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THE p-PART OF DIVISOR CLASS NUMBERS FOR CYCLOTOMIC FUNCTION FIELDS

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Abstract. In this paper, we construct explicitly an infinite family of primes P with $h_P^{\pm} \equiv 0 \pmod{q^{\deg P}}$, where $h_P^{\pm}$ are the plus and minus parts of the divisor class number of the P-th cyclotomic function field over $\mathbb{F}_q(T)$. By using this result and Dirichlet's theorem, we give a condition of $A, M \in \mathbb{F}_q[T]$ such that there are infinitely many primes P satisfying with $h_P^{\pm} \equiv 0 \pmod{p^e}$ and $P \equiv A \pmod{M}$.

1. Introduction

Let p be prime. Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements. Let $k = \mathbb{F}_q(T)$ be the rational function field over $\mathbb{F}_q$, and let $\mathbb{A} = \mathbb{F}_q[T]$ be the associated polynomial ring. We denote by $\mathbb{P}$ the set of all monic irreducible polynomials in $\mathbb{A}$. For a monic polynomial $N \in \mathbb{A}$, let K_N, K_N^+ be the N-th cyclotomic function field, and its maximal real subfield, respectively. Let h_N (resp. h_N^+) be the divisor class number of K_N (resp. K_N^+), and $h_N^- = h_N/h_N^+$.

For a positive integer n, we consider the infinity of the set of primes

$$H^{\pm}(n) = \left\{ P \in \mathbb{P} \mid h_P^{\pm} \equiv 0 \pmod{n} \right\}.$$

Goss [3] found Kummer's criterion for function fields, and proved that ${}^{\#}H^-(p) = \infty$ when $q = p \geq 3$. Feng [1] extended Goss's results and showed that ${}^{\#}H^{\pm}(p) = \infty$ for a general q. Yaouanc [11] used elliptic curves over finite fields to prove that ${}^{\#}H^-(q) = \infty$. He also showed that there exist infinitely many primes $P \in \mathbb{P}$ such that $h(\mathcal{O}_P^+) \equiv 0 \pmod{q}$, where $h(\mathcal{O}_P^+)$ is the ideal class number for K_P^+. This result implies ${}^{\#}H^+(q) = \infty$ because $h_P^+ \equiv 0 \pmod{h(\mathcal{O}_P^+)}$. More recently, Lee and Lee [6] gave a lower bound on the p-rank of the divisor class group for K_P^+, and proved that ${}^{\#}H^+(p^{p(p-1)}) = \infty$ when $q = p$.

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Our first goal of this paper is to construct explicitly an infinite family of primes $P \in \mathbb{P}$ with

$$h_P^{\pm} \equiv 0 \pmod{q^{\deg P}} \tag{1}$$

(see Theorem 3.3 and Corollary 3.5). As a corollary of this result, we have

$$\begin{cases} {}^{\#}H^{\pm}(p^e) = \infty & \text{if } q \neq 2, \\ {}^{\#}H^{+}(p^e) = \infty & \text{if } q = 2 \end{cases} \tag{2}$$

for any positive integer e.

Secondly, we prove a much stronger form of (2). For $A, M \in \mathbb{A}$ and a positive integer n, we define

$$H^{\pm}(A, M, n) = \left\{P \in \mathbb{P} \mid P \equiv A \pmod{M},\ h_P^{\pm} \equiv 0 \pmod{n}\right\}.$$

Then we have:

Theorem 1.1. *Let $A, M \in \mathbb{A}$ with*

$$\deg M \geq 1, \quad \gcd(M, T^p - T) = \gcd(M, A) = 1.$$

We further assume that there exits a polynomial $B \in \mathbb{A}$ such that $A \equiv B(T^p - T) \pmod{M}$. Then, for any positive integer e, we have

$$\begin{cases} {}^{\#}H^{\pm}(A, M, p^e) = \infty & \textit{if } q \neq 2, \\ {}^{\#}H^{+}(A, M, p^e) = \infty & \textit{if } q = 2. \end{cases} \tag{3}$$

This paper is organized as follows. In Section 2, based on the idea of Lee-Lee [6], we give lower bounds on the p-parts of divisor class numbers for cyclotomic function fields. In Section 3, we use these lower bounds to construct an infinite family of primes $P \in \mathbb{P}$ satisfying with (1). By using this result and Dirichlet's theorem, we prove Theorem 1.1.

2. Lower bounds of divisor class numbers

For a positive integer n, the nth Goss-Bernoulli number is defined by

$$B_n(T) = \begin{cases} \sum_{i=0}^{\infty} s_i(n) & \text{if } n \not\equiv 0 \mod q-1, \\ \sum_{i=0}^{\infty} -is_i(n) & \text{if } n \equiv 0 \mod q-1. \end{cases}$$

Here,

$$s_i(n) = \sum_{A \in \mathbb{A}(i)} A^n,$$

where $\mathbb{A}(i)$ is the set of all monic polynomials in $\mathbb{A}$ of degree i. We put

$$l(n) = a_0 + a_1 + \cdots + a_{d-1},$$

where $a_0 + a_1 q + \cdots + a_{d-1} q^{d-1}$ is the q-adic expansion of n.

Lemma 2.1 (cf. [2] Proposition 2.11). *If $i > l(n)/(q-1)$, then $s_i(n) = 0$. In particular, $B_n(T)$ is a polynomial in $\mathbb{A}$.*

Lemma 2.2 (cf. [2] Lemma 6.1)**.** *If* $n \equiv 0 \pmod{q-1}$, *then we have*

$$\sum_{i=0}^{\infty} s_i(n) = 0.$$

Lemma 2.3.

(1) *If* $n = (q-1) + q^e$ $(e = 1, 2, \ldots)$, *then* $B_n(T) = 1 - (T^{q^e} - T)$.

(2) *If* $n = (q-1) + (q-1)q^e$ $(e = 1, 2, \ldots)$, *then* $B_n(T) = 1 - (T^{q^e} - T)^{q-1}$.

Proof. By Lemmas 2.1 and 2.2, we have

$$B_n(T) = \begin{cases} 1 + s_1(n) & \text{if } q > 2 \text{ and } n = (q-1) + q^e, \\ -s_1(n) - 2s_2(n) = 2 + s_1(n) & \text{if } n = (q-1) + (q-1)q^e. \end{cases}$$

By Theorems 4.1 and 4.2 in [5], we have

$$s_1(n) = \begin{cases} -(T^{q^e} - T) & \text{if } q > 2 \text{ and } n = (q-1) + q^e, \\ -1 - (T^{q^e} - T)^{q-1} & \text{if } n = (q-1) + (q-1)q^e. \end{cases}$$

Therefore, the result follows. □

Let $P \in \mathbb{P}$ be a prime of degree d. We denote by C_P (resp. C_P^+) the p-primary part of the divisor class group of degree 0 for K_P (resp. K_P^+). Let

$$\varphi : C_P^+ \to C_P \quad ([D] \mapsto [i_{K_P/K_P^+}(D)])$$

be the conorm map, and put $C_P^-(p) = \operatorname{coker} \varphi$ (cf. Chapter 3 in [9]).

Lemma 2.4. *The map* φ *is injective. In particular, the order of* $C_P^-(p)$ *is equal to the* p*-part of* h_P^-.

Proof. Suppose that $[D] \in \ker \varphi$. Then we have $i_{K_P/K_P^+}(D) = (\alpha)_{K_P}$ for some $\alpha \in (K_P)^\times$. Fix a generator σ of the Galois group for K_P/K_P^+. Then we see that $(\alpha^\sigma)_{K_P} = (\alpha)_{K_P}$. Hence $\alpha^{\sigma-1} \in \mathbb{F}_q^\times$, and so $\alpha^{q-1} \in (K_P^+)^\times$. We thus get $[D] = [0]$ because $\gcd(q-1, p) = 1$. □

Let W be the ring of Witt vectors of $\mathbb{A}/P\mathbb{A}$, and $\mathfrak{m}$ be its maximal ideal. Let $\omega : (\mathbb{A}/P\mathbb{A})^\times \to W$ be the Teichmüller character such that $\omega(x) \equiv x \pmod{\mathfrak{m}}$ for any $x \in (\mathbb{A}/P\mathbb{A})^\times$. Then we have the decomposition into isotypical components according to characters of $(\mathbb{A}/P\mathbb{A})^\times$:

$$C_P \otimes_{\mathbb{Z}_p} W = \bigoplus_{n=1}^{q^d-2} C_P(\omega^n)$$

(Similarly for $C_P^\pm$). It is easy to check that

$$C_P^+(\omega^n) \simeq C_P(\omega^n) \;\text{ and }\; C_P^-(\omega^n) = \{0\} \;\text{ if } n \equiv 0 \pmod{q-1},$$

and

$$C_P^+(\omega^n) = \{0\} \;\text{ and }\; C_P^-(\omega^n) \simeq C_P(\omega^n) \;\text{ if } n \not\equiv 0 \pmod{q-1}.$$

Hence we obtain

$$C_P^+ \otimes_{\mathbb{Z}_p} W \simeq \bigoplus_{\substack{n=1 \\ q-1 \mid n}}^{q^d-2} C_P(\omega^n), \tag{4}$$

$$C_P^- \otimes_{\mathbb{Z}_p} W \simeq \bigoplus_{\substack{n=1 \\ q-1 \nmid n}}^{q^d-2} C_P(\omega^n). \tag{5}$$

Goss and Sinnott [4] proved that

$$C_P(\omega^{q^d-1-n}) \neq \{0\} \iff B_n(T) \equiv 0 \pmod P \tag{6}$$

for $1 \leq n < q^d - 1$ (see Theorem 5.3.8 in [10]). By (4)-(6), we have the following lower bounds on p-ranks:

$$\mathrm{rank}_p(C_P^+) \geq l_P^+ \quad \text{and} \quad \mathrm{rank}_p(C_P^-) \geq l_P^-,$$

where

$$l_P^+ = {}^\#\{1 \leq n \leq q^d - 2 \mid n \equiv 0 \pmod{q-1},\ B_n(T) \equiv 0 \pmod P\},$$
$$l_P^- = {}^\#\{1 \leq n \leq q^d - 2 \mid n \not\equiv 0 \pmod{q-1},\ B_n(T) \equiv 0 \pmod P\}.$$

In particular, we have

$$h_P^+ \equiv 0 \pmod{p^{l_P^+}} \quad \text{and} \quad h_P^- \equiv 0 \pmod{p^{l_P^-}}. \tag{7}$$

3. Proofs of main results

For a positive integer d, we define

$$\mathbb{T}_d := \{F \in \mathbb{P} \mid \deg F = d,\ \mathrm{Tr}_{\mathbb{F}_q}(a_{d-1,F}) = -1\},$$

where $a_{i,F}$ is the coefficient of degree i in F, and Tr_E is the trace from E to $\mathbb{F}_p$ for a finite extension $E/\mathbb{F}_p$.

Lemma 3.1. $\mathbb{T}_d \neq \phi$.

Proof. It is clear if $d = 1$ or $(q, d) = (2, 2)$. So we may assume either $d \geq 2$, $q \geq 3$ or $d \geq 3$, $q = 2$. For $u \in \mathbb{F}_p$, we set

$$\mathbb{T}_d(u) := \{F \in \mathbb{P} \mid \deg F = d,\ \mathrm{Tr}_{\mathbb{F}_q}(a_{d-1,F}) = u\}.$$

By Theorem 3.25 in [7], we have

$$\sum_{u \in \mathbb{F}_p} {}^\#\mathbb{T}_d(u) = {}^\#\{F \in \mathbb{P} \mid \deg F = d\} = \frac{1}{d} \sum_{k \mid d} \mu(k) q^{\frac{d}{k}},$$

where μ is the Möbius function. This implies that

$$d \sum_{u \in \mathbb{F}_p} {}^\#\mathbb{T}_d(u) \geq q^d - 2q^{[\frac{d}{2}]} + 1,$$

where $[x]$ is the greatest integer less than or equal to x. We see that

$$d^{\#}\mathbb{T}_d(0) \leq {}^{\#}\{\alpha \in \mathbb{F}_{q^d} \mid \mathrm{Tr}_{\mathbb{F}_{q^d}}(\alpha) = 0\} = \frac{q^d}{p},$$

and ${}^{\#}\mathbb{T}_d(u) = {}^{\#}\mathbb{T}_d$ $(u \in \mathbb{F}_p^{\times})$. Hence we have

$$d(p-1)^{\#}\mathbb{T}_d \geq \left(1 - \frac{1}{p}\right) q^d - 2q^{[\frac{d}{2}]} + 1. \tag{8}$$

From the assumption of (q, d), the right-side of (8) is positive. We thus get $\mathbb{T}_d \neq \phi$. □

For a positive integer d, we define

$$I(d) := \{\alpha \in \mathbb{F}_{q^d} \mid \mathrm{Tr}_{\mathbb{F}_{q^d}}(\alpha) = 1\}.$$

Then we have:

Lemma 3.2.

(1) *If $\alpha \in I(d)$, then $T^p - T - \alpha$ is irreducible in $\mathbb{F}_{q^d}[T]$.*

(2) $T^{q^d} - T - 1 = \prod_{\alpha \in I(d)}(T^p - T - \alpha)$.

Proof. See Corollary 3.79 and Theorem 3.80 in [7]. □

Theorem 3.3. *Assume that $F \in \mathbb{T}_d$. Then the polynomial $P = F(T^p - T)$ is irreducible in $\mathbb{F}_q[T]$ of degree dp, and the following holds*:

(1) $h_P^- \equiv 0 \pmod{q^{dp}}$ *if* $q \neq 2$.

(2) $h_P^+ \equiv \begin{cases} 0 \pmod{q^{dp}} & \text{if } p \neq 2, \\ 0 \pmod{q^d} & \text{if } p = 2 \text{ and } d \geq 2. \end{cases}$

Proof. Let $\beta \in \overline{\mathbb{F}}_q$ be a root of P, and $\alpha = \beta^p - \beta$. Since $F(\alpha) = 0$ and $F \in \mathbb{T}_d$, we have $\alpha \in \mathbb{F}_q(\alpha) = \mathbb{F}_{q^d}$, and

$$\mathrm{Tr}_{\mathbb{F}_{q^d}}(\alpha) = \mathrm{Tr}_{\mathbb{F}_q}(-a_{d-1,F}) = 1.$$

Hence $\alpha \in I(d)$. By Lemma 3.2(1), we have $[\mathbb{F}_{q^d}(\beta) : \mathbb{F}_{q^d}] = p$, and so $[\mathbb{F}_q(\beta) : \mathbb{F}_q] = dp$. This implies that P is irreducible in $\mathbb{F}_q[T]$ of degree dp.

We next prove the assertion (1). Put $n = (q-1) + q^d$. From Lemma 2.3, we have

$$1 \leq n < q^{dp} - 1, \quad n \not\equiv 0 \pmod{q-1}, \quad B_n(T) = 1 - (T^{q^d} - T).$$

It follows from Lemma 3.2(2) that β is a root of $B_n(T)$. Hence we obtain $B_n(T) \equiv 0 \pmod{P}$. Suppose that $1 \leq n_1 < q^{dp} - 1$ satisfies with $n_1 \equiv np^e$ $\pmod{q^{dp} - 1}$ for some integer $e \geq 0$. Since $A^{n_1} \equiv A^{np^e} \pmod{P}$ for any $A \in \mathbb{A}$, we have $B_{n_1}(T) \equiv B_n(T)^{p^e} \equiv 0 \pmod{P}$. We thus get

$$l_P^- \geq {}^{\#}\{R(p^e n) \mid e = 0, 1, 2, \ldots\} = dpr, \tag{9}$$

where $R(x)$ is the remainder of x divided by $q^{dp} - 1$ (note $q = p^r$). By (7) and (9), we have $h_P^- \equiv 0 \pmod{q^{dp}}$.

Finally, we prove the assertion (2). Putting $n = (q-1) + (q-1)q^d$, then

$$1 \leq n < q^{dp} - 1, \quad n \equiv 0 \pmod{q-1}, \quad B_n(T) = 1 - (T^{q^d} - 1)^{q-1}.$$

By a similar discussion as above, we have $B_n(T) \equiv 0 \pmod P$, and

$$l_P^+ \geq {}^\#\{R(p^e n) \mid e = 0, 1, 2, \ldots\} = \begin{cases} dpr & \text{if } p \neq 2, \\ dr & \text{if } p = 2. \end{cases}$$

This leads the assertion (2). □

Example 3.4. Suppose that $q = 3$ and $F = T - 1 \in \mathbb{T}_1$. By Theorem 3.3, the polynomial $P = F(T^3 - T) = T^3 - T - 1$ is irreducible in $\mathbb{F}_3[T]$, and $h_P^- \equiv h_P^+ \equiv 0 \pmod{3^3}$. In fact, we find that $h_P^- = 2^{12} \cdot 3^3 \cdot 7$ and $h_P^+ = 3^9$ by PARI/GP computation.

The next result follows immediately from Lemma 3.1 and Theorem 3.3.

Corollary 3.5. *For any integer $d \geq 1$ ($d \geq 2$ if $p = 2$), there exists a prime $P \in \mathbb{P}$ of degree dp such that*

$$\begin{cases} h_P^\pm \equiv 0 \pmod{q^{dp}} & \text{if } p \neq 2, \\ h_P^\pm \equiv 0 \pmod{q^d} & \text{if } q > 2 \text{ and } p = 2, \\ h_P^+ \equiv 0 \pmod{q^d} & \text{if } q = 2. \end{cases}$$

In particular, for any positive integer e, we have

$$\begin{cases} {}^\# H^\pm(p^e) = \infty & \text{if } q \neq 2, \\ {}^\# H^+(p^e) = \infty & \text{if } q = 2. \end{cases}$$

In order to prove Theorem 1.1, we need the following form of Dirichlet's theorem.

Proposition 3.6. *Suppose that $A, M \in \mathbb{A}$ are relatively prime and* $\deg M \geq 1$. *For a positive integer d, we set*

$$\mathbb{P}_d(A, M) = \{P \in \mathbb{P} \mid P \equiv A \pmod M, \ \deg P = d\}.$$

Then

$$^\#\mathbb{P}_d(A, M) = \frac{1}{\Phi(M)} \frac{q^d}{d} + O\left(\frac{q^{\frac{d}{2}}}{d}\right),$$

where $\Phi(M)$ is the order of the multiplicative group of $\mathbb{A}/M\mathbb{A}$.

Proof. See Theorem 4.8 in [8]. □

Now we prove Theorem 1.1.

Proof. Since $\gcd(T^p - T, M) = 1$, we can choose $S \in \mathbb{A}$ and a positive integer n_0 such that

$$(T^p - T)S \equiv 1 \pmod M, \quad (T^p - T)^{n_0} \equiv 1 \pmod M.$$

We set

$$A_1(T) = B(T^{n_0-1}), \quad M_1(T) = \frac{T^{n_0} - 1}{\gcd(A_1^{n_0}, T^{n_0} - 1)}.$$

It is easy to check that A_1 and M_1 satisfy

$$A \equiv A_1(S(T)) \pmod{M}, \quad M_1(S(T)) \equiv 0 \pmod{M}, \quad \gcd(M_1, A_1) = 1.$$

Since $\gcd(M_1, T) = 1$, we can choose $A_2 \in \mathbb{A}$ such that

$$A_2 \equiv A_1 \pmod{M_1}, \quad A_2 \equiv 1 - aT \pmod{T^2},$$

where a is an element of $\mathbb{F}_q$ with $\mathrm{Tr}_{\mathbb{F}_q}(a) = 1$. Fix a positive integer $d_0 \geq 2$. By Proposition 3.6, there exists a prime $P_1 \in \mathbb{P}$ of degree d satisfying with

$$d \equiv 0 \pmod{n_0}, \quad d \geq \max\{d_0, e\}, \quad P_1 \equiv A_2 \pmod{M_1}.$$

Putting $P_2(T) = T^d P_1(1/T)$, then $P_2 \in \mathbb{T}_d$ because $P_1 \equiv 1 - aT \pmod{T^2}$. It follows from Theorem 3.3 that the polynomial $P = P_2(T^p - T)$ is irreducible in $\mathbb{A}$, and we have

$$\begin{cases} h_P^+ \equiv h_P^- \equiv 0 \pmod{p^e} & \text{if } q \neq 2, \\ h_P^+ \equiv 0 \pmod{2^e} & \text{if } q = 2. \end{cases}$$

Furthermore,

$$P \equiv (T^p - T)^d P_1(S(T)) \equiv A_1(S(T)) \equiv A \pmod{M},$$

and $\deg P = dp \geq d_0$. Hence we have Theorem 1.1. □

Example 3.7. We consider the case $q = 3$, $M = T^3 + T + 2$, and $A = T$. If $B = T + 2$, then $B(T^3 - T) \equiv A \pmod{M}$. Therefore, by Theorem 1.1, we have ${}^{\#}H^{\pm}(A, M, 3^e) = \infty$ for any positive integer e.

From now on, we focus on the case that M is irreducible.

Theorem 3.8. *Let $M \in \mathbb{P}$ and $A \in \mathbb{A}$ with*

$$\deg M \not\equiv 0 \pmod{p}, \quad \gcd(A, M) = 1.$$

Then, for any positive integer e, we have

$$\begin{cases} {}^{\#}H^{\pm}(A, M, p^e) = \infty & \text{if } q \neq 2, \\ {}^{\#}H^{+}(A, M, p^e) = \infty & \text{if } q = 2. \end{cases} \tag{10}$$

To prove Theorem 3.8, we first prove the next lemma.

Lemma 3.9. *For $a \in \mathbb{F}_q^{\times}$ and $d_0 \geq 1$, there exists a prime $F \in \mathbb{T}_d$ such that $F(0) = a$ and $d \geq d_0$.*

Proof. By Proposition 3.6, there exists a prime $P \in \mathbb{P}$ such that

$$d := \deg P \geq d_0, \quad d \not\equiv 0 \pmod{p}, \quad P \equiv a \pmod{T^q - T}.$$

Choose $z \in \mathbb{F}_q$ with $d\mathrm{Tr}_{\mathbb{F}_q}(z) = -1 - \mathrm{Tr}_{\mathbb{F}_q}(a_{d-1,P})$, and put $F(T) = P(T + z)$. Noting that

$$\mathrm{Tr}_{\mathbb{F}_q}(a_{d-1,F}) = \mathrm{Tr}_{\mathbb{F}_q}(dz + a_{d-1,P}) = -1,$$

we have $F \in \mathbb{T}_d$. Furthermore, we have $F(0) = P(z) = a$ because $P \equiv a \pmod{T^q - T}$. $\square$

Now we prove Theorem 3.8.

Proof of Theorem 3.8. Assume that $\gcd(M, T^p - T) = 1$. Let $\mathcal{R} = \mathbb{A}/M\mathbb{A}$ be the residue field of M, and let $\alpha = T^p - T \mod M \in \mathcal{R}$. Since $\mathcal{R}/\mathbb{F}_q(\alpha)$ is an Artin-Schreier extension, we have $[\mathcal{R} : \mathbb{F}_q(\alpha)] = 1$ or p. From $\deg M \not\equiv 0 \pmod{p}$, we must have $[\mathcal{R} : \mathbb{F}_q(\alpha)] = 1$. It follows that there exists $B \in \mathbb{A}$ with $A \equiv B(T^p - T) \pmod{M}$. Therefore, by Theorem 1.1, the equality (10) holds.

We next consider the case $\gcd(M, T^p - T) \neq 1$. Since M is irreducible, we have that $M = T - a$ for some $a \in \mathbb{F}_p$. Fix an integer $d_0 \geq \max\{2, e\}$. By Lemma 3.9, there exists a prime $F \in \mathbb{T}_d$ such that $F(0) = A(a)$ and $d \geq d_0$. From Theorem 3.3, the polynomial $P = F(T^p - T)$ is irreducible in $\mathbb{A}$, and

$$\begin{cases} h_P^+ \equiv h_P^- \equiv 0 \pmod{p^e} & \text{if } q \neq 2, \\ h_P^+ \equiv 0 \pmod{2^e} & \text{if } q = 2. \end{cases}$$

Furthermore, we have $\deg P = dp \geq d_0$ and $P \equiv A \pmod{M}$. Hence we obtain the equality (10). $\square$

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