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ENERGY DECAY FOR A VISCOELASTIC EQUATION WITH BALAKRISHNAN-TAYLOR DAMPING INVOLVING INFINITE MEMORY AND NONLINEAR TIME-VARYING DELAY TERMS IN DYNAMICAL BOUNDARY

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Abstract. In this paper, we study the initial-boundary value problem for viscoelastic wave equations of Kirchhoff type with Balakrishnan–Taylor damping terms in the presence of the infinite memory and external time-varying delay. For a certain class of relaxation functions and certain initial data, we prove that the decay rate of the solution energy is similar to that of relaxation function which is not necessarily of exponential or polynomial type. Also, we show another stability with g satisfying some general growth at infinity.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2. Her Γ_0 and Γ_1 are closed and disjoint, with $\Gamma_0 \neq \emptyset$, ν be the outward normal to Γ. In this paper we investigate general decay results of the energy for a viscoelastic problem with Balakrishnan-Taylor damping, infinite memory and nonlinear time varying delay terms in dynamical boundary conditions:

$$
(1)\quad \begin{aligned}
& u_{tt} - \left(a + b\|\nabla u\|^2 + \sigma \int_\Omega \nabla u \nabla u_t \mathrm{d}x\right) \Delta u \\
& \quad + \int_0^{+\infty} g(s)\Delta u(t-s)\mathrm{d}s + f(u) = 0 && \text{in } \Omega \times (0,\infty), \\
& u(x,t) = 0 && \text{on } \Gamma_1 \times (0,\infty), \\
& \frac{\partial u}{\partial \nu}(x,t) - \int_0^{+\infty} g(s) \frac{\partial}{\partial \nu} u(x,t-s)\,\mathrm{d}s \\
& \quad + \mu_1 h_1(u_t(x,t)) + \mu_2 h_2(u_t(x,t-\tau(t))) = 0 && \text{on } \Gamma_0 \times (0,+\infty),
\end{aligned}
$$

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$$\begin{aligned}
&u(x,-t)=u_0(x,t) && \text{in } \Omega\times(0,\infty),\\
&u_t(x,0)=u_1(x) && \text{in } \Omega,\\
&u_t(x,t)=j_0(x,t) && \text{on } \Gamma_0\times(-\tau(0),0),
\end{aligned}$$

where a, b, σ are fixed positive constants, $\mu_1 > 0$, $\mu_2 \neq 0$, g and f are given functions, $\tau(t) > 0$ represents the time delay. Problem (1), from the physical point of view, with $g = 0$ and $\mu_1 = \mu_2 = 0$, is the model with Balakrishnan-Taylor damping ($\sigma > 0$) which has been described initially by Balakrishnan and Taylor [1], and treated by Bass and Zes [2]. It is well known in the literature that it is related to the panel flutter equation (the "spillover" problem) and arises from a wind tunnel experiment at supersonic speeds. To a certain extent it has been studied by many authors such as [16, 21] and in some references therein. When $\mu_1 = \mu_2 = 0$ in (1) with finite memory, several authors have studied the existence of the solutions and stability of the corresponding energy. For example, Tatar and Zarai [24,25] showed polynomial/exponential decay results under the classical condition of g. Recently, Park [21] proved arbitrary decay rates without imposing the usual known relations $g'(t) \leq -\zeta(t)\, g(t)$. Considering the infinite memory, and time-varying delay term $\mu_2 h_2(u_t(x, t-\tau(t)))$ in boundary feedback, the problem is different from those of existing literature. Time delays arise in many physical, chemical, biological, thermal and economical phenomena because these phenomena depend not only on the present state but also on the past history of the system in a more complicated way (see, for example, [5, 9, 14]). In recent years, the systems with time delay effects have become an active area of research, see for example [17, 22] and the references therein. In [4], the authors showed that a small delay in boundary control is a source of instability. To stabilize a hyperbolic system involving input delay, additional conditions or control terms have been used. For instance, consider a wave equation with a delay of the form

$$u_{tt}(x,t)-\Delta u(x,t)+\mu_0\sigma(t)h_1(u_t(x,t))+\mu_1\sigma(t)h_2(u_t(x,t-\tau(t)))=0. \tag{2}$$

Nicaise and Pignotti [18] proved that the energy of the problem is exponentially stable when $\sigma(t) = 1$, μ_0, $\mu_1 > 0$, $\tau(t) = \tau$ (constant), and $h_1(v) = h_2(v) = v$. On the other hand, the case of time-varying delay in the wave equation in 1-dimensional space has been studied recently by Nicaise et al. In [20] the authors proved an exponential stability result under the condition $0 < \mu_2 < \sqrt{1-d}\mu_1$, where the function $\tau(t)$ satisfies $\tau'(t) \leq d, \forall t > 0$ for the constant $d < 1$. Several authors studied a nonlinear viscoelastic wave equation with strong damping, time-varying delay, and dynamical boundary conditions, in which they proved a general decay result for the energy, from which the usual exponential and polynomial decay rates only occur in particular cases. For the related problems, we also refer to [3, 12]. For Balakrishnan-Taylor problem with delay, Jum-Ran Kang et al. [11] studied the following equations with Dirichlet boundary value,

$$u_{tt}-\left(a+b\|\nabla u\|^2+\sigma\int_\Omega \nabla u\nabla u_t \mathrm{d}x\right)\Delta u \tag{3}$$

$$+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_1 h_1\left(u_t\left(x,t\right)\right)+\mu_2 h_2\left(u_t\left(x,t-\tau\left(t\right)\right)\right)=0.$$

By dropping the restriction $\mu_2>0$, the authors investigate the general decay rates of energy for Problem (3) by establishing suitable Lyapunov functionals that are equivalent to the corresponding energy, and they improve those existing results. Jianghao Hao in [10] examined the following system

$$\begin{aligned}(4)\qquad & u_{tt}-\left(a+b\|\nabla u\|^2+\sigma\int_\Omega \nabla u\nabla u_t\mathrm{d}x\right)\Delta u\\ &+\int_0^t g(t-s)\Delta u(s)\mathrm{d}s+\mu_0 u_t\left(x,t\right)+\mu_1 u_t\left(x,t-\tau\left(t\right)\right)+f\left(u\right)=0,\end{aligned}$$

and they established general decay estimates of the energy to the solution. Guesmia [7] studied the following infinite memory problem

$$(5)\qquad u_{tt}(t)+Au(t)+\int_0^\infty g(s)Bu(t-s)\mathrm{d}s=0,\quad \forall t>0,$$

and established a general decay estimate of the energy with g satisfying the following general growth at infinity

$$\int_0^{+\infty}\frac{g(s)}{G^{-1}\left(-g'(s)\right)}\mathrm{d}s+\sup_{x\in\mathbb{R}_+}\frac{g(s)}{G^{-1}\left(-g'(s)\right)}<+\infty$$

with $G:\mathbb{R}_+\to\mathbb{R}_+$ is an increasing strictly convex function of class $C^1\left(\mathbb{R}_+\right)\cap C^2((0,\infty))$ and $G(0)=G'(0)=0$ and $\lim_{t\to+\infty}G'(t)=+\infty$. In [8], the author proved the exponential stability of (5), in the case $A=B$ and in the presence of delay term $\mu u_t(t-\tau)$, $\mu\in\mathbb{R}^*$, for problems of a past history see also [13,15]. Recently, Pignotti [23] considered the following problem

$$u_{tt}+Au(t,x)-\int_0^{+\infty}\mu(s)\Delta u(t-s,x)\mathrm{d}s+b(t)u_t(t-\tau,x)=0,$$

and he established a general decay estimate. To the best of our knowledge, this is the first result dealing with equation (1) subject to the interaction of the infinite memory term with Balakrishnan–Taylor damping and external time-varying delay type and presenting general decay. Motivated by these results, and by constructing Lyapunov functionals which are equivalent to the corresponding energy, we will investigate some general decay rates of energy for Problem (1). The first fundamental stability result is given without imposing any restrictive growth assumption on the function g and damping term, and the second result is given with a relation between the damping term and relaxation function. This paper is organized as follows. In Section 2, we give some assumptions that will be needed for our work and state the main results. We establish the two general decay results of the energy in Section 3.

2. Preliminaries and main results

In this section, we present some material that we shall use in order to present our results. We use the notation

$$(u,v)=\int_\Omega u(x,t)v(x,t)\mathrm{d}x \quad\text{and}\quad (u,v)_{\Gamma_0}=\int_{\Gamma_0} u\,(x)\,v\,(x)\,\mathrm{d}\Gamma,$$

and we mean by $\|\cdot\|_2$ the $L^2(\Omega)$ norm, and by $\|\cdot\|_{\Gamma_0}$ the $L^2(\Gamma_0)$ norm. Also we denote by

$$H^1_{\Gamma_1}(\Omega)=\left\{u\in H^1(\Omega):u=0\text{ on }\Gamma_1\right\},$$

the closed subspace of $H^1(\Omega)$ equipped with the norm equivalent to the usual norm in $H^1(\Omega)$. The Poincaré inequality holds on $H^1_{\Gamma_1}(\Omega)$, i.e., there exits a constant C_* such that:

$$\forall u\in H^1_{\Gamma_1}(\Omega),\ \|u(t)\|_2\le C_*\|\nabla u(t)\|_2,$$

and there exists a constant $\bar{C}_*>0$ such that

$$\|u\|_{\Gamma_0}\le\bar{C}_*\|\nabla u\|_2\ \text{ for all } u\in H^1_{\Gamma_1}(\Omega).$$

For studying Problem (1), we will need the following assumptions.

(H1) Hypotheses on g: $g:\mathbb{R}^+\to\mathbb{R}^+$ is a bounded C^1 function satisfying

$$g(0)>0,\ l_0=\int_0^\infty g(s)\mathrm{d}s<a. \tag{6}$$

(H2) Hypotheses on f: Concerning the source term $f(u)$, we assume that

$$f(0)=0,\ |f(u)-f(v)|\le C\left(1+|u|^p+|v|^p\right)|u-v|,$$

where C is a constant, and p satisfies

$$\begin{cases} p>0, & 1\le n\le 4,\\ 0<p<\frac{4}{n-4}, & n\ge 5.\end{cases}$$

We denote $F(z)=\int_0^z f(s)\mathrm{d}s$ and assume that

$$0\le F(s)\le sf(s),\ s\in\mathbb{R}.$$

(H3) Hypotheses on h_1, h_2: $h_1:\mathbb{R}\to\mathbb{R}$ is a non-decreasing function of the class $C(\mathbb{R})$ such that there exist positive constants $r<1$, α_1, α_2 satisfying

$$\alpha_1|s|\le|h_1(s)|\le\alpha_2|s|\ \text{ for } |s|\ge r. \tag{7}$$

Moreover, assume that there exists a convex increasing function $H_1:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2((0,\infty))$ satisfying

$$H_1(0)=0, \tag{8}$$

$$H_1 \text{ is linear on } (0,r],\ \text{or } H_1'(0)=0 \text{ and } H_1''(t)>0 \text{ on } (0,r], \tag{9}$$

$$h_1^2(s) \leq H_1^{-1}\left(sh_1(s)\right) \quad \text{for } |s| \leq r, \tag{10}$$

$h_2 : \mathbb{R} \to \mathbb{R}$ is an odd non-decreasing function of the class $C^1(\mathbb{R})$ such that there exist positive constants α_i, $i = 3, 4, 5$, satisfying

$$|h_2'(s)| \leq \alpha_3 \quad \text{for } s \in \mathbb{R}, \tag{11}$$

$$\alpha_4 s h_2(s) \leq H_2(s) \leq \alpha_5 s h_1(s) \quad \text{for } s \in \mathbb{R}, \tag{12}$$

where $H_2(s) = \int_0^s h_2(t)dt$.

(H4) Hypotheses on $\boldsymbol{\tau}\,(\cdot)$: For the time-varying delay τ, we assume that $\tau \in W^{2,\infty}([0,T])$, $\forall T > 0$ and there exist positive constants τ_0, τ_1 and d satisfying

$$0 < \tau_0 \leq \tau\,(t) \leq \tau_1, \;\; \tau'\,(t) \leq d < 1, \;\; \forall t > 0. \tag{13}$$

(H5) Hypotheses on $\boldsymbol{\mu_1}$, $\boldsymbol{\mu_2}$: The weight of dissipation and the delay satisfy

$$0 < |\mu_2| < \frac{\alpha_4(1-d)}{\alpha_5\,(1-\alpha_4 d)}\mu_1. \tag{14}$$

In order to deal with the delay feedback term, motivated by the cited works, we introduce the following new dependent variable η, for studying Problem (1):

$$\eta^t(x,s) = z(x,s) = u(x,t) - u(x,t-s), \;\; s, t \in \mathbb{R}_+.$$

Moreover, as in [19], we define

$$z(x,\rho,t) := z(\rho,t) = u_t(x, t-\rho\tau(t)), \;\; (x,\rho,t) \in \Gamma_0 \times (0,1) \times (0,\infty).$$

Therefore, Problem (1) takes the form

$$\begin{aligned}
& u_{tt} - \left(a - l_0 + b\|\nabla u\|^2 + \sigma \int_\Omega \nabla u \nabla u_t \mathrm{d}x\right)\Delta u \\
& - \int_0^{+\infty} g(s)\Delta\eta^t(s)\mathrm{d}s + f(u) = 0 && \text{in } \Omega \times (0,\infty), \\
& u(x,t) = \eta^t(x,s) = 0 && \text{on } \Gamma_1 \times (0,\infty), \\
& (a - l_0)\frac{\partial u}{\partial \nu}(x,t) + \int_0^t g\,(t-s)\,\frac{\partial}{\partial\nu}\eta^t\,(x,s)\,\mathrm{d}s \\
& + \mu_1 h_1\left(u_t\,(x,t)\right) + \mu_2 h_2\left(z\,(x,1,t)\right) = 0 && \text{on } \Gamma_0 \times (0,+\infty), \\
& \tau(t) z_t + (1 - \tau'(t)\rho) z_\rho = 0 && \text{in } \Gamma \times (0,1) \times (0,\infty), \\
& \eta^t_t(x,s) + \eta^t_s(x,s) = u_t(x,t) && \text{in } \Omega \times (0,\infty) \times (0,\infty),
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
& z(x,0,t) = u_t(x,t) \text{ in } \Omega \times (0,\infty), \\
& z(x,\rho,0) = z_0\,(x,\rho) = g_0(x, -\rho\tau(0)) \text{ in } \Omega \times (-\tau\,(0)\,,0), \\
& u(x,-t) = u_0(x,t), \\
& \eta^0(x,s) = \eta_0(x,s) = u_0(0) - u_0(s) \;\; x \in \Omega, \;\; t,s \in (0,\infty),
\end{aligned} \tag{16}$$

$$u_t(x,0) = u_1(x) \text{ in } \Omega.$$

Let us recall the original Jensen's inequality which plays an essential role in investigating the decay effect for solutions as the following lemma shows.

Lemma 2.1 (Jensen's Inequality)**.** *If H is a convex function on $[a,b]$, $h : D \to [a,b]$ and q are integrable functions on D, $q(x) \geq 0$ and $\int_D q(x)\mathrm{d}x = Q > 0$, then*

$$H\left(\frac{1}{Q}\int_D h(x)q(x)\mathrm{d}x\right) \leq \frac{1}{Q}\int_D H(h(x))q(x)\mathrm{d}x.$$

2.1. The well-posedness of Problem (1)

In this section, we give the existence and uniqueness results for Problem (15) using the semigroup theory. Introducing the vector function $\Phi = (u, u_t, \eta^t, w)^T$, Problem (15) can be rewritten

$$\begin{cases} \Phi'(t) - \mathrm{A}\Phi(t) = 0, \ t > 0, \\ \Phi(0) = \Phi_0 = (u_0, u_1, \eta_0, w_0)^T, \end{cases} \tag{17}$$

where the operator A is defined by

$$\mathrm{A}\begin{pmatrix} u \\ \varphi \\ v \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ \left(a - l_0 + b\|\nabla u\|^2 + \sigma\int_\Omega \nabla u \nabla u_t \mathrm{d}x\right)\Delta u + \int_0^{+\infty} g(s)\Delta v(s)\mathrm{d}s - f(u) \\ -v_s + \varphi \\ -\frac{(1-\tau'(t)\rho)}{\tau(t)} z_\rho \end{pmatrix} \tag{18}$$

with domain

(19) $D(\mathrm{A})$

$$= \left\{ \begin{array}{c} (u,\varphi,v,z) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2_g((0,+\infty), H^1_{\Gamma_0}(\Omega)) \times L^2\left((0,1), L^2(\Omega)\right) : \\ u \in H^1_{\Gamma_0}(\Omega), \ \varphi \in H^1_{\Gamma_0}(\Omega), \ z_\rho \in L^2\left((0,1), L^2(\Omega)\right), \\ v \in L^2((0,+\infty), H^1_{\Gamma_0}(\Omega)), \ v_s \in L^2((0,+\infty), H^1_{\Gamma_0}(\Omega)), \ z(0) = \varphi(x), \ v(0) = 0, \\ \frac{\partial u}{\partial \nu} + \int_0^t g(s)\gamma_1 v_s \mathrm{d}s = -\mu_1 h_1(\varphi) - \mu_2 h_2(z) \end{array} \right\},$$

where $\gamma_1 : H^1(\Omega) \to H^{-1/2}(\Gamma)$ is the Neumann trace map, and $L^2_g((0,+\infty), H^1_{\Gamma_0}(\Omega))$ denotes the Hilbert space $H^1_{\Gamma_0}(\Omega)$-valued functions on $\mathbb{R}_+$ endowed with the inner product

$$[\varkappa, \psi]_{L^2_g((0,+\infty), H^1_{\Gamma_0}(\Omega))} = \int_\Omega \int_0^{+\infty} g(s)\nabla\varkappa(s)\nabla\psi(s)\mathrm{d}s\mathrm{d}x.$$

Let

$$\mathrm{H} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \times L^2_g((0,+\infty), H^1_{\Gamma_0}(\Omega)) \times L^2\left((0,1), L^2(\Omega)\right)$$

be the Hilbert space equipped with the following inner product

$$\begin{aligned} (\Phi, \widetilde{\Phi})_{\mathrm{H}} = {} & (a - l_0)\int_\Omega \nabla u \nabla\widetilde{u}\mathrm{d}x + \int_\Omega\int_0^{+\infty} g(s)\nabla v\nabla\widetilde{v}\mathrm{d}s\mathrm{d}x \\ & + \frac{\xi}{2}\int_{\Gamma_0}\int_{t-\tau(t)}^t e^{\lambda(\rho - t)} z\widetilde{z}\mathrm{d}\rho\mathrm{d}\Gamma, \end{aligned} \tag{20}$$

where ζ is a positive constant such that

$$\frac{\mu_2}{\sqrt{1-d}} < \xi < 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}. \tag{21}$$

By using the approaches from [19] with the ones from [6], we can prove that the operator A generates a strongly continuous semigroup on H, i.e., (for a sufficiently large constant $\lambda > 0$, the operator $(\mathrm{A} - \lambda I)$ is dissipative), and the following well-posedness theorem holds.

Theorem 2.2. *Assume that* (6) *and* (H1)-(H5) *hold. Then for the given* $\Phi_0 \in \mathrm{H}$*, there exists a unique weak solution* $\Phi \in C(\mathbb{R}^+, \mathrm{H})$ *of Problem* (1)*. Moreover, if* $\Phi_0 \in D(\mathrm{A})$*, then* $\Phi \in C(\mathbb{R}_+, \mathrm{H}) \cap C(\mathbb{R}_+, D(\mathrm{A}))$.

3. Main general theorems

The purpose of this paper is to give two general theorems concerning to the asymptotic stability of solutions for Problem (1). First, we suppose that the kernel function g satisfies

$$\gamma(t) > 0,\ g'(t) \leq -\gamma(t)\, g(t) \quad \text{for all } t \geq 0, \tag{22}$$

where γ is nonincreasing differentiable function $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$. Then we have the first general theorem:

Theorem 3.1. *Under the conditions of Theorem 2.2 and* (22) *holds, there exist positive constants* ω, κ, t_0, *and* ϵ_0 *such that the energy for Problem* (15) *satisfies*

$$E(t) \leq \kappa H^{-1}\left\{\omega\left(1 + \int_{t_0}^{t} \gamma(s)\mathrm{d}s\right)\right\} \ \textit{for } t \geq t_0 \tag{23}$$

with

$$H(t) = \int_t^1 \frac{1}{H_0(s)}\mathrm{d}s \tag{24}$$

and

$$H_0(t) = \begin{cases} t & \textit{if } H_1 \textit{ is linear on } [0,r], \\ tH_1'(\epsilon_0 t) & \textit{if } H_1'(0) = 0 \textit{ and } H_1''(t) > 0 \textit{ on } (0,r]. \end{cases} \tag{25}$$

Second, we suppose that there exists a strictly convex and increasing function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2([0,\infty))$ satisfying $G(0) = G'(0) = 0$ and $\lim_{t\to+\infty} G'(t) = +\infty$ such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}\mathrm{d}s + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{26}$$

Then we have the second general theorem:

Theorem 3.2. *Let* U *be the solution of* (15). *Assume that* (21) *and* (26) *hold. Then, there exist positive constants* σ, σ_1, σ_2, ϵ_0 *and* δ_5 *such that the solution energy of* (1) *satisfies*

$$E(t) \leq \sigma \mathscr{L}_1^{-1}\left(\sigma_1 t+\sigma_2\right) \text{ for all } t \geq 0, \tag{27}$$

where

$$\mathscr{L}_1(t)=\int_t^1 \frac{\mathrm{d} s}{s H_1'\left(\epsilon_0 s\right) G'\left(\delta_5 s\right)}.$$

3.1. Technical lemmas

In this subsection we present for rather technical lemmas that we need to complete the proof of Theorem 3.1 and Theorem 3.2. Let us define the modified energy functional E associated with Problem (15) by

$$\begin{aligned} E(t) &\\ =&\frac{1}{2}\left\|u_t(t)\right\|_2^2+\frac{1}{2}\left(a-l_0\right)\|\nabla u\|_2^2+\int_{\Omega} F(u)\, dx \\ &+\frac{\xi}{2} \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{\lambda(s-t)} H_2(z(x, 0, s))\, \mathrm{d} s \mathrm{d} \Gamma+\frac{1}{2}\left(g \circ \nabla \eta^t\right)(t), \end{aligned} \tag{28}$$

where

$$\left(g \circ \nabla \eta^t\right)(t)=\int_0^{+\infty} \int_{\Omega} g(s)\left|\nabla \eta^t(s)-\nabla \eta^t(t)\right|^2 \mathrm{d} x \mathrm{d} s.$$

The following three lemmas are essential to prove the main result given in Theorem 2.2.

Lemma 3.3. *Let* (u, z) *be the solution of* (15). *Then, for some two positive constants* β_1 *and* β_2, *we have*

$$\begin{aligned} E'(t) &\\ \leq &-\beta_1 \int_{\Gamma_0} h_1\left(u_t\right) u_t \mathrm{d} \Gamma-\beta_2 \int_{\Gamma_0} h_2(z(1, t)) z(1, t) \mathrm{d} \Gamma \\ &-\frac{\lambda \xi}{2} \int_{\Gamma_0} \int_{t-\tau(t)}^t e^{\lambda(s-t)} H_2\left(u_t(x, s)\right) \mathrm{d} s \mathrm{d} \Gamma+\frac{1}{2}\left(g' \circ \nabla \eta^t\right)(t) \\ &-\frac{1}{2} g(t)\|\nabla u(t)\|_2^2 . \end{aligned} \tag{29}$$

Proof. Multiplying the equation $(15)_1$ by u_t, integrating over Ω, and multiplying the equation $(15)_4$ by $\zeta z e^{-\lambda \tau(t) \rho}$, and integrating the result over $(0,1) \times \Gamma_0$ with respect to ρ and x using integration by parts and adding them up we obtain

$$\begin{aligned} E'(t) &\\ =&-\sigma\left(\frac{1}{2} \frac{d}{d t}\|\nabla u(t)\|_2^2\right)^2-\mu_1 \int_{\Gamma_0} h_1\left(u_t\right) u_t \mathrm{d} \Gamma-\mu_2 \int_{\Gamma_0} h_2(z(1, t)) u_t \mathrm{d} x \end{aligned} \tag{30}$$

$$
\begin{aligned}
&-\frac{\lambda\xi}{2}\int_{\Gamma_0}\int_{t-\tau(t)}^{t} e^{\lambda(s-t)} H_2\left(u_t(x,s)\right) \mathrm{d}s\mathrm{d}\Gamma+\frac{\xi}{2}\int_{\Gamma_0} H_2\left(u_t(x,t)\right)\mathrm{d}\Gamma \\
&-\frac{\xi}{2}\int_{\Gamma_0} e^{-\lambda\tau(t)}\left(1-\tau'(t)\right) H_2\left(u_t(x,t-\tau(t))\right)\mathrm{d}\Gamma \\
&+\frac{1}{2}\left(g'\circ\nabla\eta^t\right)(t)-\frac{1}{2}\left\|\nabla u(t)\right\|_2^2 g(t).
\end{aligned}
$$

From (12) and (13), using $z(1,t)=u_t(t-\tau(t))$, we see that

$$
\begin{aligned}
(31)\quad &-\frac{\xi}{2}\int_{\Gamma_0} e^{-\lambda\tau(t)}\left(1-\tau'(t)\right) H_2\left(u_t(x,t-\tau(t))\right)\mathrm{d}\Gamma+\frac{\xi}{2}\int_{\Gamma_0} H_2\left(u_t(x,t)\right)\mathrm{d}\Gamma \\
\leq\ &-\frac{\xi\alpha_4}{2}e^{-\lambda\tau_1}\left(1-\tau'(t)\right)\int_{\Gamma_0} h_2(z(x,1,t))z(x,1,t)\mathrm{d}\Gamma \\
&+\frac{\xi\alpha_5}{2}\int_{\Gamma_0} h_1\left(u_t(x,t)\right)u_t(x,t)\mathrm{d}\Gamma \\
\leq\ &-\frac{\xi\alpha_4}{2}e^{-\lambda\tau_1}(1-d)\int_{\Gamma_0} h_2(z(1,t))z(1,t)\mathrm{d}\Gamma+\frac{\xi\alpha_5}{2}\int_{\Gamma_0} h_1\left(u_t\right)u_t\mathrm{d}\Gamma.
\end{aligned}
$$

To estimate the second term in the right hand side of (30), let G^* be the conjugate function of the convex function G defined by

$$
(32)\qquad G^*(s)=\sup_{t\geq 0}(st-G(t)).
$$

Then G^* is a Legendre transform of G which is given by:

$$
(33)\qquad G^*(s)=s\left(G'\right)^{-1}(s)-G\left(\left(G'\right)^{-1}(s)\right)\ \forall s\geq 0,
$$

and satisfies the inequality

$$
(34)\qquad st\leq G^*(s)+G(t)\quad\text{for } s,t\geq 0.
$$

Taking the definition of H_2 into account and (33), we get

$$
(35)\qquad H_2^*(s)=sh_2^{-1}(s)-H_2\left(h_2^{-1}(s)\right)\quad\text{for } s\geq 0.
$$

Using (35), we can easily check that

$$
\begin{aligned}
(36)\quad &-\mu_2\int_{\Gamma_0} h_2\left(z(1,t)\right)u_t\mathrm{d}\Gamma \\
&\leq\ |\mu_2|\int_{\Gamma_0}\left(h_2(z(x,1,t))z(x,1,t)-H_2(z(x,1,t))+H_2\left(u_t(x,t)\right)\right)\mathrm{d}\Gamma,
\end{aligned}
$$

which, together with (12), leads to

$$
\begin{aligned}
(37)\quad &-\mu_2\int_{\Gamma_0} h_2\left(z(1,t)\right)u_t\mathrm{d}\Gamma \\
&\leq\ |\mu_2|\left(1-\alpha_4\right)\int_{\Gamma_0} h_2(z(1,t))z(1,t)\mathrm{d}\Gamma+|\mu_2|\,\alpha_5\int_{\Gamma_0} h_1\left(u_t\right)u_t\mathrm{d}\Gamma.
\end{aligned}
$$

Substituting (31) and (37) into (30) yields

$$\begin{aligned}
E'(t) \leq & -\left(\mu_1 - \frac{\xi\alpha_5}{2} - |\mu_2|\,\alpha_5\right)\int_{\Gamma_0} h_1\left(u_t\right) u_t \mathrm{d}\Gamma \\
& -\left(\frac{\xi\alpha_4}{2} e^{-\lambda\tau_1}(1-d) - |\mu_2|\,(1-\alpha_4)\right)\int_{\Gamma_0} h_2(z(1,t))z(1,t)\mathrm{d}\Gamma \\
& +\frac{1}{2}\left(g'\circ\nabla\eta^t\right)(t) - \frac{1}{2}\|\nabla u(t)\|_2^2\, g\,(t) \\
& -\frac{\lambda\xi}{2}\int_{\Gamma_0}\int_{t-\tau(t)}^{t} e^{\lambda(s-t)} H_2\left(u_t(x,s)\right)\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}$$

Putting $\beta_1 = \mu_1 - \frac{\xi\alpha_5}{2} - |\mu_2|\,\alpha_5 > 0$ and $\beta_2 = \frac{\xi\alpha_4}{2}e^{-\lambda\tau_1}(1-d) - |\mu_2|\,(1-\alpha_4) > 0$, we complete the proof of Lemma 3.3. □

Next, let us define the perturbed energy by

$$\mathrm{L}(t) = ME(t) + \epsilon\Psi(t) + \Phi(t) + \mathcal{E}(t), \tag{38}$$

where M is a positive constant to be chosen later, and

$$\begin{aligned}
\Psi(t) &= \int_{\Omega} u_t(t)u(t)\mathrm{d}x + \frac{\sigma}{4}\|\nabla u(t)\|_2^4, \\
\Phi(t) &= -\int_{\Omega} u_t(t)\int_0^{+\infty} g(s)\eta^t(s)\mathrm{d}x\mathrm{d}s, \\
\mathcal{E}(t) &= \int_{\Gamma_0}\int_{t-\tau(t)}^{t} e^{(s-t)} H_2\left(u_t(x,s)\right)\mathrm{d}s\mathrm{d}\Gamma.
\end{aligned}$$

The functional L is equivalent to the energy function E by the following lemma.

Lemma 3.4. *For $M > 0$ large enough, there exist two positive constants C_1 and C_2 such that*

$$C_1 E(t) \leq \mathrm{L}(t) \leq C_2 E(t), \;\; t \geq 0.$$

Proof. Integrating by parts using Young's inequality and Poincare's Theorem, we have

$$\begin{aligned}
|\Psi(t)| &\leq \frac{1}{2}\|u_t\|^2 + \frac{1}{2}C_*\|\nabla u\|^2 + \frac{\sigma}{4}\|\nabla u(t)\|_2^4 \\
&\leq \frac{1}{2}\|u_t\|^2 + \frac{C_*}{2l}\left(a - l_0\right)\|\nabla u\|^2 + \frac{\sigma}{4}\|\nabla u(t)\|_2^4 \leq cE(t), \\
|\Phi(t)| &\leq \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(\int_0^{+\infty} g(s)\|u(t) - u(t-s)\|\mathrm{d}s\right)^2 \\
&= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(\int_0^{+\infty} g(s)\|u(t) - u(t-s)\|\mathrm{d}s\right)^2 \\
&\leq \frac{1}{2}\|u_t\|^2 + \frac{a-l_0}{2}C_*\left(g\circ\nabla\eta^t\right) \leq cE(t)
\end{aligned}$$

and

$$|\mathcal{E}(t)| \leq cE(t).$$

Choosing $M > 0$ large, we obtain

$$|\mathrm{L}(t) - ME(t)| \leq cE(t),$$

the proof of Lemma 3.4 is concluded. □

Lemma 3.5. *There exist positive constants C_3, C_4, C_5 and $t_0 > 0$ such that*

$$\mathrm{L}'(t) \leq -C_3 E(t) + C_4 \|h_1(u_t)\|_{\Gamma_0}^2 + C_5 (g \circ \nabla \eta^t)(t), \ t \geq t_0. \tag{39}$$

Proof. Using Problem (15), we have

$$\begin{aligned}
(40)\quad & \psi'(t) \\
&= \int_\Omega u_{tt} u \mathrm{d}x + \int_\Omega u_t^2 \mathrm{d}x + \sigma \|\nabla u(t)\|_2^2 \int_\Omega \nabla u \nabla u_t \mathrm{d}x \\
&= \int_\Omega \left(\left(a + b\|\nabla u\|^2 + \sigma \int_\Omega \nabla u \nabla u_t \mathrm{d}x \right) \Delta u - \int_0^{+\infty} g(s) \Delta u(t-s) \mathrm{d}s - f(u) \right) u \mathrm{d}x \\
&\quad + \int_{\Gamma_0} \{ -\mu_1 h_1(u_t) - \mu_2 h_2(u_t(t-\tau(t))) \} u \mathrm{d}\Gamma + \int_\Omega u_t^2 \mathrm{d}x \\
&= -a \|\nabla u\|_2^2 - b \|\nabla u\|_2^4 + \int_\Omega \int_0^{+\infty} g(s) \nabla u(t-s) \mathrm{d}s \nabla u(t) \mathrm{d}x \\
&\quad - \mu_1 \int_{\Gamma_0} h_1(u_t) u \mathrm{d}\Gamma - \int_\Omega f(u) u \mathrm{d}x - \mu_2 \int_{\Gamma_0} h_2(u_t(t-\tau(t))) u \mathrm{d}\Gamma \\
&\quad + \int_\Omega u_t^2 \mathrm{d}x + \int_\Omega \int_0^{+\infty} g(s) \nabla u(s) \mathrm{d}s \nabla u(t) \mathrm{d}x.
\end{aligned}$$

By using Hölder inequality and Young's inequality, the second term on the right-hand side of (40) is estimated as follows.

$$\begin{aligned}
(41)\quad & \int_\Omega \int_0^{+\infty} g(s) \nabla u(t-s) \mathrm{d}s \nabla u \mathrm{d}x \\
&\leq \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \left| \int_0^{+\infty} g(s) \nabla u(t-s) \mathrm{d}s \right|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_\Omega \int_0^{+\infty} g(s) \mathrm{d}s \int_0^{+\infty} g(s) |\nabla u(t-s)|^2 \mathrm{d}s \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \left(\int_\Omega |\nabla u|^2 \mathrm{d}x \int_0^{+\infty} g(s) \mathrm{d}s \right)^{\frac{1}{2}} \left(\int_\Omega \int_0^{+\infty} g(s) |\nabla u(t-s)|^2 \mathrm{d}s \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x \int_0^{+\infty} g(s) \mathrm{d}s + \frac{1}{2} \int_\Omega \int_0^{+\infty} g(s) |\nabla u(t-s)|^2 \mathrm{d}s \mathrm{d}x \\
&\leq \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x \int_0^{+\infty} g(s) \mathrm{d}s
\end{aligned}$$

$$+\frac{1}{2}\int_{\Omega}\int_{0}^{+\infty} g(s)|\nabla u(t-s)-\nabla u(t)+\nabla u(t)|^{2}\mathrm{d}s\mathrm{d}x.$$

We use Young's inequality and (H1) to obtain for every $\eta > 0$

$$\begin{aligned}
(42)\quad & \frac{1}{2}\int_{\Omega}\int_{0}^{+\infty} g(s)[\nabla u(t-s)-\nabla u(t)+\nabla u(t)]^{2}\mathrm{d}s\mathrm{d}x \\
&\leq \frac{1}{2}\int_{\Omega}\int_{0}^{+\infty} g(s)\left((\nabla u(t-s)-\nabla u(t))^{2}+2|\nabla u(t-s)-\nabla u(t)\|\nabla u|+|\nabla u|^{2}\right)\mathrm{d}s\mathrm{d}x \\
&= \frac{1}{2}\int_{\Omega}\int_{0}^{+\infty} g(s)|\nabla u(t-s)-\nabla u(t)|^{2}\mathrm{d}s\mathrm{d}x+\frac{1}{2}\int_{\Omega}\int_{0}^{+\infty} g(s)|\nabla u|^{2}\mathrm{d}s\mathrm{d}x \\
&\quad+\int_{\Omega}\int_{0}^{+\infty} g(s)\left|\nabla u(t-s)-\nabla u(t)\right|\left|\nabla u\right|\mathrm{d}s\mathrm{d}x \\
&\leq \frac{1}{2}(g\circ\nabla\eta^{t})(t)+\frac{1}{2}\int_{0}^{+\infty} g(s)\mathrm{d}s\int_{\Omega}|\nabla u|^{2}\mathrm{d}x \\
&\quad+\frac{\eta}{2}\int_{0}^{+\infty} g(s)\mathrm{d}s\int_{\Omega}|\nabla u|^{2}\mathrm{d}x+\frac{1}{2\eta}(g\circ\nabla\eta^{t})(t) \\
&\leq \frac{1}{2}(1+\eta)\int_{0}^{+\infty} g(s)\mathrm{d}s\int_{\Omega}|\nabla u|^{2}\mathrm{d}x+\frac{1}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla\eta^{t})(t) \\
&\leq (1+\eta)\frac{(a-l_{0})}{2}\int_{\Omega}|\nabla u|^{2}\mathrm{d}x+\frac{1}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla\eta^{t})(t).
\end{aligned}$$

Combining (41) and (42) we get

$$\begin{aligned}
&\int_{\Omega}\int_{0}^{+\infty} g(s)\nabla u(t-s)\mathrm{d}s\nabla u\left(t\right)\mathrm{d}x \\
&\leq \frac{(a-l_{0})}{2}\int_{\Omega}|\nabla u|^{2}\mathrm{d}x+\frac{(a-l_{0})}{2}(1+\eta)\int_{\Omega}|\nabla u|^{2}\mathrm{d}x+\frac{1}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla\eta^{t})(t) \\
&= (2+\eta)\frac{(a-l_{0})}{2}\|\nabla u\|^{2}+\frac{1}{2}\left(1+\frac{1}{\eta}\right)(g\circ\nabla\eta^{t})(t).
\end{aligned}$$

By taking $\eta = \frac{l_0}{a-l_0}$, we infer that

$$(43)\qquad \int_{\Omega}\int_{0}^{+\infty} g(s)\nabla u(t-s)\mathrm{d}s\nabla u\mathrm{d}x \leq \left(a-\frac{l_{0}}{2}\right)\|\nabla u\|_{2}^{2}+\frac{a}{2l_{0}}(g\circ\nabla\eta^{t})(t).$$

For the third and forth terms, Young's inequality gives

$$\begin{aligned}
(44)\qquad & \mu_{1}\int_{\Gamma_{0}} h_{1}\left(u_{t}\right)u\mathrm{d}\Gamma+\mu_{2}\int_{\Gamma_{0}} h_{2}\left(u_{t}(t-\tau(t))\right)u\mathrm{d}\Gamma \\
&\leq \mu_{1}\int_{\Gamma_{0}}\left|h_{1}\left(u_{t}\right)\right|\left|u\right|\mathrm{d}x+\mu_{2}\int_{\Gamma_{0}}\left|h_{2}\left(u_{t}(t-\tau(t))\right)\right|\left|u\right|\mathrm{d}x \\
&\leq \mu_{1}\|u\|_{\Gamma_{0}}\left\|h_{1}\left(u_{t}\right)\right\|_{\Gamma_{0}}+\mu_{2}\|u\|_{\Gamma_{0}}\|h_{2}\left(u_{t}(t-\tau(t))\right)\|_{\Gamma_{0}}
\end{aligned}$$

$$\leq \eta\|u\|_{\Gamma_0}^2 + \frac{\mu_1^2}{4\eta}\left\|h_1\left(u_t\right)\right\|_{\Gamma_0}^2 + \eta\|u\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta}\left\|h_2\left(u_t(t-\tau(t))\right)\right\|_{\Gamma_0}^2$$

$$\leq \eta C_*^2\|\nabla u\|^2 + \frac{\mu_1^2}{4\eta}\left\|h_1\left(u_t\right)\right\|_{\Gamma_0}^2 + \eta C_*^2\|\nabla u\|^2 + \frac{\mu_2^2}{4\eta}\left\|h_2\left(u_t(t-\tau(t))\right)\right\|_{\Gamma_0}^2$$

$$= 2\eta C_*^2\|\nabla u\|^2 + \frac{\mu_1^2}{4\eta}\left\|h_1\left(u_t\right)\right\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta}\left\|h_2\left(z(1,t)\right)\right\|_{\Gamma_0}^2.$$

We use (H2) to obtain

$$-\int_\Omega f(u)u\mathrm{d}x \leq -\int_\Omega F(u)\mathrm{d}x.$$

Substituting these estimates into (40), we get

$$\begin{aligned}(45)\quad \psi'(t) \leq\ & -a\left|\left|\nabla u\right|\right|_2^2 - b\left|\left|\nabla u\right|\right|_2^4 + \frac{2a-l_0}{2}\|\nabla u\|^2 + \frac{a}{2l_0}(g\circ\nabla\eta^t)(t)\\ & + \|u_t\|^2 + 2\eta C_*^2\|\nabla u\|^2 + \frac{\mu_1^2}{4\eta}\left\|h_1\left(u_t\right)\right\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta}\left\|h_2\left(z(1,t)\right)\right\|_{\Gamma_0}^2\\ \leq\ & \|u_t\|^2 + \frac{\mu_1^2}{4\eta}\left\|h_1\left(u_t\right)\right\|_{\Gamma_0}^2 + \frac{\mu_2^2}{4\eta}\left\|h_2\left(z(1,t)\right)\right\|_{\Gamma_0}^2 + \frac{a}{2l_0}(g\circ\nabla\eta^t)(t)\\ & - \left(\frac{l_0}{2} - 2\eta C_*^2\right)\|\nabla u\|^2 - \int_\Omega F(u)\mathrm{d}x - b\left|\left|\nabla u\right|\right|_2^4.\end{aligned}$$

Besides

$$\begin{aligned}(46)\quad & \phi'(t)\\ =\ & -l_0\left\|u_t(t)\right\|^2 - \int_0^{+\infty} g'(s)\left(u(t)-u(t-s), u_t(t)\right)\mathrm{d}s\\ & + \left(a + b\left|\left|\nabla u\right|\right|_2^2\right)\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s), \nabla u(t))\mathrm{d}s\\ & + \sigma\left(\nabla u(t), \nabla u_t(t)\right)\int_0^{+\infty} g(s)(\nabla u(t) - \nabla u(t-s), \nabla u(t))\mathrm{d}s\\ & - \int_0^{+\infty} g(s)\left(\nabla u(t) - \nabla u(t-s), \int_0^{+\infty} g(s)\nabla u(t-s)\mathrm{d}s\right)\mathrm{d}s\\ & + \int_0^{+\infty} g(s)\int_{\Gamma_0}\left(u(t)-u(t-s)\right)\left(\mu_1 h_1\left(u_t(t)\right) + \mu_2 h_2(z(1,t))\right)\mathrm{d}\Gamma\mathrm{d}s\\ :=\ & I_1 + l_2 + I_3 + I_4 + I_5 + I_6.\end{aligned}$$

We now going to estimate the $I_j\ (1 \leq j \leq 6)$ terms in (46). Taking into account that $\|\nabla u\|_2^2 \leq \frac{2}{a-l_0}E(0)$, applying Young's inequality and employing a usual computation we have for every $\eta > 0$

$$|I_2| \leq \eta\left\|u_t(t)\right\|^2 - \frac{g(0)}{4\eta}C_*\left(g'\circ\nabla\eta^t\right)(t),$$

$$
\begin{aligned}
|I_3| \leq\ & \left(a + b\,||\nabla u||_2^2\right)\left(\eta\|\nabla u(t)\|^2 + \frac{(a-l_0)}{4\eta}(g\circ\nabla\eta^t)(t)\right)\\
\leq\ & a\eta\|\nabla u(t)\|^2 + b\eta\,||\nabla u||_2^4 + \left\{\frac{a(a-l_0)}{4\eta} + \frac{bE\,(0)}{2\eta}\right\}(g\circ\nabla\eta^t)(t),\\
|I_4| \leq\ & \frac{\sigma}{2}\left|\frac{\mathrm{d}}{\mathrm{d}t}\,||\nabla u||_2^2\right|\int_0^{+\infty} g(s)\,\|\nabla u(t)\|\,\|\nabla u(t)-\nabla u(t-s)\|\,\mathrm{d}s\\
\leq\ & \frac{\eta\sigma}{2}\left(||\nabla u||_2\,\frac{\mathrm{d}}{\mathrm{d}t}\,||\nabla u||_2^2\right)^2 + \frac{\sigma\,(a-l_0)}{8\eta}(g\circ\nabla\eta^t)(t)\\
\leq\ & \frac{\eta\sigma E\,(0)}{a-l_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,||\nabla u||_2^2\right)^2 + \frac{\sigma\,(a-l_0)}{8\eta}(g\circ\nabla\eta^t)(t),\\
|I_5| =\ & \int_\Omega\left(\int_0^{+\infty} g(s)(\nabla u(t)-\nabla u(t-s))\mathrm{d}s\right)\\
& \left(\int_0^{+\infty} g(s)(\nabla u(t)-\nabla u(t-s)-\nabla u(t))\mathrm{d}s\right)\mathrm{d}x\\
\leq\ & \int_\Omega\left(\int_0^{+\infty} g(s)|\nabla u(t)-\nabla u(t-s)|\mathrm{d}s\right)\\
& \left(\int_0^{+\infty} g(s)(|\nabla u(t)-\nabla u(t-s)|+|\nabla u(t)|)\mathrm{d}s\right)\mathrm{d}x\\
\leq\ & \eta\int_\Omega\left(\int_0^{+\infty} g(s)\left(|\nabla u(t)-\nabla u(t-s)|+|\nabla u(t)|\right)\mathrm{d}s\right)^2\mathrm{d}x\\
& + \frac{1}{4\eta}\int_\Omega\left(\int_0^{+\infty} g(s)|\nabla u(t)-\nabla u(t-s)|\mathrm{d}s\right)^2\mathrm{d}x\\
\leq\ & \left(2\eta + \frac{1}{4\eta}\right)\int_\Omega\left(\int_0^{+\infty} g(s)|\nabla u(t)-\nabla u(t-s)|\mathrm{d}s\right)^2\mathrm{d}x\\
& + 2\eta\int_\Omega\left(\int_0^{+\infty} g(s)|\nabla u(t)|\mathrm{d}s\right)^2\mathrm{d}x\\
\leq\ & \left(2\eta + \frac{1}{4\eta}\right)(a-l_0)(g\circ\nabla\eta^t)(t) + 2\eta(a-l_0)^2\|\nabla u(t)\|^2,
\end{aligned}
$$

and

$$
\begin{aligned}
|I_6| \leq\ & \eta\mu_1\,\|h_1\,(u_t(t))\|_{\Gamma_0}^2 + \eta\,|\mu_2|\,\|h_2\,(z(1,t))\|_{\Gamma_0}^2\\
& + \left\{\frac{\mu_1(1-l)\bar{C}_*}{4\eta} + \frac{|\mu_2|\,(1-l)\bar{C}_*}{4\eta}\right\}(g\circ\nabla\eta^t)(t).
\end{aligned}
$$

Plugging these estimates into (46), we get

(47) $\phi'(t)$

$$
\leq\ -\left(l_0-\eta\right)\|u_t\|^2 - \frac{g(0)}{4\eta}C_*\left(g'\circ\nabla\eta^t\right)(t)
$$

$$
\begin{aligned}
&+\eta\left\{1+2(a-l_0)^2\right\}\|\nabla u(t)\|^2+\eta b\|\nabla u(t)\|^4+\frac{\eta\sigma E(0)}{a-l_0}\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2\\
&+\left\{\frac{a(a-l_0)}{4\eta}+\left(2\eta+\frac{1}{4\eta}\right)(a-l_0)+\frac{\sigma(a-l_0)}{8\eta}\right.\\
&\quad\left.+\frac{bE(0)}{2\eta}+\frac{(\mu_1+|\mu_2|)(a-l_0)\bar{C}_*}{4\eta}\right\}(g\circ\nabla\eta^t)(t)\\
&+\eta\mu_1\|h_1(u_t(t))\|_{\Gamma_0}^2+\eta|\mu_2|\|h_2(z(1,t))\|_{\Gamma_0}^2.
\end{aligned}
$$

Moreover, analogous to (31) we see that

$$
\begin{aligned}
\mathcal{E}'(t)\leq\ &-\mathcal{E}(t)-\alpha_4e^{-\tau_1}(1-d)\int_{\Gamma_0}h_2(z(1,t))z(1,t)\mathrm{d}\Gamma\\
&+\alpha_5\int_{\Gamma_0}h_1(u_t)u_t\mathrm{d}\Gamma.
\end{aligned}\tag{48}
$$

From (38), (29), (45), (47) and (48), we have

$$
\begin{aligned}
&\mathrm{L}'(t)\\
&\leq\ -(l_0-\eta-\epsilon)\|u_t\|^2+\left(\frac{M}{2}-\frac{g(0)}{4\eta}C_*\right)\left(g'\circ\nabla\eta^t\right)(t)\\
&\quad+\left[\eta\left\{a+2(a-l_0)^2\right\}-\epsilon\left\{a-(1+\eta)(a-l)-2\eta\right\}\right]\|\nabla u(t)\|^2\\
&\quad+(\eta-\epsilon)b\|\nabla u(t)\|^4+\left(C+\frac{\epsilon}{4\eta}\right)\left(g\circ\nabla\eta^t\right)(t)\\
&\quad+\left(\eta\mu_1+\frac{\epsilon\mu_1^2}{4\eta}\bar{C}_*\right)\|h_1(u_t)\|_{\Gamma_0}^2-\left(\frac{M\sigma}{4}-\frac{\eta\sigma E(0)}{a-l_0}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2\\
&\quad-\int_\Omega F(u)\mathrm{d}x+\left(\eta|\mu_2|+\frac{\epsilon\mu_2^2}{4\eta}\bar{C}_*\right)\|h_2(z(1,t))\|_{\Gamma_0}^2\\
&\quad-(M\beta_1-\alpha_5)\int_{\Gamma_0}h_1(u_t)u_t\mathrm{d}\Gamma\\
&\quad-\left(M\beta_2+\alpha_4(1-d)e^{-\tau_1}\right)\int_{\Gamma_0}h_2(z(1,t))z(1,t)\mathrm{d}\Gamma-\mathcal{E}(t).
\end{aligned}\tag{49}
$$

Making use of (11), we find

$$
\|h_2(z(1,t))\|_{\Gamma_0}^2\leq\alpha_3\int_{\Gamma_0}h_2(z(1,t))z(1,t)\mathrm{d}\Gamma.
$$

Owing to (13), it is seen that

$$
\begin{aligned}
-\mathcal{E}(t)&\leq-\tau(t)\int_{\Gamma_0}\int_0^1e^{-\tau(t)}H_2z(x,\rho,t)\mathrm{d}\rho\mathrm{d}\Gamma\\
&\leq-e^{-\tau_1\rho}\int_{\Gamma_0}\int_0^1H_2(z(x,\rho,t))\mathrm{d}\rho\mathrm{d}\Gamma.
\end{aligned}
$$

Applying these to (49), we get

$$\begin{aligned}
\mathrm{L}'(t) \leq & -(l_0-\eta-\epsilon)\|u_t\|^2 + \left(\frac{M}{2}-\frac{g(0)}{4\eta}C_*\right)\left(g'\circ\nabla\eta^t\right)(t) \\
& -\left[\epsilon\{(a-(1+\eta)(a-l_0)-2\eta\}-\eta\left\{a+2(a-l_0)^2\right\}\right]\|\nabla u(t)\|^2 \\
& +\left(c+\frac{\epsilon}{4\eta}\right)(g\circ\nabla\eta^t)(t)+\left(\eta\mu_1+\frac{\epsilon\mu_1^2}{4\eta}\bar{C}_*\right)\|h_1(u_t(t))\|_{\Gamma_0}^2 \\
& -(\epsilon-\eta)\,b\|\nabla u(t)\|^4 - e^{-\tau_1\rho}\int_{\Gamma_0}\int_0^1 H_2(z(x,\rho,t))\mathrm{d}\rho\mathrm{d}\Gamma \\
& -(M\beta_1-\alpha_5)\int_{\Gamma_0} h_1(u_t)\,u_t\mathrm{d}\Gamma - \left(\frac{M\sigma}{4}-\frac{\eta\sigma E(0)}{a-l_0}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_2^2\right)^2 \\
& -\int_\Omega F(u)\mathrm{d}x - \left\{M\beta_2+\alpha_4(1-d)e^{-\tau_1}-\alpha_3\left(\eta|\mu_2|+\frac{\epsilon\mu_2^2}{4\eta}\bar{C}_*\right)\right\} \\
& \times\int_{\Gamma_0} h_2(z(1,t))z(1,t)\mathrm{d}\Gamma \quad \text{for all } t\geq t_0.
\end{aligned}$$

At this point, we choose $\epsilon>0$ small enough such that $l_0-\epsilon>0$, and then we pick $\eta>0$ sufficiently small such that

$$\begin{aligned}
& a-(1+\eta)(a-l_0)-2\eta>0, \\
& \epsilon\{(a-(1+\eta)(a-l_0)-2\eta\}-\eta\left\{a+2(a-l_0)^2\right\}>0, \\
& l_0-\epsilon-\eta>0, \\
& \epsilon-\eta>0.
\end{aligned}$$

Then we choose $M>0$ so large such that

$$\begin{aligned}
& \frac{M}{2}-\frac{g(0)}{4\eta}C_*>0,\ M\beta_1-\alpha_5>0,\ \frac{M\sigma}{4}-\frac{\eta\sigma E(0)}{a-l_0}>0, \\
& M\beta_2+\alpha_4(1-d)e^{-\tau_1}-\alpha_3\left(\eta|\mu_2|+\frac{\epsilon\mu_2^2}{4\eta}\bar{C}_*\right)>0,
\end{aligned}$$

and we complete the proof. □

The following lemma plays an essential role in the proof of Theorem 3.1, which can be proved by repeating the same arguments of Guesmia in [7].

Lemma 3.6. *Suppose that* (26) *holds. Then, there exists* $\beta>0$ *such that* $\forall\delta_0>0$ *and* $t\in\mathbb{R}_+$, *we have*

$$G'(\delta_0E(t))\left(g\circ\nabla\eta^t\right)(t)\leq-\beta E'(t)+\beta\delta_0G'(\delta_0E(t))\,E(t). \tag{50}$$

With this preparation, we are ready to prove Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1. Multiplying (39) by $\gamma(t)$, we have from (22) and (29) that

$$\gamma(t)\mathrm{L}'(t)\leq-C_3\gamma(t)E(t)+C_4\gamma(t)\|h_1(u_t(t))\|_{\Gamma_0}^2+C_5\gamma(t)\left(g\circ\nabla\eta^t\right)(t) \tag{51}$$

$$\leq -C_3\gamma(t)E(t)+C_4\gamma(t)\left\|h_1\left(u_t(t)\right)\right\|_{\Gamma_0}^2-C_5\left(g'\circ\nabla\eta^t\right)(t)$$
$$\leq -C_3\gamma(t)E(t)+C_4\gamma(t)\left\|h_1\left(u_t(t)\right)\right\|_{\Gamma_0}^2-2C_5E'(t) \quad \text{for } t\geq t_0.$$

Now, we define

$$\mathcal{L}(t)=\gamma(t)\mathrm{L}(t)+2C_5E(t).$$

As γ is nonincreasing, we see from (51) that

$$\mathcal{L}'(t)\leq\gamma'(t)\mathrm{L}(t)-C_3\gamma(t)E(t)+C_4\gamma(t)\left\|h_1\left(u_t(t)\right)\right\|_{\Gamma_0}^2 \tag{52}$$
$$\leq -C_3\gamma(t)E(t)+C_4\gamma(t)\left\|h_1\left(u_t(t)\right)\right\|_{\Gamma_0}^2 \quad \text{for } t\geq t_0.$$

In order to obtain desired results, we needed to estimate the term

$$\gamma(t)\left\|h_1\left(u_t(t)\right)\right\|_{\Gamma_0}^2$$

in (52). For this, let

$$\Gamma_0^1=\{x\in\Gamma_0:|u_t|>r|\} \text{ and } \Gamma_0^2=\{x\in\Gamma_0:|u_t|\leq r\}.$$

For $\delta_1=\frac{\alpha_2\gamma(0)}{\beta_1}$, (7) and (29) imply that

$$\gamma(t)\int_{\Gamma_0^1}\left|h_1\left(u_t\right)\right|^2\mathrm{d}\Gamma\leq\alpha_2\gamma(0)\int_{\Gamma_0^1}u_th_1\left(u_t\right)\mathrm{d}\Gamma\leq-\delta_1E'(t). \tag{53}$$

Two cases are distinguished:

Case 1 : $\boldsymbol{H_1}$ is linear on $[\mathbf{0},\boldsymbol{r}]$: According to (7) and (10), we can easily check that there exist $\delta_2>0$ and $\delta_3>0$ such that

$$\delta_2|s|\leq h_1(s)\,|\leq\delta_3|\,s| \quad \text{for all } |s|\leq r,$$

and thus, for $\delta_4=\frac{\alpha_3\gamma(0)}{\beta_1}$,

$$\gamma(t)\int_{\Gamma_0^2}\left|h_1\left(u_t\right)\right|^2\mathrm{d}\Gamma\leq\delta_3\gamma(t)\int_{\Gamma_0^2}u_th_1\left(u_t\right)\mathrm{d}\Gamma \tag{54}$$
$$\leq\delta_3\gamma(0)\int_{\Gamma_0^2}u_th_1\left(u_t\right)\mathrm{d}\Gamma$$
$$\leq-\delta_4E'(t).$$

We substitute (53) and (54) into (52), we get

$$\left(\mathcal{L}(t)+\delta E(t)\right)'\leq-C_3\gamma(t)E(t)=-C_7\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \quad \text{for } t\geq t_0, \tag{55}$$

where $\delta=C_4\left(\delta_1+\delta_4\right)$, $C_7=C_3E(0)$ and H_0 is the function given in (25).

Case 2 : $\boldsymbol{H_1(0)=0}$ and $\boldsymbol{H_1''>0}$ on $(\mathbf{0},\boldsymbol{r}]$: From (10) and (29) it follows that

$$\gamma(t)\int_{\Gamma_0^2}h_1\left(u_t\right)^2\mathrm{d}\Gamma\leq\gamma(t)\int_{\Gamma_0^2}H_1^{-1}\left(u_th_1\left(u_t\right)\right)\mathrm{d}\Gamma$$
$$\leq\gamma(t)\left|\Gamma_0^2\right|H_1^{-1}\left(\frac{1}{\left|\Gamma_0^2\right|}\int_{\Gamma_0^2}u_th\left(u_t\right)\mathrm{d}\Gamma\right)$$

$$\leq \gamma(t)\left|\Gamma_0^2\right| H_1^{-1}\left(-\frac{E'(t)}{\beta_1\left|\Gamma_0^2\right|}\right),$$

where the Jensen's inequality (2.1) for a concave function with $D=\Gamma_0^2$, $q(x)=1$, $H=H_1$ and $f(x)=H_1^{-1}\left(u_t(x)h_1\left(u_t(x)\right)\right)$ in the second inequality is used.

Adapting this and (53) to (52), for $\widetilde{\delta}=C_4\delta_1$ and $C_6=C_4\left|\Gamma_0^2\right|$, we get

$$(\mathcal{L}(t)+\widetilde{\delta}E(t))' \leq -C_3\gamma(t)E(t)+C_6\gamma(t)H_1^{-1}\left(-\frac{E'(t)}{\beta_1\left|\Gamma_0^2\right|}\right). \tag{56}$$

For $0<\epsilon_0<r$ and $c_0>0$, the inequalities (56), (34), together with (33), drive to

$$\begin{aligned}
&\left\{H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)(\mathcal{L}(t)+\tilde{\delta}E(t))+c_0E(t)\right\}' \qquad (57)\\
&= \epsilon_0\frac{E'(t)}{E(0)}H_1''\left(\epsilon_0\frac{E(t)}{E(0)}\right)(\mathcal{L}(t)+\tilde{\delta}E(t))+H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)(\mathcal{L}(t)+\tilde{\delta}E(t))'\\
&\quad + c_0E'(t)\\
&\leq -C_3\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)E(t)+C_6\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)H_1^{-1}\left(-\frac{E'(t)}{\beta_1\left|\Gamma_0^2\right|}\right)\\
&\quad + c_0E'(t)\\
&\leq -C_3\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)E(t)+C_6\gamma(t)H_1^*\left(H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\right)\\
&\quad -\frac{C_6\gamma(t)}{\beta_1\left|\Gamma_0^2\right|}E'(t)+c_0E'(t)\\
&= -C_3\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)E(t)+C_6\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\epsilon_0\frac{E(t)}{E(0)}\\
&\quad -C_6\gamma(t)H_1\left(\epsilon_0\frac{E(t)}{E(0)}\right)-\frac{C_6\gamma(t)}{\beta_1\left|\Gamma_0^2\right|}E'(t)+c_0E'(t)\\
&\leq -\left(C_3E(0)-C_6\epsilon_0\right)\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\frac{E(t)}{E(0)}+\left(c_0-\frac{C_6\gamma(0)}{\beta_1\left|\Gamma_0^2\right|}\right)E'(t).
\end{aligned}$$

Taking ϵ_0 sufficiently small such that $C_3E(0)-C_6\epsilon_0>0$ and choosing $c_0>0$ suitably such that $c_0-\frac{C_6\gamma(0)}{\beta_1\left|\Gamma_0^2\right|}>0$, we obtain

$$\begin{aligned}
&\left\{H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)(\mathcal{L}(t)+\tilde{\delta}E(t))+c_0E(t)\right\}' \qquad (58)\\
&\leq -C_8\gamma(t)H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\frac{E(t)}{E(0)}\\
&= -C_8\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \quad \text{for } t\geq t_0,
\end{aligned}$$

where $C_8 = C_3E(0) - C_6\epsilon_0$ is a positive constant. Now, let

$$\tilde{\mathcal{L}(t)} = \begin{cases} \mathcal{L}(t)+\delta E(t) & \text{if } H_1 \text{ is linear on } [0,r], \\ H_1'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)(\mathcal{L}(t)+\tilde{\delta}E(t))+C_0E(t) & \text{if } H_1'(0)=0 \text{ and } H_1''>0 \text{ on } (0,r]. \end{cases} \tag{59}$$

Then from (55) and (58), we see that

$$\tilde{\mathcal{L}'(t)} \leq -C_9\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \quad \text{for } t \geq t_0, \tag{60}$$

where $C_9 = \min\{C_7, C_8\}$. Since $\tilde{\mathcal{L}(t)}$ is equivalent to $E(t)$, there exist two positive constants α_3 and α_4 such that

$$\alpha_3\tilde{\mathcal{L}(t)} \leq E(t) \leq \alpha_4\tilde{\mathcal{L}(t)}. \tag{61}$$

Let us define

$$\mathcal{J}(t) = \alpha_3\frac{\tilde{\mathcal{L}(t)}}{E(0)}. \tag{62}$$

It is to be noted that

$$\mathcal{J}(t) \leq \frac{E(t)}{E(0)} < 1 \quad (\text{see } (61)). \tag{63}$$

From (62), (60), (63) and the fact that H_0 is increasing, we arrive at

$$\mathcal{J}'(t) \leq -\frac{\alpha_3C_9}{E(0)}\gamma(t)H_0\left(\frac{E(t)}{E(0)}\right) \leq -C_{10}\gamma(t)H_0(\varepsilon(t)), \tag{64}$$

where $C_{10} = \frac{\alpha_3C_9}{E(0)}$.

Integrating this over (t_0, t) and using $H'(t) = -\frac{1}{H_0(t)}$ (see (24)), we observe that

$$H(\mathcal{J}(t)) - H\left(\mathcal{J}\left(t_0\right)\right) \geq C_{10}\int_{t_0}^{t}\gamma\left(s\right)\mathrm{d}s.$$

Thanks to the fact H^{-1} is decreasing, we infer

$$\mathcal{J}(t) \leq H^{-1}\left(H(\mathcal{J}(0)) + C_{10}\int_{t_0}^{t}\gamma(s)\mathrm{d}s\right) \text{ for } t \geq t_0.$$

This completes the proof from the equivalent relation of $\mathcal{J}$ and E. □

Proof of Theorem 3.2. The following two cases are distinguished:
Case 1 : H_1 is linear on $[0, r]$: Then according to (7) and (10), we deduce that

$$c_1|s| \leq |h_1(s)| \leq c_2|s| \quad \text{for all } s \in \mathbb{R}.$$

Hence, by applying (29) the estimate (39) becomes

$$\mathrm{L}_1'(t) \leq -C_3E(t) - C_{11}E'(t) + C_5\left(g \circ \nabla\eta^t\right)(t), \ \forall t \geq 0.$$

C_{11} is a positive constant, which yields

$$\mathrm{L}_2'(t) \leq -C_3E(t) + C_5\left(g\circ\nabla\eta^t\right)(t),\ \forall t\geq 0, \tag{65}$$

where the function $\mathrm{L}_2 = \mathrm{L}_1 + C_{11}E$ is equivalent to E. Multiplying (65) by $G'(\delta_0 E(t))$ and applying (50), we get

$$G'(\delta_0E(t))\,\mathrm{L}_2'(t) \leq -\left(C_3 - C_5\beta\delta_0\right)G'(\delta_0E(t))\,E(t) - C_5\beta E'(t),\ \forall t\geq 0.$$

By taking δ_0 small enough so that $C_{12} = C_3 - C_5\beta\delta_0 > 0$, we obtain

$$G'(\delta_0E(t))\,\mathrm{L}_2'(t) + C_5\beta E'(t) \leq -C_{12}G'(\delta_0E(t))\,E(t),\ \forall t\geq 0.$$

Let $\mathrm{L}_3(t) = G'(\delta_0E(t))\,\mathrm{L}_2(t) + C_5\beta E(t)$ and take into account the fact that $G'(\delta_0E(t))$ is nonincreasing. Then we reach at

$$\mathrm{L}_3 \sim E \ \text{ and } \ \mathrm{L}_3'(t) \leq -C_{13}G'(\delta_1\mathrm{L}_3(t))\,\mathrm{L}_3(t).$$

The last inequality leads to

$$\left(\mathscr{L}\left(\mathrm{L}_3(t)\right)\right)' \geq C_{14},$$

where $\mathscr{L}(t) = \int_t^1 \mathrm{d}s/\left(sG'(\delta_1 s)\right)$ on $(0,1]$. Integrating the previous inequality on $(0,t)$, by using the property of G, we infer

$$\mathrm{L}_3(t) \leq \mathscr{L}^{-1}\left(C_{15}t + C_{16}\right) \ \text{for all } t\geq 0,$$

where C_{15} and C_{16} are positive constants. Thanks to $\mathrm{L}_3 \sim E$, we get the desired result. In this case, we have $H_1(s) = cs$.

Case 2 : H_1 is nonlinear: Supposing in this case that $H_1'(0) = 0$, $H_1'' > 0$ on $[0,r]$, since H_1 is convex and increasing, H_1^{-1} is concave and increasing, by (28), the reversed Jensens inequality for concave function, and (10), it comes

$$\begin{aligned}
\int_{\Gamma_0} h_1^2(u_t(t))\,\mathrm{d}\Gamma &= \int_{\Gamma_0^1} h_1^2(u_t(t))\,\mathrm{d}\Gamma + \int_{\Gamma_0^2} h_1^2(u_t(t))\,\mathrm{d}\Gamma\\
&\leq \int_{\Gamma_0} u_t(t)h_1(u_t(t))\,\mathrm{d}\Gamma + \int_{\Gamma_0^2} H_1^{-1}\left(u_th_1(u_t)\right)\mathrm{d}\Gamma\\
&\leq -cE'(t) + cH_1^{-1}\left(\frac{1}{|\Gamma_0^2|}\int_{\Gamma_0^2} u_th_1(u_t)\,\mathrm{d}\Gamma\right).
\end{aligned}$$

Then (39) is rewritten as

$$\begin{aligned}
\mathscr{F}'(t) \leq\ & -C_3E(t) + C_5\left(g\circ\nabla\eta^t\right)(t)\\
& + cC_4H_1^{-1}\left(\frac{1}{|\Gamma_0^2|}\int_{\Gamma_0^2} u_1h_1(u_t)\,\mathrm{d}\Gamma\right)\ \forall t\geq 0,
\end{aligned} \tag{66}$$

where $\mathscr{F}(t) = \mathrm{L}_1(t) + cC_4E(t)$, which is equivalent to E. Now, for $\epsilon_0 > 0$ and $\alpha > 0$, let us denote

$$\mathscr{F}_1(t) = H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)\mathscr{F}(t) + \alpha E(t),\ \forall t\geq t_0. \tag{67}$$

By (39) and the fact that $E' < 0$, $H_1' \geq 0$, $H_1'' \geq 0$, we obtain $\mathscr{F}_1 \sim E$, and

$$\begin{aligned}
(68)\quad \mathscr{F}_1'(t) &= \epsilon_0 \frac{E'(t)}{E(0)} H_1'' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathscr{F}(t) + H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right)' \mathscr{F}(t) + \alpha E'(t) \\
&\leq -C_3 E(t) H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C_5 \left(g \circ \nabla \eta^t\right)(t) H_1 \left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\
&\quad + c C_4 H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) H_1^{-1} \left(\frac{1}{|\Gamma_0^2|} \int_{\Gamma_0^2} u_t h_1(u_t)\, \mathrm{d}\Gamma\right) + \alpha E(t).
\end{aligned}$$

Since H_1^* is the Legendre transform of the convex function H_1 defined by (32), the following inequality

$$(69)\qquad H_1^* = s\left(H_1'\right)^{-1}(s) - H_1\left[\left(H_1'\right)^{-1}(s)\right] \leq s\left(H_1'\right)^{-1}(s),\ \forall s \geq 0,$$

holds. Using (28) and (68), (69) and (34) with $s = H_1'\left(\epsilon_0(E(t)/E(0))\right)$, $t = H_1^{-1}\left(\frac{1}{|\Gamma_0^2|}\int_{\Gamma_0^2} u_t h_1(u_t)\, d\Gamma\right)$ and $G = H_1$, we obtain

$$\begin{aligned}
\mathscr{F}_1'(t) \leq\ & -\left(C_3 E(0) - c C_4 \epsilon_0\right) \frac{E(t)}{E(0)} H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) \\
& - \left(c C_4 - \alpha\right) E'(t) + C_5 \left(g \circ \nabla \eta^t\right)(t) H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right).
\end{aligned}$$

Hence, with a suitable choice of ϵ_0 and α, we obtain

$$(70)\qquad \mathscr{F}_1'(t) \leq -\alpha_6 \frac{E(t)}{E(0)} H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) + C_5 \left(g \circ \nabla \eta^t\right)(t) H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right).$$

Multiplying (70) by $G'\left(\delta a_0 E(t)\right)$ and taking (50) into consideration, we get for all $t \geq 0$

$$\begin{aligned}
& G\left(\delta_0 E(t)\right) \mathscr{F}_1'(t) \\
\leq\ & -\alpha_6 \frac{E(t)}{E(0)} H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) G'\left(\delta_0 E(t)\right) \\
& - C_5 H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) E'(t) + C_5 \beta \delta_0 G'\left(\delta_0 E(t)\right) H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) E(t) \\
\leq & -\alpha_6 \frac{E(t)}{E(0)} H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) G'\left(\delta_0 E(t)\right) - \alpha_7 E(t) \\
& + C_5 \beta \delta_0 G'\left(\delta_0 E(t)\right) H_1' \left(\epsilon_0 \frac{E(t)}{E(0)}\right) E(t).
\end{aligned}$$

Let $\mathscr{F}_2(t) = G'(\delta_0 E(t)) \mathscr{F}_1(t) + \alpha_7 E(t)$ and using the fact that $G'' > 0$ and $E' < 0$, we determine two positive constants γ_1, γ_2 such that

$$(71)\qquad \gamma_1 \mathscr{F}_2(t) \leq E(t) \leq \gamma_2 \mathscr{F}_2(t)$$

and

$$\mathscr{F}_2'(t) \leq -\left(\alpha_6 - C_5\beta\delta_5\right)\frac{E(t)}{E(0)}H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)G'\left(\delta_5\frac{E(t)}{E(0)}\right),$$

where $\delta_5 = \delta_0 E(0)$. Taking δ_0 small enough so that $\alpha_3 = \alpha_6 - C_5\beta\delta_5 > 0$ in the previous estimate, we get

$$\begin{aligned}\mathscr{F}_2'(t) &\leq -\alpha_8\frac{E(t)}{E(0)}H_1'\left(\epsilon_0\frac{E(t)}{E(0)}\right)G'\left(\delta_5\frac{E(t)}{E(0)}\right)\\ &= -\alpha_8\mathscr{L}_2\left(\frac{E(t)}{E(0)}\right),\end{aligned}$$

where $\mathscr{L}_2(\mathrm{s}) = sH_1'\left(\epsilon_0 s\right)G\left(\delta_5 s\right)$. Finally, we let

$$\mathscr{F}_3(t) = \gamma_1\frac{\mathscr{F}_2(t)}{E(0)}.$$

By using (71), $\mathscr{F}_3$ is equivalent to E and for some $\sigma_1 > 0$, we have

$$\mathscr{F}_3'(t) \leq -\sigma_1\mathscr{L}_2\left(\mathscr{F}_3(t)\right),\ \forall t \geq 0.$$

Simple integration of the previous inequality on $(0, t)$ yields

$$\mathscr{F}_3(t) \leq \mathscr{L}_1^{-1}\left(\sigma_1 t + \sigma_2\right),\ \forall t \geq 0,$$

where $\mathscr{L}_1(s) = \int_s^1 (\mathrm{d}\alpha/\mathscr{L}_2(\alpha))$, $s \in (0,1]$ and σ_2 is a positive constant. Employing the fact that $\mathscr{F}_3 \sim E$, our result is deduced. □

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