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CONVERGENCE OF SEQUENCES IN GENERALIZED TOPOLOGICAL SPACES VIA FILTER

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Abstract. In this paper a generalization of convergent sequences in connection with generalized topologies and filters is given. Additionally, properties such as uniqueness, behavior related to continuous functions are established and notions relative to product spaces.

1. Introduction

The sequences of real numbers play a crucial role in mathematics due to the assistance provided when proving theorems and topological properties, as well as in applied mathematics. To name a few instances, sequences are key regarding the characterization of continuous functions or compact subsets in metrizable spaces. In applied mathematics, sequences of real numbers are used extensively in fields such as numerical analysis, scientific computing, and optimization. For instance, to prove the existence of solutions related to some numerical equations through the fixed-point theorem or iterative methods like the divide and conquer algorithm, in asymptotic notation, which allows estimating the efficiency of an algorithm and they are often used to approximate solutions to mathematical problems that cannot be solved exactly, such as the solutions to differential equations or optimization problems [12], the relevance of the matter cannot be denied.

A broad group of experts in mathematics have made contributions regarding the generalization of the notion of convergence using many structures such as ideals, filters, semi-open sets, and generalized topologies [1, 2, 8, 13, 15–19, 21]. These approaches have many applications in different areas of mathematics, for example, the notion of Painleve-Kuratowski convergence for a sequence of sets [16] was used to study the convergence of the strict efficient and weak efficient solution sets [20].

In 1937, Henri Cartan, one of the founders of the Bourbaki group, introduced the concepts of filter and ultrafilter [5, 6], Bourbaki's exposition of General Topology [3] relied heavily on the concepts of filters, ideals, and convergence

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spaces. In fact, these concepts were some of the main cornerstones of Bourbaki's approach to mathematics in general, which emphasized the importance of rigor and abstraction in mathematical reasoning. In 2000, Csaszar [10] introduced the notion of generalized topology as a collection that is stable under unions, allowing a more flexible notion of openness that includes sets that may not be open in the traditional sense. Csaszar's work on generalized topology has had a significant impact on the field of topology and has led to the development of other related structures. In this article, we consider (X, μ) to be a generalized topological space and $\mathcal{F}$ a filter on $\mathbb{N}$ and we shall define when the sequence (x_n) in X is $\mu_{\mathcal{F}}$-convergent to a point $x \in X$. We will show that by choosing the particular cases of (X, μ) and filters, we could recover a well-known classical concepts of convergence. We will prove the uniqueness of the limit when certain separation axioms are assumed and the behavior of these sequences under continuous functions. We will also give conditions to establish characterizations of generalized closure in terms of the convergence of sequences using ultrafilters, and we will also show that to study the convergence of sequences in product of generalized topological spaces using filters, it is enough to study the convergence of each coordinate sequence.

2. Preliminaries

Let $X \neq \emptyset$ be any set and 2^X denote the power set of X. A subfamily μ of 2^X is said to be a generalized topology (briefly GT) on X if $\emptyset \in \mu$ and μ is closed under arbitrary union. We call the pair (X, μ) a generalized topological space (briefly GTS). The elements of μ are called μ-open sets and the complements are called μ-closed sets. If $\mathbf{B} \subset 2^X$ such that $\emptyset \in \mathbf{B}$, then all unions of some elements of $\mathbf{B}$ is a GT on X which will be denoted by $\mu(\mathbf{B})$ and $\mathbf{B}$ is said to be a base for $\mu(\mathbf{B})$. A generalized topology μ is said to be strong if $X \in \mu$. For $A \subseteq X$, the largest μ-open set contained in A is called the μ-interior of A and is denoted by $i_\mu(A)$. The smallest μ-closed set containing A is called the μ-closure of A and is denoted by $c_\mu(A)$. In Lemma 2.1 of [9], it is established that $x \in c_\mu(A)$ in a GTS (X, μ) if and only if $U \cap A \neq \emptyset$ for every μ-open set U containing x.

A GTS (X, μ) is called μ-Hausdorff if for any pair of points $x, y \in X$, $x \neq y$, there exist $U, V \in \mu$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. A GTS $(X.\mu)$ is said to be that satisfies the generalized interior nonempty property (briefly g.i.n.e property) [4] if for any finite collection $\{U_i\}_{i=1}^n$ of nonempty μ-open sets such that $\bigcap_{i=1}^n U_i \neq \emptyset$, satisfies $i_\mu\left(\bigcap_{i=1}^n U_i\right) \neq \emptyset$. Let (X, μ) and (Y, λ) be GT spaces, a function $f : (X, \mu) \to (Y, \lambda)$ is said to be (μ, λ)-continuous [10] if the inverse image of every λ-open subset of Y is a μ-open subset of X. A subset A of a GTS (X, μ) is called μ_h-set if for every μ-open set U such that $U \neq \emptyset$ and $U \neq X$ is true that $A \subset i_\mu(A \cup U)$. A GTS (X, μ) is called μ-compact if every μ-open cover has a finite subcover.

Definition 1. Let X be a nonempty set. A nonempty family $\mathcal{F}$ of subsets of X is a filter on X if it satisfies the following properties:

(1) $\emptyset \notin \mathcal{F}$.
(2) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
(3) If $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

Example 2.1. The following are examples of filters, for more details you can review [14].

(1) Given an infinite set X, the Frechet filter is defined as $\mathcal{F}_r = \{F \subset X : X - F \text{ is finite}\}$.
(2) Given an infinite set X, the principal filter of a nonempty subset A, is defined as $\mathcal{F}_A = \{F \subset X : A \subseteq F\}$. When A is a unitary set $\{x\}$, the principal filter is denoted by $\mathcal{F}_x$.
(3) If $\mathcal{I}$ is an ideal on X, then the collection $\mathcal{F}_{\mathcal{I}} = \{A \subset X : X - A \in \mathcal{I}\}$ is a filter on X.
(4) Given a subset A of $\mathbb{N}$, the density of A, denoted by $d(A)$, is defined as:
$$d(A) = \lim_{n\to\infty} \frac{Card\,(A \cap \{1, 2, \ldots, n\})}{n}.$$
The collection $\mathcal{F}_d = \{A \subseteq \mathbb{N} : d(A) = 1\}$ is a filter on $\mathbb{N}$ called density filter.

Definition 2. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two filters on the same set X, $\mathcal{F}_2$ is said to be finer than $\mathcal{F}_1$ if $\mathcal{F}_1 \subset \mathcal{F}_2$. If also $\mathcal{F}_1 \neq \mathcal{F}_2$, then $\mathcal{F}_2$ is said to be strictly finer than $\mathcal{F}_1$.

We said that a filter $\mathcal{F}$ is free if $\mathcal{F}_r \subset \mathcal{F}$. If a filter is not free, then we said the filter is fixed. The filter $\mathcal{F}_d$ is free because $\mathcal{F}_r \subseteq \mathcal{F}_d$ and the principal filter is fixed.

Definition 3. Let $X \neq \emptyset$ be any set and $\mathcal{F}$ a filter on X. If a filter $\mathcal{F}$ has the property that there is no filter on X which is strictly finer than a filter $\mathcal{F}$, $\mathcal{F}$ is called an ultrafilter on X.

Observe that if X is an infinite set, each point x of X uniquely will determine an ultrafilter $\mathcal{F}_x$. The free ultrafilters exist only with the help of Zorn's Lemma. You can find the proof of the following three results in [7].

Theorem 2.2. *Each filter is contained in an ultrafilter.*

Theorem 2.3. *Let X be a nonempty set. Then each family of subsets of X that satisfies the finite intersection property is contained in an ultrafilter.*

Theorem 2.4. *Let X be any set and $\mathcal{F}$ a filter on X. Then the following statements are equivalent*:

(1) *$\mathcal{F}$ is an ultrafilter.*
(2) *For each subset A of X, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.*

(3) *For each subset A of X such that $A \notin \mathcal{F}$, there exists $F \in \mathcal{F}$ such that $F \cap A = \emptyset$.*

If $\mathcal{F}$ is a fixed ultrafilter on an infinite set X, then it is possible to find $x \in X$ such that $\mathcal{F} = \mathcal{F}_x$.

We will now consider the very special case of ultrafilters on the set of naturals numbers, denoted by $\beta(\mathbb{N})$. Each $m \in \mathbb{N}$ is identified with its main ultrafilter $\mathcal{F}_m$, that is, it is possible to create a one-to-one relationship between $\mathbb{N}$ and the unitary ultrafilters on $\mathbb{N}$. So $\mathbb{N}^* = \beta(\mathbb{N}) - \mathbb{N}$ can be considered as the collection of all free ultrafilters on $\mathbb{N}$

3. $\mu_{\mathcal{F}}$-convergence and some basic properties

In this section, we introduce the notion of a $\mu_{\mathcal{F}}$-convergent sequence to a point in a generalized topological space (X, μ) and study some properties associated with this concept. We start with a lemma that will be used in the proof of the next results.

Lemma 3.1. *Let (X, μ) be a GTS that satisfies g.i.n.e. property such that every singleton is a μ_h-set and $x \in X$ be a point in the μ-closure of the set $A = \{x_n : n \in \mathbb{N}\}$. If $x_n \neq x$ for all $n \in \mathbb{N}$, then the family $\{\{n \in \mathbb{N} : x_n \in V\} : V \in \mu;\ \ x \in V\}$ satisfies the finite intersection property.*

Proof. Suppose that the family $\{\{n \in \mathbb{N} : x_n \in V\} : V \in \mu;\ \ x \in V\}$ does not satisfies the finite intersection property. Then there exists a finite subcollection $\{\{n \in \mathbb{N} : x_n \in V_i\} : V_i \in \mu;\ \ x \in V_i\}_{i=1}^k$ such that:

$$\bigcap_{i=1}^{k} \{n \in \mathbb{N} : x_n \in V_i\} = \emptyset.$$

It follows that $\{n \in \mathbb{N} : x_n \in \bigcap_{i=1}^k V_i\} = \emptyset$, and then, for all $n \in \mathbb{N}$ we have $x_n \notin \bigcap_{i=1}^k V_i$. Note that $i_\mu\left(\bigcap_{i=1}^k V_i\right) \neq \emptyset$ because of $x \in \bigcap_{i=1}^k V_i$ and X satisfies the g.i.n.e property.

Since $i_\mu\left(\bigcap_{i=1}^k V_i\right)$ is non-empty μ-open and every singleton is a μ_h-set then $i_\mu\left(\bigcap_{i=1}^k V_i \bigcup \{x\}\right)$ is a μ-open set containing x and

$$\begin{aligned} i_\mu\left(\bigcap_{i=1}^{k} V_i \bigcup \{x\}\right) \bigcap A \subset \left(\bigcap_{i=1}^{k} V_i \bigcup \{x\}\right) \bigcap A \\ &= \left(\left(\bigcap_{i=1}^{k} V_i\right) \bigcap A\right) \bigcup \left(\{x\} \bigcap A\right) \\ &= \emptyset. \end{aligned}$$

Finally, using the fact that $x_n \neq x$ for all $n \in \mathbb{N}$, we obtain that $x \notin cl_\mu(A)$, which is a contradiction. ☐

In generalized topological spaces, the finite intersection of open sets is not necessarily an open set. The property g.i.n.e. guarantees that even when the finite intersection of open sets is not open, it is always possible to obtain a non-empty open set contained in the intersection. The following examples show us that the hypothesis in the previous lemma can not be removed.

Example 3.2. Let $\mathbb{R}$ be the set of all real numbers with the usual topology, $\mu = \{U \subset \mathbb{R} : U \subset cl\,(int(U))\}$ and $\mathcal{F}$ a filter on $\mathbb{N}$, note that the collection μ doesn't have g.i.n.e. property. Consider the sequence (x_n) defined as $x_n := \frac{(-1)^n}{n}$. Then $0 \in c_\mu\left(\{x_n : n \in \mathbb{N}\}\right)$, consider the family

$$\mathcal{A} = \{\{n \in \mathbb{N} : x_n \in V\} : V \in \mu;\;\; 0 \in V\}.$$

Note that:

- $\{n \in \mathbb{N} : x_n \in [0, 1/2)\} = \{2k : k \in \mathbb{N}\} \in \mathcal{A}$.
- $\{n \in \mathbb{N} : x_n \in (-1/3, 0]\} = \{2k + 1 : k \in \mathbb{N}\} \in \mathcal{A}$.

However

$$\{n \in \mathbb{N} : x_n \in [0, 1/2)\} \cap \{n \in \mathbb{N} : x_n \in (-1/3, 0]\} = \emptyset.$$

So the family $\mathcal{A}$ doesn't have the finite intersection property.

Example 3.3. Let $\mathbb{R}$ be the set of all real numbers,

$$\mu = \{\emptyset, \mathbb{R}, \{0\}\} \cup \{\mathbb{R} - \{y\} : y \in \mathbb{R} \text{ and } y \neq 0\},$$

note that the collection μ satisfies g.i.n.e. property but the singleton don't are μ_h-sets because $\{0\} \in \mu$ but is not true that $\{1\} \subset i_\mu(\{0\} \cup \{1\}) = \{0\}$. Consider the sequence (x_n) defined as:

$$x_n = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

Then $3 \in c_\mu\left(\{x_n : n \in \mathbb{N}\}\right)$, consider the family

$$\mathcal{A} = \{\{n \in \mathbb{N} : x_n \in V\} : V \in \mu;\;\; 3 \in V\}.$$

Note that:

- $\{n \in \mathbb{N} : x_n \in \mathbb{R} - \{1\}\} = \{2k : k \in \mathbb{N}\} \in \mathcal{A}$.
- $\{n \in \mathbb{N} : x_n \in \mathbb{R} - \{2\}\} = \{2k + 1 : k \in \mathbb{N}\} \in \mathcal{A}$.

However

$$\{n \in \mathbb{N} : x_n \in \mathbb{R} - \{1\}\} \cap \{n \in \mathbb{N} : x_n \in \mathbb{R} - \{2\}\} = \emptyset.$$

So the family $\mathcal{A}$ doesn't have the finite intersection property.

Now we shall give our main definition, which provides us with a generalization of the notion of convergence sequences.

Definition 4. Let (X, μ) be a GTS and $\mathcal{F}$ a filter on $\mathbb{N}$. A sequence (x_n) in X is said to be $\mu_\mathcal{F}$-convergent to a point $x \in X$ if for every μ-open set U such that $x \in U$, $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$. In this case, the point x is called the $\mu_\mathcal{F}$-limit of (x_n) and is denoted by $\mu_\mathcal{F}\text{-}\lim x_n = x$.

Observe that choosing particular cases of generalized topologies and filters, we recover well known classical concepts of convergence as is shown:

(1) If μ is a topology and $\mathcal{F}$ is the Frechet filter $\mathcal{F}_r$, then the $\mu_{\mathcal{F}}$-convergent sequences are the convergent sequences in usual sense.
(2) If μ is a metric topology and $\mathcal{F}$ is the filter $\mathcal{F}_{\mathcal{I}}$ for some ideal $\mathcal{I}$ on X, then the $\mu_{\mathcal{F}}$-convergent sequences are the $\mathcal{I}$-convergent sequences [17].
(3) If μ is a metric topology and $\mathcal{F}$ is the density filter $\mathcal{F}_d$, then the $\mu_{\mathcal{F}}$-convergent sequences are the statistically convergent sequences [8].
(4) If μ is the collection of semi open sets and $\mathcal{F}$ is the filter $\mathcal{F}_{\mathcal{I}}$ for some ideal $\mathcal{I}$ on X, then the $\mu_{\mathcal{F}}$-convergent sequences are the S-$\mathcal{I}$-convergent sequences [13].
(5) If μ is a GT and $\mathcal{F}$ is the Frechet filter $\mathcal{F}_r$, then the $\mu_{\mathcal{F}}$-convergent sequences are the μ-convergent sequences [19].
(6) If μ is a topology and $\mathcal{F}$ is a filter on $\mathbb{N}$, then the $\mu_{\mathcal{F}}$-convergent sequences are the convergent sequences in filter.

Theorem 3.4. *Let (X,μ) be a GTS and $\mathcal{F}$ a filter on $\mathbb{N}$. If $\mathcal{F}$ is a free filter, then every μ-convergent sequence is a $\mu_{\mathcal{F}}$-convergent sequence. The converse is true if $\mathcal{F}$ is the Frechet filter.*

Proof. Suppose that μ-$\lim x_n = x$, and choose $U \in \mu$ such that $x \in \mu$. Then there exists a positive integer n_0 such that $x_n \in U$ for all $n \geq n_0$. It follows that: $\{n \in \mathbb{N} : n \geq n_0\} \subset \{n \in \mathbb{N} : x_n \in U\}$. Since $\{n \in \mathbb{N} : n \geq n_0\} \in \mathcal{F}_r$ and $\mathcal{F}$ is a free filter, we obtain that $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$ and we conclude that $\mu_{\mathcal{F}}$-$\lim x_n = x$.

Conversely, suppose that $\mu_{\mathcal{F}_r}$-$\lim x_n = x$, and choose $U \in \mu$ such that $x \in \mu$. Then $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}_r$. So, the set $\{n \in \mathbb{N} : x_n \notin U\}$ is finite. Take $n_0 = \max\{n \in \mathbb{N} : x_n \notin U\}$. It follows that, if $n \geq n_0$, then $x_n \in U$. In consequence, we obtain that μ-$\lim x_n = x$. □

There exists sequences that are $\mu_{\mathcal{F}}$-convergent but not μ-convergent as we can see in the next example.

Example 3.5. Let $\mathbb{R}$ be the set of all real numbers,

$$\mu = \{\emptyset, \mathbb{R}, \{0\}\} \cup \{\mathbb{R} - \{y\} : y \in \mathbb{R} \text{ and } y \neq 0\}$$

and $\mathcal{F}_d$ the density filter defined in Example 2.1. Consider the sequence (x_n) defined as:

$$x_n = \begin{cases} 2, & n \neq 2^k \text{ for all } k \in \mathbb{N}, \\ 1, & n = 2^k \text{ for some } k \in \mathbb{N}. \end{cases}$$

Note that (x_n) does not μ-converge to any element of $\mathbb{R}$. However, $\mu_{\mathcal{F}_d}$-$\lim x_n = 1$. Indeed, choose $U \in \mu$ such that $1 \in U$, then we analyze the following two cases:

- If $2 \in U$, then $\{n \in \mathbb{N} : x_n \in U\} = \mathbb{N} \in \mathcal{F}_d$.
- If $2 \notin U$, then $\{n \in \mathbb{N} : x_n \in U\} = \mathbb{N} - \{2^n : n \in \mathbb{N}\} \in \mathcal{F}_d$.

In both cases, we conclude that $\mu_{\mathcal{F}_d}$-$\lim x_n = x$.

Theorem 3.6. *Let (X,μ) be a GTS and $\mathcal{F}$ a filter on $\mathbb{N}$. If X is a μ-Hausdorff space and (x_n) is a $\mu_{\mathcal{F}}$-convergent sequence, then its $\mu_{\mathcal{F}}$-limit is unique.*

Proof. Suppose that $\mu_{\mathcal{F}}$-lim $x_n = x$ and $\mu_{\mathcal{F}}$-lim $x_n = y$ for some $x, y \in X$ with $x \neq y$. Then there exist $U, V \in \mu$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. Using Definition 4, $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$ and $\{n \in \mathbb{N} : x_n \in V\} \in \mathcal{F}$. It follows that

$$\{n \in \mathbb{N} : x_n \in U \cap V\} = \{n \in \mathbb{N} : x_n \in U\} \cap \{n \in \mathbb{N} : x_n \in V\} \in \mathcal{F}.$$

Since $\mathcal{F}$ is a filter on $\mathbb{N}$, then $\{n \in \mathbb{N} : x_n \in U \cap V\} \neq \emptyset$, in consequence, there exists a positive integer $n_0 \in \{n \in \mathbb{N} : x_n \in U \cap V\}$, and then $x_{n_0} \in U \cap V$, contradiction. □

It is widely known that a sequence (x_n) in a topological space X converges to a point $x \in X$ if and only if there exists an ultrafilter $p \in \beta(\mathbb{N})$ such that the sequence converges to x with respect to p. This provides a powerful characterization of convergence using ultrafilters, and allows us to study convergence properties of sequences in a more abstract and general way. The next theorem shows us that the reciprocal proposition of the aforementioned statement is true in generalized topological spaces.

Theorem 3.7. *Let (X,μ) be a GTS and $\mathcal{F}$ a filter on $\mathbb{N}$. If (x_n) is a sequence on X such that $x = \mu_{\mathcal{F}}$-lim x_n for some $x \in X$, then $x \in cl_\mu(\{x_n : n \in \mathbb{N}\})$.*

Proof. Suppose that $x = \mu_{\mathcal{F}}$-lim x_n. Denote by $A = \{x_n : n \in \mathbb{N}\}$ and consider that $x \notin cl_\mu(A)$, then there exists $U \in \mu$ such that $x \in U$ and then $U \cap A = \emptyset$. Since $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$ and $U \cap A = \emptyset$, then $\{n \in \mathbb{N} : x_n \in U\} = \emptyset \in \mathcal{F}$. The last one is a contradiction, therefore $x \in cl_\mu(A)$. □

Corollary 3.8. *Let (X,μ) be a GTS, $\mathcal{F}$ a filter on $\mathbb{N}$ and $A \subset X$. If (x_n) is a sequence on A such that $x = \mu_{\mathcal{F}}$-lim x_n for some $x \in X$, then $x \in cl_\mu(A)$.*

Proof. It is a direct consequence of Theorem 3.7. □

Unlike what happens in topological spaces, we need to include additional hypothesis to ensure that the reciprocal of Theorem 3.7 comes to be true.

Theorem 3.9. *Let (X,μ) be a GTS that satisfies the g.i.n.e. property, every singleton is a μ_h-set, and $x \in X$ such that $x_n \neq x$ for all $n \in \mathbb{N}$. If $x \in cl_\mu(\{x_n : n \in \mathbb{N}\})$, then there exists an ultrafilter $p \in \beta(\mathbb{N})$ such that $x = \mu_p$-lim x_n.*

Proof. Suppose that $x \in cl_\mu(\{x_n : n \in \mathbb{N}\})$. Then by Lemma 3.1, the following family

$$\mathcal{A} = \{\{n \in \mathbb{N} : x_n \in V\} : V \in \mu,\ \ x \in V\}$$

satisfies the finite intersection property, and then by Theorem 2.3, there exists an ultrafilter $p \in \beta(\mathbb{N})$ such that $x = \mu_p$-lim x_n. □

The following result concerns the stability of convergence of sequences under continuous functions, although the proof is very simple, it is included for completeness.

Theorem 3.10. *Let (X,μ) and (Y,λ) be two GTS, let $f:(X,\mu)\to(Y,\lambda)$ be a (μ,λ)-continuous function, and let $\mathcal{F}$ be a filter on $\mathbb{N}$. If $x=\mu_{\mathcal{F}}$-lim x_n, then $f(x)=\lambda_{\mathcal{F}}$-lim $f(x_n)$.*

Proof. Suppose that $x=\mu_{\mathcal{F}}\text{-}\lim x_n$ and let V be a λ-open set such that $f(x)\in V$. Since f is (μ,λ)-continuous $f^{-1}(V)\in\mu$ and $x\in f^{-1}(V)$, therefore,

$$\{n\in\mathbb{N}:f(x_n)\in V\}=\{n\in\mathbb{N}:x_n\in f^{-1}(V)\}\in\mathcal{F}.$$

In consequence, $f(x)=\lambda_{\mathcal{F}}\text{-}\lim\ f(x_n)$. □

The following example shows that the converse of the above theorem need not be true.

Example 3.11. Let $\mathbb{R}$ be the set of all real numbers,

$$\mu=\{\emptyset,\mathbb{R},\{0\}\}\cup\{\mathbb{R}-\{y\}:y\in\mathbb{R}\ \text{ and }\ y\neq 0\},$$

$\mathcal{F}$ any filter and consider the sequence (x_n), where $x_n=n$. The function $f:(\mathbb{R},\mu)\to(\mathbb{R},\mu)$ defined as $f(x)=0$ is (μ,μ)-continuous. Observe that for all $U\in\mu$, $0\in\mu$ and then $f(0)=0$. It is clear that:

$$\{n\in\mathbb{N}:f(x_n)\in U\}=\mathbb{N}\in\mathcal{F}.$$

So, $f(0)=\mu_{\mathcal{F}}\text{-}\lim\ f(x_n)$. However, (x_n) does not $\mu_{\mathcal{F}}$-converge to 0. Indeed, $\{0\}\in\mu$ and

$$\{n\in\mathbb{N}:x_n\in\{0\}\}=\emptyset\notin\mathcal{F}.$$

And this conclude the example.

In generalized compact topological spaces, it can be guaranteed that for any sequence, it converges to a point for every ultrafilter.

Theorem 3.12. *Let (X,μ) be a GTS such that X is a μ-compact space and (x_n) a sequence with infinite rank on X, then for all $p\in\mathbb{N}^*$ there exists $x\in X$ such that $x=\mu_p$-lim x_n.*

Proof. Let X be a μ-compact space; let (x_n) be a sequence with infinite rank on X and let $p\in\mathbb{N}^*$. Consider the family of all μ-closed subsets of X described as follows: $\mathcal{A}=\{cl_\mu(\{x_n:n\in A\}):A\in p\}$. It is easy to see that $\mathcal{A}$ satisfies the finite intersection property. In fact, take $cl_\mu(\{x_n:n\in A_1\})$ and $cl_\mu(\{x_n:n\in A_2\})\in\mathcal{A}$, since $A_1,A_2\in p$, then $A_1\cap A_2\in p$ and therefore, $A_1\cap A_2\neq\emptyset$. Note that:

$$\begin{aligned}\emptyset&\neq\{x_n:n\in A_1\cap A_2\}\\&\subseteq cl_\mu\left(\{x_n:n\in A_1\}\cap\{x_n:n\in A_2\}\right)\\&\subset cl_\mu\left(\{x_n:n\in A_1\}\right)\cap cl_\mu\left(\{x_n:n\in A_2\}\right).\end{aligned}$$

Now using the μ-compactness of X, we obtain that $\bigcap_{A\in p}cl_\mu(\{x_n:n\in A\})\neq\emptyset$.

Now take $x \in \bigcap_{A\in p} cl_\mu(\{x_n : n \in A\})$. We want to prove that $x = \mu_p$-lim x_n. In order to do that, consider the following sets $V \in \mu$, such that $x \in V$ and $\{n \in \mathbb{N} : x_n \in V\}$. Since for each $A \in p$, we have that $x \in cl_\mu(\{x_n : n \in A\})$, then $V \cap \{x_n : n \in A\} \neq \emptyset$, that is $\{n \in \mathbb{N} : x_n \in V\} \cap A \neq \emptyset$ for all $A \in p$. Using Theorem 2.4, we obtain that $\{n \in \mathbb{N} : x_n \in V\} \in p$, and then $x = \mu_p$-lim x_n. □

Now we will show that our notion of convergence can be extended to the product of generalized topological spaces, and that convergence is determined by the convergence of each coordinate. Let $\Delta \neq \emptyset$ be an index set and for each $\alpha \in \Delta$ and let (X_α, μ_α) be a strong GTS. We said that a set G_α satisfies the P_α property if $G_\alpha \in \mu_\alpha$ and $G_\alpha = X_\alpha$ except for finitely many values of $\alpha \in \Delta$. With this notation we can define the following collection:

$$\mathbf{B} = \{\Pi_{\alpha\in\Delta} G_\alpha : G_\alpha \text{ satisfies the } P_\alpha \text{ property}\}.$$

The GT μ having $\mathbf{B}$ as a basis is the generalized product topology of $X = \Pi_{\alpha\in\Delta} X_\alpha$. In Proposition 2.7 of [11], it is established that μ is strong and each projection π_α is (μ, μ_α)-continuous.

Theorem 3.13. *Let (X, μ) be the product GT, where $X = \Pi_{\alpha\in\Delta} X_\alpha$; let (X_α, μ_α) be a strong GTS for each $\alpha \in \Delta$, let $\mathcal{F}$ be a filter on $\mathbb{N}$ and let (x_n) be a sequence in X. Then $x = \mu_{\mathcal{F}}$-lim x_n if and only if $\pi_\alpha(x) = \mu_{\mathcal{F}}$-lim $\pi_\alpha(x_n)$ for every $\alpha \in \Delta$.*

Proof. Suppose that $x = \mu_{\mathcal{F}}$-lim x_n. Since $\pi_\alpha : X \to X_\alpha$ is (μ, μ_α)-continuous for every $\alpha \in \Delta$, it follows from Theorem 3.10 that $\pi_\alpha(x) = \mu_{\mathcal{F}}$-lim $\pi_\alpha(x_n)$ for every $\alpha \in \Delta$.

Reciprocally, take any μ-open set W such that $x \in W$, then there exists a neighborhood basic B such that $x \in B \subset W$. It is clear that

$$B = \pi_{\alpha_1}^{-1}(U_{\alpha_1}) \cap \cdots \cap \pi_{\alpha_m}^{-1}(U_{\alpha_m}),$$

where for each i, $1 \leq i \leq m$; $x_{\alpha_i} \in U_{\alpha_i}$. Since $\pi_{\alpha_i}(x) = \mu_{\mathcal{F}}$-lim $\pi_{\alpha_i}(x_n)$, then

$$\{n \in \mathbb{N} : \pi_{\alpha_i}(x_n) \in U_{\alpha_i}\} \in \mathcal{F}.$$

Therefore,

$$\begin{aligned}\{n \in \mathbb{N} : x_n \in B\} &= \{n \in \mathbb{N} : \pi_{\alpha_1}(x_n) \in U_{\alpha_n}\} \cap \cdots \cap \{n \in \mathbb{N} : \pi_{\alpha_m}(x_n) \in U_{\alpha_m}\} \\ &\in \mathcal{F}.\end{aligned}$$

Due to $\{n \in \mathbb{N} : x_n \in B\} \subset \{n \in \mathbb{N} : x_n \in W\}$, we conclude that

$$\{n \in \mathbb{N} : x_n \in W\} \in \mathcal{F},$$

and the proof of the theorem is complete. □

4. Conclusions

We have introduced the notion of $\mu_{\mathcal{F}}$-convergent sequences in the sense of generalized topology and filter. An example of $\mu_{\mathcal{F}}$-convergent but not μ-convergent sequence was presented, showing thus that this new notion is more general than the notion of convergent sequence in generalized topological spaces. The uniqueness of the limit was proved under the assumption of certain separation axioms. Conditions were established to characterize the generalized closure in terms of ultrafilters.

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