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COLORING LINKS BY THE SYMMETRIC GROUP OF DEGREE THREE

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Abstract. We consider the number of colors for colorings of links by the symmetric group S_3 of degree 3. For knots, such a coloring corresponds to a Fox 3-coloring, and thus the number of colors must be 1 or 3. However, for links, there are colorings by S_3 with 4 or 5 colors. In this paper, we show that if a 2-bridge link admits a coloring by S_3 with 5 colors, then the link also admits such a coloring with only 4 colors.

1. Introduction

One of the most well-known invariants of knots and links would be the Fox 3-coloring, originally introduced by R. Fox. For example, it is described in [1, Chap. VI, Exercises, 6, pp. 92–93]. In this exercise, readers are asked to show that a knot has a diagram which is 3-colorable if and only if the knot group can be mapped homomorphically onto the symmetric group of degree 3. In view of this, as a generalization of the Fox 3-coloring, we consider the colorings of links by the symmetric group of degree 3, which we denote by S_3.

Definition. Let D be a diagram of a link. We call a map $\Gamma : \{\text{arcs of } D\} \to S_3 \setminus \{e\}$ an S_3-*coloring* on D if it satisfies $\Gamma(x)\Gamma(y) = \Gamma(z)\Gamma(x)$ (respectively, $\Gamma(y)\Gamma(x) = \Gamma(x)\Gamma(z)$) at a positive (resp. negative) crossing on D, where x denotes the over arc, y and z the under arcs at the crossing supposing z is the under arc before passing through the crossing and y is the other.

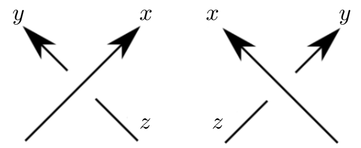

Figure 1. Crossing conditions for S_3-coloring

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The image $\Gamma(a)$ of an arc a on D by an S_3-coloring Γ is said to be a *color* on a with respect to Γ.

Note that an S_3-coloring on a diagram D of a link L gives a representation $G_L \to S_3$ of the link group $G_L = \pi_1(S^3 - L)$ of L, and conversely, a representation of G_L to S_3 gives an S_3-coloring on any diagram D of a link L.

Actually, for knots, such an S_3-coloring corresponds to a Fox 3-coloring, as stated in [1, Chap. VI, Exercises, 6, pp. 92–93]. Thus the number of colors for such colorings must be 1 or 3. However, for links, there exist colorings by S_3 with 4 or 5 colors. See the example below. (See the next section for details.)

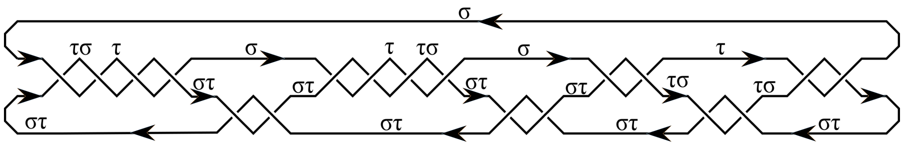

FIGURE 2. A link diagram with an $(S_3, 4)$-coloring

Focusing the number of colors, in this paper, we call an S_3-coloring Γ an (S_3, n)-*coloring* if Γ uses n colors for an integer $n \in \{1, 2, 3, 4, 5\}$. An $(S_3, 1)$-coloring is said to be a *trivial* S_3-coloring. A link L is said to be S_3-*colorable* (resp. (S_3, n)-*colorable*) if L has a diagram which admits a non-trivial S_3-coloring (resp. an (S_3, n)-coloring). Then, for links, the following holds.

Proposition 1.1. *Any $(S_3, 4)$-colorable link is also $(S_3, 5)$-colorable. Precisely, if a link L has a diagram which admits an S_3-coloring with 4 colors, then L also has another diagram which admits an S_3-coloring with 5 colors.*

On the other hand, one can ask if the converse does hold: Is an $(S_3, 5)$-colorable link always $(S_3, 4)$-colorable? It seems to expect too much naively, but there are some results on the Fox coloring related to this question. For example, it is known that if a knot K is Fox 5-colorable, then K has a diagram which admits a Fox 5-coloring with only 4 colors [6]. Also the second author [4] and independently M. Zhang, X. Jin and Q. Deng [7] proved that if a link L is $\mathbb{Z}$-colorable, then L has a diagram which admits a $\mathbb{Z}$-coloring with only 4 colors.

About the question above, in this paper, we obtain the following for 2-bridge links.

Theorem 1.2. *Any $(S_3, 5)$-colorable 2-bridge link L is $(S_3, 4)$-colorable.*

In the next section, we describe the local behavior of S_3-colorings on links preparing lemmas. Then, in Section 3, we give a proof of Theorem 1.2. By Theorem 1.2, all the $(S_3, 5)$-colorable 2-bridge links are $(S_3, 4)$-colorable. Some of them actually are also $(S_3, 3)$-colorable, but some others are not. In the last section, among 2-bridge links, we determine the double twist links and the torus links that are $(S_3, 4)$-colorable but not $(S_3, 3)$-colorable.

2. Local behavior of S_3-colorings

Throughout the paper, we set a presentation of S_3 as $\langle \sigma, \tau \mid \sigma^2 = \tau^2 = e, \sigma\tau\sigma = \tau\sigma\tau \rangle$, where e denotes the identity element of S_3. Then, note that $S_3 = \{e, \sigma, \tau, \sigma\tau\sigma, \sigma\tau, \tau\sigma\}$ as a set.

In this section, we observe the local behavior of S_3-colorings on links, and prepare some lemmas used in the remaining sections.

Let L be a link with a diagram D. Suppose that D admits a non-trivial S_3-coloring Γ. At a crossing of D, let x denote the over arc, y and z the under arcs at the crossing supposing y is the under arc before passing through the crossing and z is the other. See Figure 1. Then the possible colors of the arcs x, y, z assigned by Γ can be summarized in the following table.

TABLE 1. Colors on y when the colors on x and z are assigned

Color on z / Color on x	σ	τ	$\sigma\tau\sigma$	$\sigma\tau$	$\tau\sigma$
σ	σ	$\sigma\tau\sigma$	τ	$\tau\sigma$	$\sigma\tau$
τ	$\sigma\tau\sigma$	τ	σ	$\tau\sigma$	$\sigma\tau$
$\sigma\tau\sigma$	τ	σ	$\sigma\tau\sigma$	$\tau\sigma$	$\sigma\tau$
$\sigma\tau$	$\sigma\tau\sigma$ / τ	σ / $\sigma\tau\sigma$	τ / σ	$\sigma\tau$	$\tau\sigma$
$\tau\sigma$	τ / $\sigma\tau\sigma$	$\sigma\tau\sigma$ / σ	σ / τ	$\sigma\tau$	$\tau\sigma$

In the table above, α / β means that the color on y is α (resp. β) if the crossing is positive (resp. negative).

Remark 2.1. Also, from Table 1, we see that any link with at least 2 components admits an S_3-coloring with 2 colors $\{\sigma\tau, \tau\sigma\}$. See Figure 3 for example.

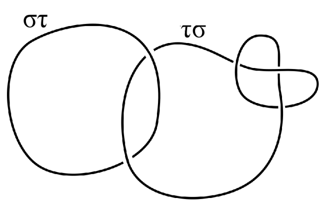

FIGURE 3. An S_3-colored link with 2 colors

The next is our fundamental lemma, which we will use implicitly and repeatedly. It follows from Table 1.

Lemma 2.2. *Let Γ be an S_3-coloring on a diagram D of a link L. Then the set of the colors on arcs of D corresponding to one component of L are either a subset of $\{\sigma, \tau, \sigma\tau\sigma\}$ or a subset of $\{\sigma\tau, \tau\sigma\}$. The set $\{\sigma, \tau, \sigma\tau\sigma\}$ or $\{\sigma\tau, \tau\sigma\}$ for a component of L is unchanged by modifying the diagram and the coloring by Reidemeister moves.*

Proof. From Table 1, if one of the under arcs at a crossing of a link diagram is colored by one of $\{\sigma, \tau, \sigma\tau\sigma\}$ or one of $\{\sigma\tau, \tau\sigma\}$ by an S_3-coloring, then the other under arc is also. Thus the first statement holds. One can check the local behavior of S_3-colorings by Reidemeister moves to keep the set of colors on the related arcs. This implies the second statement. $\square$

We remark that this lemma can be derived from considering the conjugacy classes of S_3 or the conjugate quandle structure of S_3.

For a diagram D of a knot, there is a one-to-one correspondence between a non-trivial Fox 3-coloring on a diagram D and an $(S_3, 3)$-coloring on D as follows.

Lemma 2.3. (i) *For a non-splittable $(S_3, 3)$-colorable link L, the set of colors for an $(S_3, 3)$-coloring on a diagram of L is $\{\sigma, \tau, \sigma\tau\sigma\}$.*

(ii) *For a knot K, there is a one-to-one correspondence between a Fox 3-coloring on a diagram D and an S_3-coloring on D of K. Thus a knot K is S_3-colorable if and only if K is Fox 3-colorable. In particular, if a knot is (S_3, n)-colorable, then $n = 1$ or 3.*

Proof. (i) Suppose that a diagram D of a link L admits an $(S_3, 3)$-coloring Γ. From Lemma 2.2, the set of colors on each component of the link are either of $\{\sigma, \tau, \sigma\tau\sigma\}$ or $\{\sigma\tau, \tau\sigma\}$. Let α, β, γ be the three colors used by Γ. If $\alpha \in \{\sigma, \tau, \sigma\tau\sigma\}$ and $\beta, \gamma \in \{\sigma\tau, \tau\sigma\}$, then, by Table 1, the arc colored by α is constantly an over arc, or an under arc at the crossing with the over arc colored by α, a contradiction. Thus the component with an arc colored by α is splittable from the other components, implying that L is splittable. Similarly, the same argument applies for the case $\alpha \in \{\sigma\tau, \tau\sigma\}$ and $\beta, \gamma \in \{\sigma, \tau, \sigma\tau\sigma\}$. Thus, if L is non-splittable, the set of 3 colors for Γ is $\{\sigma, \tau, \sigma\tau\sigma\}$.

(ii) Suppose that K is Fox 3-colorable, i.e., K has a diagram D of a knot K admitting a non-trivial S_3-coloring. Then, by Lemma 2.2, the set of colors appearing are either from $\{\sigma, \tau, \sigma\tau\sigma\}$ or from $\{\sigma\tau, \tau\sigma\}$. If an arc on D could have a color from $\{\sigma\tau, \tau\sigma\}$, since K has only one component, then, by Table 1, the coloring uses only one color on D, that is, the coloring is trivial, contradicting Γ is non-trivial. It follows that the set of colors for the coloring must be from $\{\sigma, \tau, \sigma\tau\sigma\}$. In this case, seeing Table 1, we note that all the 3 colors appear or only single color appears at each of the crossings of D. Thus, if the coloring Γ is non-trivial, then Γ must use 3 colors. By replacing the colors $\{\sigma, \tau, \sigma\tau\sigma\}$ to $\{0, 1, 2\}$, we can verify by Table 1 that a Fox 3-coloring on D can be obtained from Γ. Conversely, one can obtain an S_3-coloring from a Fox 3-coloring on a knot diagram by setting the colors $\{0, 1, 2\}$ to the colors $\{\sigma, \tau, \sigma\tau\sigma\}$. See Table 1 again. □

Remark 2.4. For splittable links, the lemma above does not hold. See Figure 4 for example.

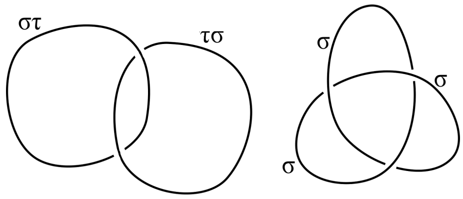

FIGURE 4. An $(S_3, 3)$-coloring on a splittable 2-component link with the three colors $\{\sigma, \sigma\tau, \tau\sigma\}$

From Lemma 2.3, a knot K is S_3-colorable if and only if K is Fox 3-colorable. In particular, if a knot is (S_3, n)-colorable, then $n = 1$ or 3.

On the other hand, if a link L has at least 2 components, then L can be (S_3, n)-colorable with $n \geq 4$, as illustrated in Figure 2 for an example.

For such S_3-colorings with 4 or 5 colors, we have the following.

Lemma 2.5. *Let L be a non-splittable link and D a diagram of L. Suppose that D admits an $(S_3, 4)$-coloring or an $(S_3, 5)$-coloring, say Γ.*

(i) *The set of colors of Γ contains at least 2 colors from $\{\sigma, \tau, \sigma\tau\sigma\}$ and 2 colors from $\{\sigma\tau, \tau\sigma\}$.*

(ii) *The S_3-coloring induced from Γ on a diagram of L obtained by Reidemeister moves from D has at least 4 colors.*

Proof. Suppose that D admits an $(S_3, 4)$-coloring or an $(S_3, 5)$-coloring, say Γ. Since L has at least two components by Lemma 2.3(ii), one of which is colored by Γ with $\{\sigma, \tau, \sigma\tau\sigma\}$, and the other is colored by Γ with $\{\sigma\tau, \tau\sigma\}$ by Lemma 2.2.

(i) Suppose for a contradiction that Γ uses only one color, say γ, from $\{\sigma\tau, \tau\sigma\}$. Then, by Table 1, the arc colored by γ is constantly an over arc, or an under arc at the crossing with the over arc colored by γ. This means that the component is splittable, and it contradicts that L is non-splittable. Thus the set of colors of Γ contains at least 2 colors from $\{\sigma, \tau, \sigma\tau\sigma\}$ and 2 colors from $\{\sigma\tau, \tau\sigma\}$.

(ii) Let Γ' be the S_3-coloring induced from Γ on a diagram of L obtained by Reidemeister moves from D. Then, by Lemma 2.2, such sets of colors on the components are unchanged by Reidemeister moves, and so, Γ' has at least one color in $\{\sigma, \tau, \sigma\tau\sigma\}$ and one color in $\{\sigma\tau, \tau\sigma\}$. Moreover, since L is non-splittable, there exists at least one crossing where the pair of the colors above appear. Then, by Table 1, there has to be one more color at the crossing.

Thus Γ' uses at least 3 colors with one color in $\{\sigma, \tau, \sigma\tau\sigma\}$ and one color in $\{\sigma\tau, \tau\sigma\}$. It follows from Lemma 2.3, together with above, the coloring Γ' is not an $(S_3, 3)$-coloring. Therefore, if D admits an $(S_3, 4)$-coloring or an $(S_3, 5)$-coloring, then any S_3-coloring on a diagram of L obtained by Reidemeister moves from D with the coloring has at least 4 colors. □

Now we give a proof of Proposition 1.1.

Proof of Proposition 1.1. Let L be an $(S_3, 4)$-colorable link and D a diagram of L with an $(S_3, 4)$-coloring Γ.

If L is non-splittable, then there exist 2 colors in $\{\sigma, \tau, \sigma\tau\sigma\}$ and 2 colors in $\{\sigma\tau, \tau\sigma\}$ on D from Lemma 2.5(i). Let $\alpha \in \{\sigma, \tau, \sigma\tau\sigma\}$ be the color which Γ does not use. Consider an arc on D colored by $\beta, \gamma \in \{\sigma, \tau, \sigma\tau\sigma\}$ with $\beta, \gamma \neq \alpha$. Then one can deform D and Γ to a diagram with a coloring so that α appears by using Reidemeister move II repeatedly, as illustrated in Figure 5. Then the coloring so obtained uses five colors by Lemma 2.5(ii).

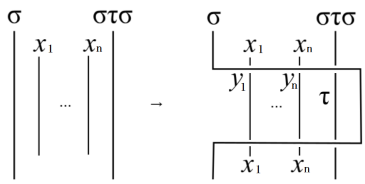

FIGURE 5. Making τ appear from $\{\sigma, \sigma\tau\sigma, \sigma\tau, \tau\sigma\}$

When L is splittable, we also have to consider the case that there exists 3 colors in $\{\sigma, \tau, \sigma\tau\sigma\}$ and 1 color in $\{\sigma\tau, \tau\sigma\}$ on D. In this case, let $\alpha \in \{\sigma\tau, \tau\sigma\}$ be a color which Γ does not use. On the other hand, D contains an arc colored by $\beta \in \{\sigma\tau, \tau\sigma\}$ with $\beta \neq \alpha$. Then one can deform D with the coloring to a diagram with a coloring with α by using Reidemeister move II repeatedly, as illustrated in Figure 6. Then the coloring so obtained uses five colors by Lemma 2.5(ii).

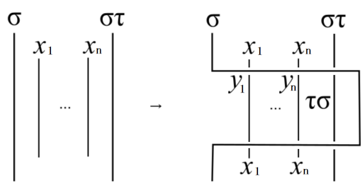

FIGURE 6. Making $\tau\sigma$ appear from $\{\sigma, \tau, \sigma\tau\sigma, \sigma\tau\}$

□

3. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2. Recall that it is known that a 2-bridge link always has a Conway diagram $C(2a_1, 2b_1, \ldots, 2b_m, 2a_{m+1})$ depicted in Figure 7. See [3, Chapter 2] about the 2-bridge links and the Conway diagrams (called "Conway's normal form" in the book) for example. In the following, we always assume that $a_i \neq 0$ and $b_j \neq 0$ and all the 2-bridge links are oriented as illustrated in Figure 7.

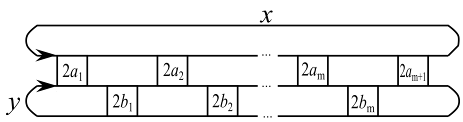

FIGURE 7. A Conway diagram $C(2a_1, 2b_1, \ldots, 2b_m, 2a_{m+1})$

We first show the following lemma.

Lemma 3.1. *The Conway diagram* $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ *of a* 2*-bridge link* L *admits an* $(S_3, 4)$*-coloring if* $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2$ *holds for the diagram.*

Note that the last congruent equation is equivalent to that the linking number of the two components of a two-bridge link is even.

Proof of Lemma 3.1. We try to construct an $(S_3, 4)$-coloring on a Conway diagram $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ from the left end of the diagram.

We fix colors on arcs x, y in Figure 7 as σ and $\sigma\tau$ respectively. Then, let us try to make a coloring by setting the color on the arc next to the right of a colored arc by using Table 1. Repeatedly perform this procedure from left to right.

First we see the colors in the twist regions with $2a_i$ crossings $(1 \leq i \leq m+1)$. Since $2a_i$ is even, pairs of colors before and after $2a_i$ crossings are the same or another color pair. Precisely, if a_i is even, the pairs of colors before and after $2a_i$ crossings are coincide. If a_i is odd, the pairs of colors before and after $2a_i$ crossings are distinct, but in a fixed pattern. For example, if a pair of colors $\{\sigma, \sigma\tau\}$ appears before the twist, then the pairs of colors on the parallel arcs during the twist are $\{\sigma, \sigma\tau\}$ or $\{\tau, \tau\sigma\}$ alternately as illustrated in Figure 8. In particular, during the twists, only 4 colors can appear.

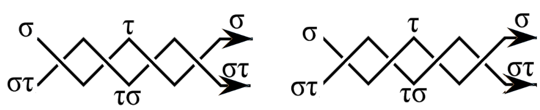

FIGURE 8. Colors in the twist with $2a_i$ crossings

Next we see the colors in the twist regions with $2b_j$ crossings ($1 \leq j \leq m$). On the arcs in the twist with $2b_j$ crossings, just two colors $\sigma\tau, \tau\sigma$ appear. Moreover, the colors on the parallel arcs before and after the twisting are the same. See Figure 9.

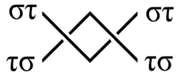

FIGURE 9. Colors in the twist with $2b_i$ crossings

From this procedure, checking the right-end of the diagram, we can obtain an S_3-coloring on the diagram if and only if $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2$ holds. By the construction, the coloring so obtained uses only 4 colors. □

Proof of Theorem 1.2. Let L be an $(S_3, 5)$-colorable 2-bridge link. By Reidemeister moves, we deform a diagram D of L with an $(S_3, 5)$-coloring to a Conway diagram $D_C = C(2a_1, 2b_1, \ldots, 2b_m, 2a_{m+1})$ as shown in Figure 7 with the induced S_3-coloring Γ. By Lemma 2.5(i) and (ii), the coloring Γ uses at least 2 colors from $\{\sigma, \tau, \sigma\tau\sigma\}$ and 2 colors from $\{\sigma\tau, \tau\sigma\}$. Moreover, by Lemma 2.2, the arcs contained in one component have the colors either from $\{\sigma, \tau, \sigma\tau\sigma\}$ or from $\{\sigma\tau, \tau\sigma\}$.

Now we consider the colors on the arcs x and y in Figure 7 by Γ.

When $\Gamma(x) \in \{\sigma, \tau, \sigma\tau\sigma\}$ and $\Gamma(y) \in \{\sigma\tau, \tau\sigma\}$, then, by retaking the colors if necessary, the coloring is completely the same as that constructed in the proof of Lemma 3.1. That is, Γ is an $(S_3, 4)$-coloring on the diagram, and $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2$ must hold.

Consider the case that $\Gamma(x) \in \{\sigma\tau, \tau\sigma\}$ and $\Gamma(y) \in \{\sigma, \tau, \sigma\tau\sigma\}$. Then one can deform the diagram and the coloring to Γ' so that $\Gamma'(x) \in \{\sigma, \tau, \sigma\tau\sigma\}$ and $\Gamma'(y) \in \{\sigma\tau, \tau\sigma\}$ by Reidemeister moves. Precisely, it is achieved by rotating the interior part of the thin line inside-out, keeping the exterior part of the line fixed as illustrated in Figure 10. Also see [5, Chapter 9].

After such modifications, in the same way as the proof of Lemma 3.1, we see that the condition $\Sigma_{i=1}^{m+1}|a_i| \equiv 0 \pmod 2$ have to be satisfied, and Γ' is an $(S_3, 4)$-coloring on the diagram.

It concludes that if a 2-bridge link L is $(S_3, 5)$-colorable, then L has a Conway diagram $C(2a_1, 2b_1, 2a_2, 2b_2, \ldots, 2b_m, 2a_{m+1})$ satisfying $\sum_{i=1}^{m+1} |a_i| \equiv 0 \pmod 2$, and the diagram admits an $(S_3, 4)$-coloring, i.e., the link L is $(S_3, 4)$-colorable. This completes the proof of Theorem 1.2. □

4. Examples

From Theorem 1.2, any $(S_3, 5)$-colorable 2-bridge link is $(S_3, 4)$-colorable. Among such $(S_3, 4)$-colorable links, there exists some of the links which is also $(S_3, 3)$-colorable and the others are not. In this section, we collect some

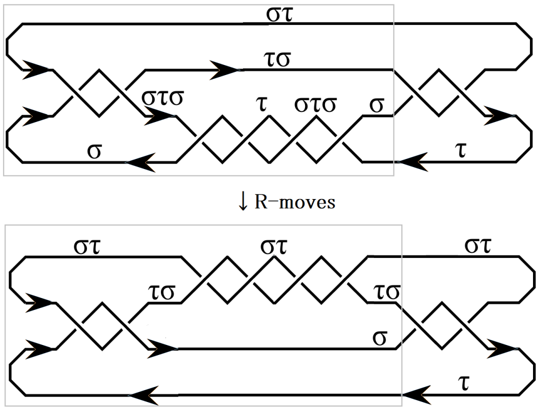

FIGURE 10. Reidemeister moves to $\Gamma'(x) \in \{\sigma, \tau, \sigma\tau\sigma\}$

examples of S_3-colorings for 2-bridge links, and in particular, consider double twist links. One of the simplest 2-bridge links would be 2-bridge torus links, that are the torus links with only two strands.

Example 4.1 (The torus link $T(2, q)$)**.** By Theorem 1.2, the torus link $T(2, q)$ is $(S_3, 4)$-colorable if and only if $q \equiv 0 \pmod 4$. The next figure depicts a torus link with $(S_3, 4)$-coloring which is not $(S_3, 3)$-colorable.

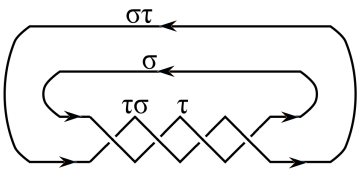

FIGURE 11. Torus link $T(2, 4)$ with an $(S_3, 4)$-coloring

In fact, by using Table 1, one can see that the standard torus diagram of $T(2, q)$ (Figure 11) is $(S_3, 4)$-colorable if and only $q \equiv 0 \pmod 4$ and $T(2, q)$ is $(S_3, 3)$-colorable if and only if $q \equiv 0 \pmod 3$. See Figures 8 and 12.

Also, by [2], the determinant of $T(2, q)$ is q, and so, $T(2, q)$ is Fox 3-colorable, equivalently, is $(S_3, 3)$-colorable if and only if $q \not\equiv 0 \pmod 3$.

For example, the torus link $T(2, 12)$ is (S_3, n)-colorable for $n = 3, 4, 5$.

Next, we consider *double-twist links*, which are the links admitting the diagrams shown in Figure 14.

An example of the double twist link with $(S_3, 4)$-coloring which is not $(S_3, 3)$-colorable is depicted in Figure 15.

Actually, for double twist links, we have the following.

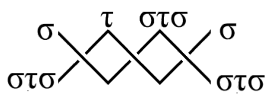

FIGURE 12. Twists with $\{\sigma, \tau, \sigma\tau\sigma\}$

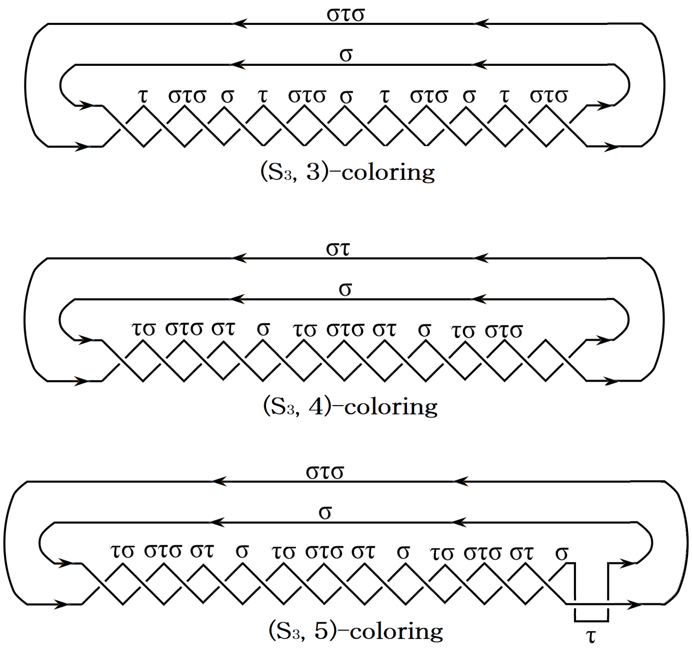

FIGURE 13. S_3-colorings for $T(2,12)$

Proposition 4.2. *A double twist link $J(k,l)$ depicted in Figure 14 is $(S_3,4)$-colorable if and only if $kl \equiv 3 \pmod 4$, and is $(S_3,3)$-colorable if and only if $kl \equiv 2 \pmod 3$.*

Proof. To see which $J(k,l)$ is $(S_3,4)$-colorable, we need to consider Conway diagrams to apply Theorem 1.2, but here, we directly consider the diagram D of $J(k,l)$ shown in Figure 14.

First we show that D is $(S_3,4)$-colorable if $kl \equiv 3 \pmod 4$. We set colors $\Gamma(x)$, $\Gamma(y)$ of arcs x, y on Figure 14 as $\Gamma(x) = \sigma, \Gamma(y) = \sigma\tau$. Then the pair

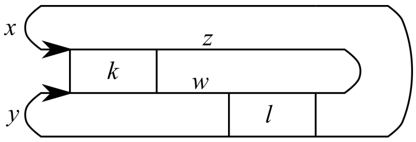

FIGURE 14. A diagram of a double twist link $J(k,l)$

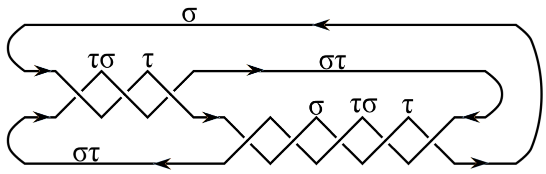

FIGURE 15. Double twist link $J(3,5)$ with an $(S_3,4)$-coloring

of colors $(\Gamma(z),\Gamma(w))$ on arcs (z,w) is fixed as $(\tau\sigma,\sigma)$ with $k \equiv 1 \pmod 4$, or $(\sigma\tau,\tau)$ with $k \equiv 3 \pmod 4$ to make a coloring on D by Table 1. For the case of $k \equiv 1 \pmod 4$, $l \equiv 3 \pmod 4$ also holds, and so D is S_3-colorable as $(\Gamma(x),\Gamma(y),\Gamma(z),\Gamma(w)) = (\sigma,\sigma\tau,\tau\sigma,\sigma)$. See Figure 16. Note that $\sigma\tau\sigma$ does not appear during the twists, that is, the coloring is an $(S_3,4)$-coloring.

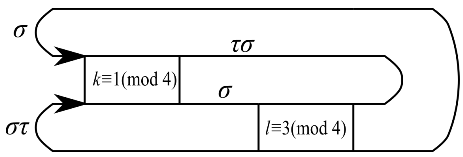

FIGURE 16. A diagram of a double twist link $J(k,l)$

In the same way, in the case of $k \equiv 3 \pmod 4$, D is shown to be $(S_3,4)$-colorable.

Conversely, suppose that $J(k,l)$ is $(S_3,4)$-colorable. In the same argument as the proof of Theorem 1.2, the diagram D of $J(k,l)$ admits a S_3-coloring such that the arcs contained in one component are all colored by either of $\{\sigma,\tau,\sigma\tau\sigma\}$ or $\{\sigma\tau,\tau\sigma\}$. Then, as above, by seeing the colors on the arcs from the left end, one can check that the condition $kl \equiv 3 \pmod 4$ is necessary.

For $(S_3,3)$-colorability, again, by [2], the determinant of $J(k,l)$ is shown to be $1+kl$, and so, $J(k,l)$ is Fox 3-colorable, equivalently, is $(S_3,3)$-colorable if and only if $kl \equiv 2 \pmod 3$. $\square$

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