on invariant regions and compact semigroup methods:

$$\begin{cases} \frac{\partial u}{\partial t} - a\Delta u - b\Delta v = f\left(t, x, u, v, w, \nabla u, \nabla v, \nabla w\right), & \text{in } Q_T, \\ \frac{\partial v}{\partial t} - c\Delta u - a\Delta v - b\Delta w = g\left(t, x, u, v, w, \nabla u, \nabla v, \nabla w\right), & \text{in } Q_T, \\ \frac{\partial w}{\partial t} - c\Delta v - a\Delta w = h\left(t, x, u, v, w, \nabla u, \nabla v, \nabla w\right), & \text{in } Q_T, \\ \frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \text{ or } u = v = w = 0, & \text{in } \Sigma_T, \\ u\left(0, x\right) = u_0\left(x\right), \; v\left(0, x\right) = v_0\left(x\right), \; w\left(0, x\right) = w_0\left(x\right), & \text{in } \Omega. \end{cases} \tag{1}$$

where Ω is an open bounded domain of a class C^1 in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $Q_T =]0, T[\times \Omega$, $\Sigma_T =]0, T] \times \partial\Omega$, $T > 0$, and the Laplacian operator on $L^1(\Omega)$ with Dirichlet or Neumann boundary conditions that are denoted by Δ [1, 2, 3]. The constants a, b and c are positive constants satisfying the condition

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$\sqrt{2}a \geq (b+c)$, which reflects the parabolicity of the system and makes the matrix of diffusion:

$$A = \begin{pmatrix} a & b & 0 \\ c & a & b \\ 0 & c & a \end{pmatrix},$$

to be positive definite with observing that the eigenvalues λ_1, λ_2, and λ_3 ($\lambda_1 < \lambda_2 < \lambda_3$) of its transposed are positive. In this regard of the 2×2 reaction-diffusion systems with diagonal matrix, we find that Alikakos [4] studied the global existence of the solutions for the semilinear system where:

$$f(u,v) = -g(u,v) = -uv^{\sigma}, \text{ and } 1 < \sigma < \frac{n+2}{n}.$$

Masuda [5] showed that the solutions exist globally to this system for every $\sigma > 1$. Haraux and Youkana [6] generalized the results for the semilinearities of $f(u,v) = g(u,v) = -u\Psi(v)$. Recently Kouachi and Youkana [7] generalized the method of Haraux and Youkana to the triangular case, i.e. when $b = 0$. In [8, 9], the authors obtained a global existence of the solutions for the coupled reaction-diffusion for the semilinear system with diagonal by ordering 2 and m of the matrix of diffusion coefficients. Rebiai and Benachour [10] treated the case of the system of the reaction-diffusion via the full matrix of diffusion coefficients with nonlinearities of exponential growth. Our presented article is based on the papers [11, 12, 13, 14, 15, 16, 17, 18]. In particular, we show the global existence and uniqueness results for the reaction-diffusion system with a tridiagonal matrix of diffusion coefficients coupled with critical growth with respect to $|\nabla U|$ $(m = 3)$.

Our methods are based on compact semigroup methods and invariant areas. Polynomial growth is intended to be the nonlinear response term. We truncate system (1), then provide appropriate estimates and demonstrate the approximating problem's convergence. Ultimately, by assuming that the starting data is as the following area, we will outline the formulation of the key results and submission:

$$\Sigma = \left\{(u_0, v_0, w_0) \in \mathbb{R}^3 \text{ such that } \sqrt{2\mu}\,|v_0| \leq u_0 + \mu w_0, \text{ and } u_0 \leq \mu w_0\right\}, \quad (2)$$

where $\mu = \frac{b}{c}$ such that

$$\begin{cases} f, g, h :]0, T[\, \times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{3N} \to \mathbb{R} \text{ are measurable,} \\ f, g, h : \mathbb{R}^3 \times \mathbb{R}^{3N} \to \mathbb{R} \text{ are locally Lipschitz continuous,} \end{cases} \quad (3)$$

and

$$\begin{cases} \left(cf - \sqrt{2bc}g + bh\right)\left(t, x, -\frac{\sqrt{2bc}}{c}v - \frac{b}{c}w, v, w, -\frac{\sqrt{2bc}}{c}q_2 - \frac{b}{c}q_3, q_2, q_3\right) \geq 0 \\ (-cf + bh)\left(t, x, u, v, \frac{c}{b}u, q_1, q_2, \frac{c}{b}q_1\right) \geq 0, \\ \left(cf + \sqrt{2bc}g + bh\right)\left(t, x, -\frac{\sqrt{2bc}}{c}v - \frac{b}{c}w, v, w, -\frac{\sqrt{2bc}}{c}q_2 - \frac{b}{c}q_3, q_2, q_3\right) \geq 0, \\ \text{for all } u, v, w \in \Sigma,\ q_1, q_2, q_3 \in \mathbb{R}^N \text{ and for a.e. } (t,x) \in Q_T, \end{cases} \quad (4)$$

with

$$\begin{cases} (cf++3bh)(t,x,u,v,w,q_1,q_2,q_3) \leq C\left(cu(t,x)+3bw(t,x)\right), \\ \text{for all } u,v,w \in \Sigma,\ q_1,q_2,q_3 \in \mathbb{R}^N \text{ and for a.e. } (t,x) \in Q_T. \end{cases} \tag{5}$$

In what follow, we list some lemmas that we will use in this article.

Lemma 1.1. *Let Ω be an open bounded domain in $\mathbb{R}^n$ and $X = L^1(\Omega) \cap H^2(\Omega)$. Then the operator B is m-dissipative in X and defined as follow:*

$$\begin{cases} D(B) = \left\{ u \in X,\ \ u|_{\partial\Omega} \ \ or \ \ \left.\frac{\partial u}{\partial \eta}\right|_{\partial\Omega} = 0 \right\}, \\ Bu = \Delta u,\ \forall u \in D(B). \end{cases}$$

Lemma 1.2. *Let $S(t)$ be a semigroup generating by m-dissipative operator A in the Banach space X, where F is locally Lipchitz function and $u_0 \in X$ represents the initial data, then the following problem:*

$$\begin{cases} u \in C\left([0,T], D(A)\right) \cap C^1\left([0,T], X\right), \\ \dfrac{du}{dt} - Au = F(t,x,u,\nabla u), \\ u(0) = u_0, \end{cases} \tag{6}$$

admits a unique solution u that verifies:

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau) F(\tau,.,u(\tau),\nabla u(\tau))\, d\tau,\ \forall t \in [0,T].$$

The following is how this paper is structured. The purpose of Section 2 is to review certain fundamental concepts and facts that needed to diagonalize the system. By examining the local existence, positivity and global existence of a specific solution to the primary problem, Section 3 illustrates the key findings of this work. Other key results are proven in Section 4, and the conclusion is summarized in the last section.

2. Diagonalisation of the system (1)

In this section, we will discuss the diagonalization of system 1 by supposing that the functions f, g and h are points into the region Σ on $\partial\Sigma$. Then for any (u_0, v_0, w_0) in Σ, the solution $(u(t,.), v(t,.), w(t,.))$ of problem (1) would remain in Σ for any time. The proof of this infer follows from the same way that used in [19, 20]. By multiplying $(1)_1$ by c, $(1)_2$ by $\sqrt{2bc}$ and $(1)_3$ by b, and then by adding the first result to the third one and subtracting the second result, we get $(7)_1$. Finally by subtracting the first result from the third one, we get $(7)_2$. If we add all of the three results to each other, we get the following system $(7)_3$:

$$\begin{cases} \frac{\partial u_1}{\partial t} - \lambda_1 \Delta u_1 = f_1(t,x,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3) & \text{in } Q_T, \\ \frac{\partial u_2}{\partial t} - \lambda_2 \Delta u_2 = f_2(t,x,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3) & \text{in } Q_T, \\ \frac{\partial u_3}{\partial t} - \lambda_3 \Delta u_3 = f_3(t,x,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3) & \text{in } Q_T, \\ \frac{\partial u_1}{\partial \eta} = \frac{\partial u_2}{\partial \eta} = \frac{\partial u_3}{\partial \eta} = 0 \text{ or } u_1 = u_2 = u_3 = 0 & \text{in } \Sigma_T, \\ u_1(0,x) = u_{1,0}(x),\ u_2(0,x) = u_{2,0}(x),\ u_3(0,x) = u_{3,0}(x), & \text{in } \Omega, \end{cases} \tag{7}$$

where

$$\begin{cases} u_1(t,x) = cu(t,x) - \sqrt{2bc}v(t,x) + bw(t,x) \\ u_2(t,x) = -cu(t,x) + bw(t,x) \\ u_3(t,x) = cu(t,x) + \sqrt{2bc}v(t,x) + bw(t,x), \end{cases} \tag{8}$$

for any (t,x) in Q_T and

$$\begin{aligned} f_1(t,x,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3) &= \left(cf - \sqrt{2bc}g + bh\right) \\ &\quad \times (t,x,u,v,w,\nabla u,\nabla v,\nabla w) \\ f_2(t,x,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3) &= (-cf + bh)(t,x,u,v,w,\nabla u,\nabla v,\nabla w) \\ f_3(t,x,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3) &= \left(cf + \sqrt{2bc}g + bh\right) \\ &\quad \times (t,x,u,v,w,\nabla u,\nabla v,\nabla w), \end{aligned}$$

for all (u,v,w) in Σ with

$$\begin{cases} \lambda_1 = a - \sqrt{2bc}, \\ \lambda_2 = a, \\ \lambda_3 = a + \sqrt{2bc}. \end{cases}$$

Suppose that the hypotheses (2)-(5) are satisfied, then problem (7) will be held for the following hypotheses:

$$u_{i_0} \text{ are positive functions in } L^1(\Omega), \text{for all } 1 \leq i \leq 3. \tag{9}$$

- For all $1 \leq i \leq 3$, we have

$$\begin{cases} f_i :]0,T[\times \Omega \times \mathbb{R}^3 \times \mathbb{R}^{3N} \to \mathbb{R} \text{ are measurable.} \\ f_i : \mathbb{R}^3 \times \mathbb{R}^{3N} \to \mathbb{R} \text{ are locally Lipschitz continuous.} \end{cases} \tag{10}$$

- The positivity of the solution is ensured by

$$\begin{cases} f_i(\hat{u}_i) \geq 0, \ 1 \leq i \leq 3 \text{ where } \hat{u}_1 = (t,x,0,u_2,u_3,0,p_2,p_3), \\ \hat{u}_2 = (t,x,u_1,0,u_3,p_1,0,p_3), \ \hat{u}_3 = (t,x,u_1,u_2,0,p_1,p_2,0) \\ (U,p) \in (\mathbb{R}^+)^3 \times \mathbb{R}^{3N} \text{ and for a.e. } (t,x) \in Q_T, \\ u_{i,0} \geq 0, \text{ for all } 1 \leq i \leq 3. \end{cases} \tag{11}$$

- It exists positive constant C independent of u_i, $i = \overline{1,3}$ such as:

$$\begin{cases} \sum\limits_{i=1}^{3} f_i(t,x,U,p) \leq C \sum\limits_{i=1}^{3} u_i, \\ \text{for all } (U,p) \in (\mathbb{R}^+)^3 \times \mathbb{R}^{3N} \text{ and a.e. } (t,x) \in Q_T. \end{cases} \tag{12}$$

3. Approximating problem

In this section, we will study the local existence, positivity and global existence of a unique solution for problem (13). Define the functions $u_{i_0}^n$, $i = 1,3$ for all $n > 0$. Then by using $u_{i_0}^n = \min(u_{i_0}, n)$, it is clear that $u_{i_0}^n$ verify (9), i.e.

$(u_{i_0}^n \in L^1(\Omega), \ u_{i_0}^n \geq 0, \ \forall i = \overline{1,3})$. Now, with solution $(u_{1_n}, u_{2_n}, u_{3_n})$, consider the following approximation of system (7):

$$\begin{cases} \frac{\partial u_{1_n}}{\partial t} - \lambda_1 \Delta u_{1_n} = f_1(t, x, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}) & \text{in }]0, T[\times \Omega, \\ \frac{\partial u_{2_n}}{\partial t} - \lambda_2 \Delta u_{2_n} = f_2(t, x, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}) & \text{in }]0, T[\times \Omega, \\ \frac{\partial u_{3_n}}{\partial t} - \lambda_3 \Delta u_{3_n} = f_3(t, x, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}) & \text{in }]0, T[\times \Omega, \\ \frac{\partial u_{i_n}}{\partial \eta} = 0, \text{ or } u_{i_n} = 0, \ i = \overline{1,3} & \text{on }]0, T[\times \partial\Omega, \\ u_{i_n}(0, x) = u_{i_0}^n(x) \geq 0, \ i = \overline{1,3} & \text{in } \Omega. \end{cases} \tag{13}$$

3.1. Local existence of a unique solution for problem (13). In this subsection, we convert problem (13) in the Banach space $X = \left(L^1(\Omega)\right)^3$ to an abstract first order-system of the form:

$$\begin{cases} \frac{\partial \omega_n}{\partial t} = A\omega_n + F(t, x, \omega_n, \nabla \omega_n) & \text{in } [0, T] \times \Omega, \\ \frac{\partial \omega_n}{\partial \eta} = 0 \text{ or } \omega_n = 0 & \text{in } [0, T] \times \partial\Omega, \\ \omega_n(0, .) = \omega_{0_n}(.) \in X & \text{in } \Omega. \end{cases} \tag{14}$$

Now, define $\omega_n = (u_{1_n}, u_{2_n}, u_{3_n})^t$, the operator A as $A = diag(\lambda_1 \Delta, \lambda_2 \Delta, \lambda_3 \Delta)$, where $D(A) := \{\omega_n \in X : \Delta \omega_n \in X\}$, and the function F as $F = (f_1, f_2, f_3)^t$. Then the system under consideration have a critical growth with respect to $|\nabla U|$ and $\omega_{0_n} = (u_{0_1}^n, u_{0_2}^n, u_{0_3}^n)^t$. Consequently, system (14) can be returned to the shape of system (7), and thus if $(u_{1_n}, u_{2_n}, u_{3_n})$ is a solution of (14), then it verifies the integral equations:

$$u_{i_n}(t) = S_i(t) u_{i_0}^n$$
$$+ \int_0^t S_i(t - \tau) f_i(\tau, u_{1_n}(\tau), u_{2_n}(\tau), u_{3_n}(\tau), \nabla u_{1_n}(\tau), \nabla u_{2_n}(\tau), \nabla u_{3_n}(\tau)) d\tau$$

where $S_i(t)$ is the semigroup generated by the operator $\lambda_i \Delta$, for all $i = \overline{1,3}$.

Theorem 3.1. *For $T_M > 0$, there exists a local unique solution ω_n of (14) for all $t \in [0, T_M]$.*

Proof. We know that $S_i(t)$ are semigroups of contraction and as F is locally Lipschitz in ω_n in the space X. So $\exists T_M > 0$ and a local solution ω_n of (14) on $[0, T_M]$. $\square$

3.2. Positivity of the unique solution for problem (13). In this subsection and by using the next result, we obtain the positivity of u_{i_n}, for all $1 \leq i \leq 3$.

Lemma 3.2. *Let $(u_{1_n}, u_{2_n}, u_{3_n})$ be the solution of problem (13) such that $u_{i_0}^n(x) \geq 0$, for all $i = \overline{1,3}$, $\forall x \in \Omega$. Then, $u_{i_n}(t, x) \geq 0$, for all $i = \overline{1,3}$, $\forall (t, x) \in (0, T) \times \Omega$.*

Proof. Let $\bar{u}_{i_n} = e^{-\sigma t} u_{i_n}$ and $\sigma > 0$, then we have:

$$\frac{\partial u_{i_n}}{\partial t} = e^{\sigma t} \left(\frac{\partial \bar{u}_{i_n}}{\partial t} + \sigma \bar{u}_{i_n} \right), \text{ for all } 1 \leq i \leq 3.$$

Consequently by problem (14) and for $1 \leq i \leq 3$, we have $\bar{u}_{i_n}$ is a solution of the system:

$$\begin{cases} \dfrac{\partial \bar{u}_{1_n}}{\partial t} + \sigma \bar{u}_{1_n} - \lambda_1 \Delta \bar{u}_{1_n} = e^{-\sigma t} f_1\left(t, x, \bar{u}_{1_n}, \bar{u}_{2_n}, \bar{u}_{3_n}, \nabla \bar{u}_{1_n}, \nabla \bar{u}_{2_n}, \nabla \bar{u}_{3_n}\right) \\ \dfrac{\partial \bar{u}_{2_n}}{\partial t} + \sigma \bar{u}_{2_n} - \lambda_2 \Delta \bar{u}_{2_n} = e^{-\sigma t} f_2\left(t, x, \bar{u}_{1_n}, \bar{u}_{2_n}, \bar{u}_{3_n}, \nabla \bar{u}_{1_n}, \nabla \bar{u}_{2_n}, \nabla \bar{u}_{3_n}\right) \\ \dfrac{\partial \bar{u}_{3_n}}{\partial t} + \sigma \bar{u}_{3_n} - \lambda_3 \Delta \bar{u}_{3_n} = e^{-\sigma t} f_3\left(t, x, \bar{u}_{1_n}, \bar{u}_{2_n}, \bar{u}_{3_n}, \nabla \bar{u}_{1_n}, \nabla \bar{u}_{2_n}, \nabla \bar{u}_{3_n}\right) \\ \dfrac{\partial \bar{u}_{i_n}}{\partial \eta} = 0 \text{ or } \bar{u}_{i_n} = 0\ ,\ 1 \leq i \leq 3, \text{ on }]0, T[\times \partial\Omega, \\ \bar{u}_{i_n}(0, x) = u_{i_0}^n(x) \geq 0\ ,\ 1 \leq i \leq 3 \text{ in } \Omega. \end{cases} \tag{15}$$

Let $U_0 = (t_0, x_0)$ be the minimum of $\bar{u}_{1_n}$ on $]0, T[\times \Omega$. We will show that $\bar{u}_{1_n}(U_0) \geq 0$ which will imply that $\bar{u}_{1_n} \geq 0$ on $]0, T[\times \Omega$ and then $u_n \geq 0$ on $]0, T[\times \Omega$. Suppose the contrary, namely $\bar{u}_{1_n}(U_0) < 0$. By the properties of the minimum, we can ensure that $U_0 \in]0, T] \times \Omega$ and

$$\begin{aligned} &\frac{\partial \bar{u}_{1_n}}{\partial t}(U_0) = 0,\ \nabla \bar{u}_{1_n}(U_0) = 0,\ \Delta \bar{u}_{1_n}(U_0) \geq 0 \quad \text{if } 0 < t_0 < T, \\ &\frac{\partial \bar{u}_{1_n}}{\partial t}(U_0) \leq 0,\ \nabla \bar{u}_{1_n}(U_0) = 0,\ \Delta \bar{u}_{1_n}(U_0) \geq 0 \quad \text{if } t_0 = T. \end{aligned}$$

Hence, the first equation in (15) yields to:

$$\begin{aligned} &\sigma \bar{u}_{1_n}(U_0) = -\frac{\partial \bar{u}_{1_n}}{\partial t}(U_0) + \lambda_1 \Delta \bar{u}_{1_n}(U_0) \\ &+ e^{-\sigma t_0} f_1\left(U_0, \bar{u}_{1_n}(U_0), \bar{u}_{2_n}(U_0), \bar{u}_{3_n}(U_0), 0, \nabla \bar{u}_{2_n}(U_0), \nabla \bar{u}_{3_n}(U_0)\right) \\ &\geq e^{-\sigma t_0} f_1\left(U_0, \bar{u}_{1_n}(U_0), \bar{u}_{2_n}(U_0), \bar{u}_{3_n}(U_0), 0, \nabla \bar{u}_{2_n}(U_0), \nabla \bar{u}_{3_n}(U_0)\right). \end{aligned}$$

Now, we use the structure of $\bar{u}_{1_n}(U_0)$ and hypothesis (11) to write:

$$\begin{aligned} &f_1\left(U_0, \bar{u}_{1_n}(U_0), \bar{u}_{2_n}(U_0), \bar{u}_{3_n}(U_0), 0, \nabla \bar{u}_{2_n}, \nabla \bar{u}_{3_n}\right) \\ &\quad = f_1\left(U_0, 0, \bar{u}_{2_n}(U_0), \bar{u}_{3_n}(U_0), 0, \nabla \bar{u}_{2_n}(U_0), \nabla \bar{u}_{3_n}(U_0)\right) \geq 0. \end{aligned}$$

This implies that $\bar{u}_{1_n}(U_0) \geq 0$, which is impossible by the hypotheses. Arguing in the same way for the others component $\bar{u}_{k_n}$, for all $2 \leq k \leq 3$, we obtain the positivity of u_{i_n}, for all $1 \leq i \leq 3$. □

3.3. Global existence of the unique solution for problem (13). The following result will be provided in this part to demonstrate the existence of an estimated solution to problem (13) in $L^1(\Omega)$ for every $t \geq 0$. As a result, we can then conclude that the solution $(u_{1_n}, u_{2_n}, u_{3_n})$ given by Theorem 3.1 is a global solution.

Lemma 3.3. *Let $(u_{1_n}, u_{2_n}, u_{3_n})$ be the solution of problem (13), then it exists $M(t)$, which depends only of t, such that for all $0 \leq t \leq T_M$, we have:*

$$\left\| \sum_{i=1}^{3} u_{i_n}(t) \right\|_{L^1(\Omega)} \leq M(t).$$

Proof. By adding the equations of (13), we obtain:

$$\frac{\partial}{\partial t}\sum_{i=1}^{3} u_{i_n} - \sum_{i=1}^{3} \lambda_i \Delta u_{i_n} = \sum_{i=1}^{3} f_i\left(t, x, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right).$$

By taking into account (12), we have:

$$\frac{\partial}{\partial t}\sum_{i=1}^{3} u_{i_n} - \sum_{i=1}^{3} \lambda_i \Delta u_{i_n} \leq C \sum_{i=1}^{3} u_{i_n}.$$

By integrating on Ω, we can have:

$$\frac{\partial}{\partial t}\int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx - \int_{\Omega}\sum_{i=1}^{3} \lambda_i \Delta u_{i_n} dx \leq C \int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx.$$

Now, applying the formula of Green yields:

$$\frac{\partial}{\partial t}\int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx \leq C \int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx.$$

Consequently, we get:

$$\frac{\frac{\partial}{\partial t}\int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx}{\int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx} \leq C.$$

By integrating on $[0, t]$, we find:

$$\ln \int_{\Omega}\sum_{i=1}^{3} u_{i_n} dx \bigg|_0^t \leq Ct,$$

which implies

$$\ln \frac{\int_{\Omega}\sum_{i=1}^{3} u_{i_n}(t)\, dx}{\int_{\Omega}\sum_{i=1}^{3} u_{i_0}^n dx} \leq Ct.$$

This gives:

$$\frac{\int_{\Omega}\sum_{i=1}^{3} u_{i_n}(t)\, dx}{\int_{\Omega}\sum_{i=1}^{3} u_{i_0}^n dx} \leq \exp(Ct).$$

Then, we have:

$$\int_{\Omega}\sum_{i=1}^{3}u_{i_n}(t)\,dx \leq \exp(Ct)\int_{\Omega}\sum_{i=1}^{3}u_{i_0}^{n}dx,$$

i.e.

$$\int_{\Omega}\sum_{i=1}^{3}u_{i_n}(t)\,dx \leq \exp(Ct)\int_{\Omega}\sum_{i=1}^{3}u_{i_0}dx, \text{ as if } u_{i_0}^{n} \leq u_{i_0}.$$

Let us now put:

$$M(t) = \exp(Ct)\left\|\sum_{i=1}^{3}u_{i_0}\right\|_{L^1(\Omega)},$$

As u_{i_n} are positives, then we get:

$$\left\|\sum_{i=1}^{3}u_{i_n}(t)\right\|_{L^1(\Omega)} \leq M(t),\ 0 \leq t \leq T_M.$$

□

Lemma 3.4. *For any solution* $(u_{1_n}, u_{2_n}, u_{3_n})$ *of (13), there exists a constant* $K(t)$*, which depends only of* t*, such that:*

$$\left\|\sum_{i=1}^{3}u_{i_n}\right\|_{L^1(Q_T)} \leq K(t)\left\|\sum_{i=1}^{3}u_{i_0}\right\|_{L^1(\Omega)}.$$

Proof. To prove this lemma, we use the following results given in Bonafede and Schmitt [21]). So, we introduce $\theta \in C_0^{\infty}(Q_T)$ such that $\theta \geq 0$ and a nonnegative solution $\Phi \in C^{1,2}(Q_T)$ of the following system:

$$\begin{cases} -\Phi_t - d\Delta\Phi = \theta & \text{on } Q_T, \\ \frac{\partial\Phi}{\partial\eta} = 0 \text{ or } \Phi(t,.) = 0 & \text{on } [0,T[\times\partial\Omega, \\ \Phi(T,.) = 0 & \text{on } \Omega. \end{cases}$$

For all $q \in]1,\infty[$, $q' \in]1,\infty]$ such that:

$$q \leq q' \text{ and } 2 - \left(\frac{1}{q} - \frac{1}{q'}\right)(n+2) > 0,$$

there exists a non-negative constant C, independent of θ, such that:

$$\|\Phi\|_{L^{q'}(Q_T)} \leq C\|\theta\|_{L^q(Q_T)}. \tag{16}$$

According to Bonafede and Schmitt [21], we have:

$$\int_{Q_T}S_i(t)\,u_{i_0}^{n}(x)\left(-\frac{\partial\Phi}{\partial t} - d\Delta\Phi\right)dxdt = \int_{\Omega}u_{i_0}^{n}(x)\,\Phi(0,x)\,dx,$$

and

$$\int_{Q_T} \left(\int_0^t S_i\left(t-\tau\right) f_i\left(\tau, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) d\tau \right) \times \left(-\frac{\partial \Phi}{\partial t} - d\Delta\Phi \right) dxdt = \int_{Q_T} f_i\left(\tau, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) \Phi\left(\tau, x\right) dxd\tau,$$

where

$$\int_{Q_T} \left(S_i\left(t\right) u_{i_0}^n\left(x\right) \right) \theta dxdt = \int_{\Omega} u_{i_0}^n\left(x\right) \Phi\left(0, x\right) dx, \tag{17}$$

and

$$\int_{Q_T} \left(\int_0^t S_i\left(t-\tau\right) f_i\left(\tau, ., u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) d\tau \right) \theta dxdt = \int_{Q_T} f_i\left(\tau, u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) \Phi\left(\tau, x\right) dxd\tau.$$

By multiply the equation of (3.1) by θ, integrating the result on Q_T, and then by using (17) and (3.3), we obtain:

$$\begin{aligned} \int_{Q_T} u_{i_n} \theta dxdt &= \int_{Q_T} S_i\left(t\right) u_{i_0}^n\left(x\right) \theta dxdt \\ &+ \int_{Q_T} \left(\int_0^t S_i\left(t-\tau\right) f_i\left(\tau, ., u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) d\tau \right) \theta dxdt, \\ &= \int_{\Omega} u_{i_0}^n\left(x\right) \Phi\left(0, x\right) dx \\ &+ \int_{Q_T} f_i\left(\tau, ., u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) \Phi\left(\tau, x\right) dxd\tau, \; i = \overline{1,3}. \end{aligned}$$

Therefore, we can have

$$\int_{Q_T} \sum_{i=1}^{3} u_{i_n} \theta dxdt = \int_{\Omega} \sum_{i=1}^{3} u_{i_0}^n\left(x\right) \Phi\left(0, x\right) dx + \int_{Q_T} \sum_{i=1}^{3} f_i\left(\tau, ., u_{1_n}, u_{2_n}, u_{3_n}, \nabla u_{1_n}, \nabla u_{2_n}, \nabla u_{3_n}\right) \Phi\left(\tau, x\right) dxd\tau.$$

According to (12) and as $u_{i_0}^n \leq u_{i_0}$, we can have:

$$\int_{Q_T} \sum_{i=1}^{3} u_{i_n} \theta dxdt \leq \int_{\Omega} \sum_{i=1}^{3} u_{i_0}\left(x\right) \Phi\left(0, x\right) dx + \int_{Q_T} C \sum_{i=1}^{3} u_{i_n} \Phi\left(\tau, x\right) dxd\tau.$$

By using Hölder inequality, we can deduce:

$$\begin{aligned}
&\int_{Q_T}\sum_{i=1}^{3} u_{i_n}\theta dxdt \\
&\quad\leq \left\|\sum_{i=1}^{3} u_{i_0}\right\|_{L^1(\Omega)} . \left\|\Phi\left(0,.\right)\right\|_{L^\infty(Q_T)} + C\left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)} . \left\|\Phi\right\|_{L^\infty(Q_T)} \\
&\quad\leq \left(\left\|\sum_{i=1}^{3} u_{i_0}\right\|_{L^1(\Omega)} + C\left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)}\right) . \left\|\Phi\right\|_{L^\infty(Q_T)} \\
&\quad\leq \max\left(1, C\right)\left(\left\|\sum_{i=1}^{3} u_{i_0}\right\|_{L^1(\Omega)} + \left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)}\right) . \left\|\Phi\right\|_{L^\infty(Q_T)} .
\end{aligned}$$

Also, using (16) yields:

$$\int_{Q_T}\sum_{i=1}^{3} u_{i_n}\theta dxdt \leq k_1\left(t\right)\left(\left\|\sum_{i=1}^{3} u_{i_0}\right\|_{L^1(\Omega)} + \left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)}\right) . \left\|\theta\right\|_{L^\infty(Q_T)} ,$$

where $k_1\left(t\right) \geq \max\left(c, cC\right)$. Since θ is arbitrary in $C_0^\infty\left(Q_T\right)$, then we have:

$$\left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)} \leq k_1\left(t\right)\left(\left\|\sum_{i=1}^{3} u_{i_0}\right\|_{L^1(\Omega)} + \left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)}\right).$$

Now, by taking $k\left(t\right) = \frac{k_1(t)}{1-k_1(t)}$, we find:

$$\left\|\sum_{i=1}^{3} u_{i_n}\right\|_{L^1(Q_T)} \leq k\left(t\right)\left\|\sum_{i=1}^{3} u_{i_0}\right\|_{L^1(\Omega)} .$$

□

4. Proof of the main result

In this section, we will present the main result (Theorem 4.1), and proof it to show the statement of the result for this paper.

Theorem 4.1. *Assume that the hypotheses (9)-(12) are satisfied. Then, there exists a unique solution* u_i, $i = \overline{1,3}$ *of system (7) in the sense of*

$$\left\{\begin{array}{l} u_i \in C\left(\left[0,T\right], L^1\left(\Omega\right)\right) \cap L^1\left(0, T, w_0^{1,1}\left(\Omega\right)\right),\ i = \overline{1,3}, \\ f_i\left(t, x, \omega, \nabla\omega\right) \in L^1\left(Q_T\right) \text{ for all } T > 0, \\ u_i\left(t\right) = S_i\left(t\right)u_{i_0} + \int_0^t S_i\left(t-\tau\right)f_i\left(\tau, \omega\left(\tau\right), \nabla\omega\left(\tau\right)\right)d\tau, i = \overline{1,3}, \forall t \in \left[0, T\right[, \end{array}\right. \tag{18}$$

where $\omega = \left(u_1, u_2, u_3\right)$ *and* $S_i\left(t\right)$ *are the semigroups of contractions in* $L^1\left(\Omega\right)$ *generated by* $\lambda_i\Delta$, $i = \overline{1,3}$.

Proof. Let $S(t)$ be a compact semigroup of contraction engendered by the operator $d\Delta$. Define the application L by:

$$L : (w_0, h) \to S(t)\, w_0 + \int_0^t S(t-\tau)\, h(\tau, ., w(\tau), \nabla w(\tau))\, d\tau.$$

The application L is compact $L^1(Q_T) \times L^1(Q_T)$ in $L^1(Q_T)$ because it is adding two compact applications in $L^1(Q_T)$, see the result in [11, 12, 13, 14]. Therefore, there is a subsequence $\left(u^j_{1_n}, u^j_{2_n}, u^j_{3_n}\right)$ of $(u_{1_n}, u_{2_n}, u_{3_n})$ and (u_1, u_2, u_3) of $\left(L^1(Q_T)\right)^3$ such that $\left(u^j_{1_n}, u^j_{2_n}, u^j_{3_n}\right)$ converges towards (u_1, u_2, u_3). Let us now show that $\left(u^j_{1_n}, u^j_{2_n}, u^j_{3_n}\right)$ is a solution of (3.1). We have:

$$u^j_{i_n}(t,x) = S_i(t)\, u^j_{i_0} + \int_0^t S_i(t-\tau)\, f_i\left(\tau, ., u^j_{1_n}, u^j_{2_n}, u^j_{3_n}, \nabla u^j_{1_n}, \nabla u^j_{2_n}, \nabla u^j_{3_n}\right) d\tau, \tag{19}$$

for $i = \overline{1,3}$. So it is enough to show that (u_1, u_2, u_3) verifies (18). To this aim, we should first notice that if $j \to +\infty$, we have the following limits:

$$\begin{cases} f_i\left(\tau, ., u^j_{1_n}, u^j_{2_n}, u^j_{3_n}, \nabla u^j_{1_n}, \nabla u^j_{2_n}, \nabla u^j_{3_n}\right) \to f_i(\tau, ., u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3) \\ \text{a.e, } i = \overline{1,3}, \end{cases} \tag{20}$$

and

$$u^j_{i_0} \to u_{i_0},\ i = \overline{1,3}.$$

Thus, to show that (u_1, u_2, u_3) verifies (18), it remains to show that:

$$\begin{cases} f_i\left(\tau, ., u^j_{1_n}, u^j_{2_n}, u^j_{3_n}, \nabla u^j_{1_n}, \nabla u^j_{2_n}, \nabla u^j_{3_n}\right) \to f_i(\tau, ., u_1, u_2, u_3, \nabla u_1, \nabla u_2, \nabla u_3) \\ i = \overline{1,3}, \text{ in } L^1(Q_T) \text{ when } j \to +\infty. \end{cases}$$

We integrate the equations of (13) on Q_T by taking into account that:

$$-\lambda_i \int_{Q_T} \Delta u^j_{i_n}\, dxdt = 0,\ i = \overline{1,3}.$$

Then, we have:

$$\begin{aligned} &\int_\Omega u^j_{i_n} dx - \int_\Omega u^j_{i_0} dx \\ &= \int_{Q_T} f_i\left(\tau, ., u^j_{1_n}(t), u^j_{2_n}(t), u^j_{3_n}(t), \nabla u^j_{1_n}(t), \nabla u^j_{2_n}(t), \nabla u^j_{3_n}(t)\right) dxd\tau, \end{aligned}$$

where

$$\begin{aligned} -\int_{Q_T} f_i\left(\tau, ., u^j_{1_n}(t), u^j_{2_n}(t), u^j_{3_n}(t), \nabla u^j_{1_n}(t), \nabla u^j_{2_n}(t), \nabla u^j_{3_n}(t)\right) dxd\tau \\ \leq \int_\Omega u_{i_0} dx,\ \text{ for } \ i = \overline{1,3}. \end{aligned} \tag{21}$$

Now, let us put

$$\Upsilon_{i_n} = C\sum_{i=1}^{3} u_{i_n}^j - f_i\left(t,.,u_{1_n}^j(t),u_{2_n}^j(t),u_{3_n}^j(t),\nabla u_{1_n}^j(t),\nabla u_{2_n}^j(t),\nabla u_{3_n}^j(t)\right),$$

for $i=\overline{1,3}$. It is clear that Υ_{i_n} is positives. Now, according to (12) of (21), we obtain:

$$\int_{Q_T} \Upsilon_{i_n} dx d\tau \leq C\int_{Q_T} \sum_{i=1}^{3} u_{i_n}^j dx d\tau + \int_{\Omega} u_{i_0} dx.$$

Thus, Lemma 3.4 gives us:

$$\int_{Q_T} \Upsilon_{i_n} dx d\tau < +\infty,$$

which implies:

$$\begin{aligned}\int_{Q_T} &\left| f_i\left(\tau,.,u_{1_n}^j(t),u_{2_n}^j(t),u_{3_n}^j(t),\nabla u_{1_n}^j(t),\nabla u_{2_n}^j(t),\nabla u_{3_n}^j(t)\right)\right| dx d\tau \\ &\leq C\int_{Q_T} \sum_{i=1}^{3} u_{i_n}^j dx d\tau + \int_{Q_T} \Upsilon_{i_n} dx d\tau < +\infty.\end{aligned}$$

Now, let

$$\Psi_{i_n} = \Upsilon_{i_n} + C\sum_{i=1}^{3} u_{i_n}^j,\ i=\overline{1,3},$$

then Ψ_{i_n} are positives in $L^1(Q_T)$ and furthermore we have:

$$\left| f_i\left(t,.,u_{1_n}^j(t),u_{2_n}^j(t),u_{3_n}^j(t),\nabla u_{1_n}^j(t),\nabla u_{2_n}^j(t),\nabla u_{3_n}^j(t)\right)\right| \leq \Psi_{i_n} \text{ a.e,}$$

for $i=\overline{1,3}$. Let us combine this result with (20), then by dominated convergence, we obtain:

$$f_i\left(t,.,u_{1_n}^j,u_{2_n}^j,u_{3_n}^j,\nabla u_{1_n}^j,\nabla u_{2_n}^j,\nabla u_{3_n}^j\right) \to f_i(t,.,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3)$$

in $L^1(Q_T)$. By passing in the limit $j\to+\infty$ of (19) in $L^1(Q_T)$, we find:

$$u_i(t) = S_i(t)u_{i_0} + \int_0^t S_i(t-\tau) f_i(\tau,.,u_1,u_2,u_3,\nabla u_1,\nabla u_2,\nabla u_3)\, d\tau,\ i=\overline{1,3}.$$

Then (u_1,u_2,u_3) verify (18), and consequently (u_1,u_2,u_3) is the solution of (7). Hence, we conclude by (8) that there exists a unique global solution to the system (1). $\square$

Example 4.2. The result of system (7) can be applied to the following typical example:

$$\left\{\begin{array}{ll} \dfrac{\partial u_i}{\partial t} - d_i \Delta u_i = \sum\limits_{1 \le j \le i} a_{ij} \dfrac{u_j}{\sum\limits_{1 \le k \le 3} u_k} \left|\nabla u_j\right|^2 + f_i(t,x) & \text{in } Q_T \\ \frac{\partial u_i}{\partial \eta} \text{ or } u_i = 0 & \text{on } \Sigma_T \\ u_i(0,x) = u_{i,0}\,(x) & \text{in } \Omega \end{array}\right. , \text{ for } 1 \le i \le 3$$

5. Conclusion

With the use of utilizing the compact semigroup and invariant areas techniques, this article has examined and studied invariant areas for tridiagonal reaction-diffusion systems as well as the possibility of singular global solutions.

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