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THE DIMENSION GRAPH FOR MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring and M be an R-module. The dimension graph of M, denoted by $DG(M)$, is a simple undirected graph whose vertex set is $Z(M) \setminus \mathrm{Ann}(M)$ and two distinct vertices x and y are adjacent if and only if $\dim M/(x,y)M = \min\{\dim M/xM, \dim M/yM\}$. It is shown that $DG(M)$ is a disconnected graph if and only if
(i) $\mathrm{Ass}(M) = \{\mathfrak{p}, \mathfrak{q}\}$, $Z(M) = \mathfrak{p} \cup \mathfrak{q}$ and $\mathrm{Ann}(M) = \mathfrak{p} \cap \mathfrak{q}$.
(ii) $\dim M = \dim R/\mathfrak{p} = \dim R/\mathfrak{q}$.
(iii) $\dim M/xM = \dim M$ for all $x \in Z(M) \setminus \mathrm{Ann}(M)$.
Furthermore, it is shown that $\mathrm{diam}(DG(M)) \leq 2$ and $\mathrm{gr}(DG(M)) = 3$, whenever M is Noetherian with $|Z(M) \setminus \mathrm{Ann}(M)| \geq 3$ and $DG(M)$ is a connected graph.

1. Introduction

The zero-divisor graph of a commutative ring was introduced and studied by I. Beak in [4]. Furthers, D. F. Anderson and P. S. Livingston in [2] studied the concept of zero-divisor graph on nonzero zero-divisors of a ring. Let R be a commutative ring and let $Z(R)$ denote its set of zero divisors. The zero-divisor graph of R is a graph with vertex set $Z^*(R) = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and only if $xy = 0$. The concept of zero-divisor graph has been studied by many authors, see for examples [1,3] and variations of zero-divisor graph are created by changing the vertex set, the edge condition, or both. The concept of the zero-divisor graph for rings has been generalized for modules in many papers, see for examples [5,6,8]. Let R be a commutative ring and let M be an R-module. In this paper, we associate a simple graph to the module M, denoted by $DG(M)$, with vertex set $Z(M) \setminus \mathrm{Ann}(M)$ and two distinct vertices x, y are adjacent if and only if

$$\dim M/(x,y)M = \min\{\dim M/xM, \dim M/yM\}.$$

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Note that, in general, $\dim M/(x,y)M \leq \min\{\dim M/xM, \dim M/yM\}$. Here, $\dim M$ denotes the Krull dimension of M. In Section 2, for a Noetherian R-module M we investigate the connectedness, diameter and girth of $DG(M)$. More precisely, it is shown that $\mathrm{diam}(DG(M)) \leq 2$ and $\mathrm{gr}(DG(M)) \in \{3, \infty\}$. In Section 3, we study the relationship between the dimension graph and zero-divisor graph of M.

Before we state some results, let us introduce some graphical notations. Let $G = (V(G), E(G))$ be a simple graph, $V(G)$ and $E(G)$ are called vertex set and edge set of G, respectively. Let $x, y \in V(G)$. We write $x - y$, whenever x and y are adjacent. The vertex x is called universal when it is adjacent to every other vertices. We say that G is connected if there is a path between any two distinct vertices. For vertices x and y of G, we define $\mathrm{d}(x, y)$ to be the length of a shortest path between x and y. If there is no path, then $\mathrm{d}(x, y) = \infty$. The diameter of G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(x, y) : x \text{ and } y \text{ are vertices of } G\}$. The graph G is complete if any two distinct vertices are adjacent and a complete graph with n vertices is denoted by K_n. The girth of G, denoted by $\mathrm{gr}(G)$, is the length of a shortest cycle in G ($\mathrm{gr}(G) = \infty$ if G contains no cycle).

Throughout this paper, R denotes a commutative ring with nonzero identity and M denotes a unitary R-module, $\mathrm{Ann}(M) = \{r \in R : rM = 0\}$, $Z(M) = \{r \in R : rm = 0 \text{ for some nonzero } m \in M\}$ denote the annihilator and the set of zero-divisors, respectively. Moreover, $\mathrm{Ass}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} = \mathrm{Ann}(m) \text{ for some nonzero } m \in M\}$ denotes the set of associated prime ideals of M and $\mathrm{Assh}(M) = \{\mathfrak{p} \in \mathrm{Ass}(M) : \dim M = \dim R/\mathfrak{p}\}$. For notations and terminologies not given in this paper, the reader is referred to [9].

2. Basic properties of dimension graph for modules

Let R be a commutative ring and M be an R-module. The Krull dimension of M, denoted by $\dim M$, is defined to be the supremum of lengths of chains of prime ideals of R which lay in $\mathrm{Supp}(M)$ if this supremum exists, and ∞ otherwise.

Definition. Let M be an R-module. The dimension graph of M, denoted by $DG(M)$, is a simple undirected graph associated to M with the vertex set $Z(M) \setminus \mathrm{Ann}(M)$ and a pair of distinct vertices x and y are adjacent if and only if

$$\dim M/(x,y)M = \min\{\dim M/xM, \dim M/yM\}.$$

Lemma 2.1. *Let M be an R-module and let $x \in Z(M) \setminus \mathrm{Ann}(M)$. Then the following statements are true*:

(i) *If $xM = M$, then x is a universal vertex of $DG(M)$.*
(ii) *If M is Noetherian and $x \in r(\mathrm{Ann}(M))$, then x is a universal vertex of $DG(M)$. In particular, if $|\mathrm{Ass}(M)| = 1$, then $DG(M)$ is a complete graph.*

(iii) *If* $xM \subseteq yM$ *for some* $y \in Z(R) \setminus \mathrm{Ann}(M)$*, then* x, y *are adjacent in* $DG(M)$*. In particular, if* $xy \neq 0$ *and* $x \neq xy$*, then* x, xy *are adjacent in* $DG(M)$*.*

Proof. (i) Since $M = xM = (x,y)M$ for any $y \in Z(M) \setminus \mathrm{Ann}(M)$, x and y are adjacent. Note that, if M is Noetherian, then $xM \neq M$.

(ii) Let $x \in r(\mathrm{Ann}(M))$. Then $x \in \bigcap_{\mathrm{Ann}(M) \subseteq \mathfrak{p}} \mathfrak{p}$ so $\dim M/xM = \dim M$. Assume that $y \in Z(M) \setminus \mathrm{Ann}(M)$ and $x \neq y$. It is obvious that $\dim M/(x,y)M \leq \min\{\dim M/xM, \dim M/yM\}$. Furthermore, for all $\mathfrak{p} \in \mathrm{Supp}(M)$ with $y \in \mathfrak{p}$ we have $x \in \mathfrak{p}$. Thus $\dim M/(x,y)M = \dim M/yM$ so x and y are adjacent.

(iii) Suppose that $y \in Z(R) \setminus \mathrm{Ann}(M)$ and $xM \subseteq yM$. Since $(x,y)M = yM$, we have $\dim M/(x,y)M = \dim M/yM$ so x and y are adjacent in $DG(M)$. The second part now is obvious. □

Corollary 2.2. *If* M *is a Noetherian* R*-module and* $Z(M) = r(\mathrm{Ann}(M))$*, then* $DG(M)$ *is a complete graph.*

Proof. It is an immediate consequence of Lemma 2.1(ii). □

The second part of the following example shows that the converse of Corollary 2.2 is not true, necessarily.

Example 1. (i) Let p be a prime number and consider $M = \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. Then $p\mathbb{Z} = Z(M) \neq r(\mathrm{Ann}(M)) = 0$. By Lemma 2.1(i), $DG(M)$ is a complete graph so p is a universal vertex but $p \notin r(\mathrm{Ann}(M))$.

(ii) Let k be a field and consider the residue ring R of the ring $k[X_1, X_2]$ of polynomials over k in indeterminates X_1, X_2 given by $R = k[X_1, X_2]/(X_1) \cap (X_1^2, X_2)$. For each $i = 1, 2$, let x_i denotes the natural image of X_i in R. Then it is easy to see that $DG(R)$ is a complete graph but $(x_1, x_2) = Z(R) \neq \mathrm{Nil}(R) = (x_1)$.

(iii) Let k be a field and consider the residue ring R of the ring $k[X_1, X_2, X_3]$ of polynomials over k in indeterminates X_1, X_2, X_3 given by $R = k[X_1, X_2, X_3]/(X_1) \cap (X_1^2, X_2, X_3)$. For each $i = 1, 2, 3$, let x_i denote the natural image of X_i in R. Then $DG(R)$ is not a complete graph since $\dim R/(x_1 + x_2)R = 1$, $\dim R/x_3R = 1$ and $\dim R/(x_1 + x_2, x_3)R = 0$ moreover $(x_1, x_2, x_3) = Z(R) \neq \mathrm{Nil}(R) = (x_1)$.

Theorem 2.3. *Let* M *be a Noetherian* R*-module. Then* $DG(M)$ *is disconnected if and only if the following conditions are true*:

(i) $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M) = \mathfrak{p} \cap \mathfrak{q}$*,* $Z(M) = \mathfrak{p} \cup \mathfrak{q}$ *and* $\mathrm{Ass}(M) = \{\mathfrak{p}, \mathfrak{q}\}$*.*

(ii) $\dim M = \dim R/\mathfrak{p} = \dim R/\mathfrak{q}$*.*

(iii) $\dim M/xM = \dim M$*, for all* $x \in Z(M) \setminus \mathrm{Ann}(M)$*.*

Proof. (i) Let $DG(M)$ be a disconnected graph. Then in view of Lemma 2.1(ii), $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$ and there are $x, y \in Z(M) \setminus \mathrm{Ann}(M)$ such that there is no path between them. If $xyM \neq 0$, then by the hypotheses $x \neq xy$, $y \neq xy$ and $x - xy - y$ is a path in $DG(M)$ that is a contradiction. Thus we may assume that $xyM = 0$. If there exists $z \in Z(M) \setminus (\mathrm{Ann}(xM) \cup \mathrm{Ann}(yM))$, then $x - xz - z - yz - y$ is a path in $DG(M)$ which is a contradiction. Hence, $Z(M) = \mathrm{Ann}(xM) \cup \mathrm{Ann}(yM)$. Since $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$, therefore $\mathrm{Ann}(xM) \nsubseteq \mathrm{Ann}(yM)$ and $\mathrm{Ann}(yM) \nsubseteq \mathrm{Ann}(xM)$. Hence, $\mathfrak{p} = \mathrm{Ann}(xM)$, $\mathfrak{q} = \mathrm{Ann}(yM)$ are associated prime ideals of M and $y \in \mathfrak{p} \setminus \mathfrak{q}$, $x \in \mathfrak{q} \setminus \mathfrak{p}$, see [9, Lemma 9.36]. Assume that $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t = \mathfrak{p}$ and $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_k = \mathfrak{q}$ are chains of prime ideals of R, where $\mathfrak{p}_i$ and $\mathfrak{q}_j$ belong to $\mathrm{Ass}(M)$ for all $i = 0, 1, \ldots, t$ and $j = 0, 1, \ldots, k$. Thus from $xyM = 0$ it follows that $y \in \mathfrak{p}_i$ and $x \in \mathfrak{q}_j$ for all $i = 0, 1, \ldots, t$ and $j = 0, 1, \ldots, k$. Suppose that $z \in \mathfrak{p} \cap \mathfrak{q} \setminus \mathrm{Ann}(M)$. If $z \in \mathfrak{p}_i \setminus \mathfrak{p}_{i-1}$ and $z \in \mathfrak{q}_j \setminus \mathfrak{q}_{j-1}$ for some $0 \leq i \leq t$ and $0 \leq j \leq k$, then $y - z$ and $z - x$ are two edges of $DG(M)$ so $y - z - x$ is a path which is a contradiction. Hence, $\mathfrak{p} \cap \mathfrak{q} = \mathrm{Ann}(M)$.

(ii) Note that by the previous paragraph it follows that $\dim M/xM = \dim R/\mathfrak{q}$ and $\dim M/yM = \dim R/\mathfrak{p}$. Now, we show that $\dim M/xM = \dim M/yM$. Assume that $k = \dim M/xM < t = \dim M/yM$ and look for a contradiction. Suppose that $\mathrm{Ann}(M) + Rx \subseteq \mathfrak{q} = \mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_k$ and $\mathrm{Ann}(M) + Ry \subseteq \mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_t$ are chains of prime ideals of R. Let $k = t - 1$ and $z \in \mathfrak{p}_1 \cap \mathfrak{q} \setminus \mathfrak{p}$. Then $\dim M/(y, z)M = t - 1 = \dim M/zM$ so $y - z$ is an edge of $DG(M)$. On the other hand, each nonzero element of $\mathfrak{q}$ is adjacent to x thus $x - z$ is an edge of $DG(M)$. Hence, $x - z - y$ is a path and this is a contradiction. Therefore, $\dim M/xM = \dim M/yM = \dim M$ so $\mathrm{Assh}(M) = \{\mathfrak{p}' \in \mathrm{Ass}(M) : \dim M = \dim R/\mathfrak{p}'\} = \{\mathfrak{p}, \mathfrak{q}\}$.

In the following we show that, for all $z \in Z(M) \setminus \mathrm{Ann}(M)$, either $\mathrm{Ann}(zM) = \mathrm{Ann}(xM)$ or $\mathrm{Ann}(zM) = \mathrm{Ann}(yM)$. Without loss of generality, we may assume that $\mathrm{Ann}(zM) \subseteq \mathrm{Ann}(xM)$. So $xzM \neq 0$ and $z \in \mathrm{Ann}(yM)$. If there is $w \in \mathrm{Ann}(xM) \setminus \mathrm{Ann}(zM)$, then $wz \in \mathrm{Ann}(xM) \cap \mathrm{Ann}(yM)$ contrary to the assumption. Therefore, $\mathrm{Ann}(zM) = \mathrm{Ann}(xM) = \mathfrak{p}$ so $\dim M/zM = \dim R/\mathfrak{q}$.

Conversely, assume that M satisfies the conditions (i), (ii) and (iii). Thus $\dim M = \dim R/\mathfrak{p} = \dim R/\mathfrak{q}$ and therefore elements of $\mathfrak{p} \setminus \mathrm{Ann}(M)$ make a clique and elements of $\mathfrak{q} \setminus \mathrm{Ann}(M)$ make a clique too. Let $x \in \mathfrak{q} \setminus \mathfrak{p}$ and $y \in \mathfrak{p} \setminus \mathfrak{q}$. If $\dim M/(x, y)M = \dim M$, then either $y \in \mathfrak{p}$ or $x \in \mathfrak{q}$ which is a contradiction. Thus x, y are nonadjacent vertices of $DG(M)$. Hence, $DG(M) = K_{|\mathfrak{p} \setminus \mathrm{Ann}(M)|} \cup K_{|\mathfrak{q} \setminus \mathrm{Ann}(M)|}$ and therefore $DG(M)$ is disconnect. □

Example 2. (i) Let M be an Artinian R-module and let x, y be a pair of distinct vertices of $DG(M)$ such that $xM \neq M$, $yM \neq M$. Then x and y are adjacent if and only if $(x, y)M \subset M$.

(ii) Let M be a Noetherian and Artinian R-module and let x, y be a pair of distinct vertices of $DG(M)$. Then x and y are adjacent if and only if $(x, y)M \subset M$.

(iii) Let R be a finite ring and let x, y be a pair of distinct vertices of $DG(R)$. Then x and y are adjacent if and only if $(x, y)R \subset R$. In particular, for a finite local ring R, $DG(R)$ is a complete graph.

(iv) Suppose that p, q are prime numbers and m, n are positive integers such that $m \geq 2$ or $n \geq 2$. By (iii) it is clear that $DG(\mathbb{Z}_{p^m}) = K_{p^{m-1}(p-1)}$ is a complete graph whereas $DG(\mathbb{Z}_{pq}) = K_{|p\mathbb{Z}_{pq}^*|} \cup K_{|q\mathbb{Z}_{pq}^*|}$ is a disconnected graph with two component of disconnectedness, see Theorem 2.3.

Theorem 2.4. *If M is a Noetherian R-module and $DG(M)$ is a connected graph, then* $\operatorname{diam}(DG(M)) \leq 2$.

Proof. If $r(\operatorname{Ann}(M)) \neq \operatorname{Ann}(M)$, then $\operatorname{diam}(DG(M)) \leq 2$ since $DG(M)$ has a universal vertex, see Lemma 2.1(ii). Thus we may assume that $r(\operatorname{Ann}(M)) = \operatorname{Ann}(M)$. Let $x, y \in Z(M) \setminus \operatorname{Ann}(M)$ be nonadjacent vertices of $DG(M)$. If $xyM \neq 0$, then $x - xy - y$ is a path in $DG(M)$ so $\mathrm{d}(x, y) \leq 2$. Now, suppose that $xyM = 0$. If $|\operatorname{Ass}(M)| = 1$ or $\operatorname{Ass}(M) = \{\mathfrak{p}, \mathfrak{q}\}$ and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\operatorname{Ann}(M)$ is a prime ideal so either $x \in \operatorname{Ann}(M)$ or $y \in \operatorname{Ann}(M)$ that is a contradiction. Thus we may assume that $\operatorname{Ass}(M) = \{\mathfrak{p}, \mathfrak{q}\}$ and $\mathfrak{p} \not\subseteq \mathfrak{q}$, $\mathfrak{q} \not\subseteq \mathfrak{p}$. Hence, $x \in \mathfrak{p} \setminus \mathfrak{q}$ and $y \in \mathfrak{q} \setminus \mathfrak{p}$. Now, a similar argument to that of Theorem 2.3 shows that $\mathrm{d}(x, y) \leq 2$. Suppose that $\operatorname{Ass}(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ $(n \geq 3)$ we have the following three cases:
Case 1. If $\operatorname{MinAss}(M) = \{\mathfrak{p}_1\}$, then $\operatorname{Ann}(M) = \mathfrak{p}_1$ so either $x \in \mathfrak{p}_1$ or $y \in \mathfrak{p}_1$ that is a contradiction.
Case 2. If $\operatorname{MinAss}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ and $n = \dim M = \dim R/\mathfrak{p}_1 = \dim R/\mathfrak{p}_2$, then $\operatorname{Ann}(M) = \mathfrak{p}_1 \cap \mathfrak{p}_2$ and we can assume that $x \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ and $y \in \mathfrak{p}_2 \setminus \mathfrak{p}_1$. Assume that $\mathfrak{p}_3, \mathfrak{p}_4 \in \operatorname{Ass}(M)$ are such that $\mathfrak{p}_1 \subset \mathfrak{p}_3$, $\mathfrak{p}_2 \subset \mathfrak{p}_4$ and $\dim R/\mathfrak{p}_3 = \dim R/\mathfrak{p}_4 = n-1$. Then $x-z-y$ is a path in $DG(M)$ for each $z \in \mathfrak{p}_3 \cap \mathfrak{p}_4 \setminus \mathfrak{p}_1 \cup \mathfrak{p}_2$. If $n = \dim M = \dim R/\mathfrak{p}_1 > \dim R/\mathfrak{p}_2 = t$, then we may assume that $\mathfrak{p}_1 \subset \mathfrak{p}_3$ and $\dim R/\mathfrak{p}_3 = n-1$. Now, $x-z-y$ is a path in $DG(M)$ for each $z \in \mathfrak{p}_3 \cap \mathfrak{p}_2 \setminus \mathfrak{p}_1$.
Case 3. If $\operatorname{MinAss}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$, then $\operatorname{Ann}(M) \neq \mathfrak{p}_1 \cap \mathfrak{p}_2$ and $x - z - y$ is a path in $DG(M)$ for each $z \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \setminus \operatorname{Ann}(M)$. □

Lemma 2.5. *Let M be a nonzero Noetherian R-module and let $x, y \in Z(M) \setminus \operatorname{Ann}(M)$, $xyM = 0$ and x, y are nonadjacent vertices of $DG(M)$. Then $x - z$ is an edge of $DG(M)$ for all $z \in \operatorname{Ann}(yM) \setminus \operatorname{Ann}(M)$.*

Proof. Assume that x, y are nonadjacent vertices of $DG(M)$ moreover assume that $\mathfrak{p} \in \operatorname{Assh}(M/xM)$. Thus $y \notin \mathfrak{p}$ so $\operatorname{Ann}(yM) \subseteq \mathfrak{p}$. Hence, $(x, z) \subseteq \mathfrak{p}$ for all $z \in \operatorname{Ann}(yM)$. Therefore, $\dim M/(x, z)M = \dim M/xM$ and $x - z$ is an edge of $DG(M)$. □

Theorem 2.6. *Let M be a nonzero Noetherian R-module. If $DG(M)$ is a connected graph and $|Z(M) \setminus \operatorname{Ann}(M)| \geq 3$, then* $\operatorname{gr}(DG(M)) = 3$.

Proof. If there exists $x \in Z(M) \setminus \operatorname{Ann}(M)$ such that $x^2 \neq x$ and $x^2M \neq 0$, then $x - x^2$ is an edge of $DG(M)$. Since $DG(M)$ is connected and x and x^2 have same neighbors so $DG(M)$ contains a cycle of length 3. Thus we may assume that, for all $x \in Z(M) \setminus \operatorname{Ann}(M)$, either $x^2M = 0$ or $x^2 = x$. If $|r(\operatorname{Ann}(M)) \setminus \operatorname{Ann}(M)| \geq 2$, then $DG(M)$ has two universal vertices and so $\operatorname{gr}(DG(M)) = 3$. So we may assume that $r(\operatorname{Ann}(M)) \setminus \operatorname{Ann}(M) \subseteq \{x\}$.

Case 1. Let $r(\mathrm{Ann}(M)) \setminus \mathrm{Ann}(M) = \{x\}$. Then $y^2 = y$ for all $y \in Z(M) \setminus (\mathrm{Ann}(M) \cup \{x\})$. So $y-1 \in Z(M) \setminus (\mathrm{Ann}(M) \cup \{x, y\})$. If $Z(M) = \mathrm{Ann}(M) \cup \{x, y, y-1\}$, then for each $\mathfrak{p} \in \mathrm{Ass}(M)$ we have either $\mathrm{Ann}(M) \cup \{x, y\} \subseteq \mathfrak{p} \subset Z(M)$ or $\mathrm{Ann}(M) \cup \{x, y-1\} \subseteq \mathfrak{p}$. If $\mathfrak{p} = \mathrm{Ann}(M) \cup \{x, y\}$, then $x+y \in \mathrm{Ann}(M)$ which is a contradiction. Hence, $\mathrm{Ann}(M) \cup \{x, y, y-1\} \subset Z(M)$. Assume that $z \in Z(M) \setminus (\mathrm{Ann}(M) \cup \{x, y, y-1\})$. Then $x \neq yz$ ($x = yz$ implies that $0 = x^2M = y^2z^2M = yzM = xM$ which is a contradiction). If $yzM \neq 0$, then $x - y - yz - x$ ($x - z - yz - x$) is a cycle whenever $yz \neq y$ ($yz \neq z$). When $yzM = 0$ we have $z \in \mathrm{Ann}(yM)$ so it follows that $x - z - (y-1) - x$ is a cycle by Lemma 2.5.

Case 2. Let $r(\mathrm{Ann}(M)) = \mathrm{Ann}(M)$. Then $y^2 = y$ for all $y \in Z(M) \setminus \mathrm{Ann}(M)$. To prove the assertion it is enough to show that $|\mathfrak{p} \setminus \mathrm{Ann}(M)| \geq 3$ for some $\mathfrak{p} \in \mathrm{Assh}(M)$. Assume that $\mathfrak{p} \in \mathrm{Ass}(M)$ and $\mathrm{Ann}(M) \cup \{y, y-1, z\} \subseteq Z(M)$. Then $\mathrm{Ann}(M) \cup \{y, z\} \subseteq \mathfrak{p}$ and $(y+z)M = 0$. As above $z - 1 \in Z(M) \setminus (\mathrm{Ann}(M) \cup \{y, y-1, z\})$. Hence, $\mathrm{Ann}(M) \cup \{y, y-1, z, z-1\} \subseteq Z(M)$. If $Z(M) = \mathrm{Ann}(M) \cup \{y, y-1, z, z-1\}$, then by Theorem 2.4 either y is adjacent to $y-1$ or z is adjacent to $z-1$ which is a contradiction. So $\mathrm{Ann}(M) \cup \{y, y-1, z, z-1\} \subset Z(M)$ and $|\mathfrak{p} \setminus \mathrm{Ann}(M)| \geq 3$. Hence, for all $\mathfrak{p} \in \mathrm{Assh}(M)$ we have $|\mathfrak{p} \setminus \mathrm{Ann}(M)| \geq 3$. Therefore, $DG(M)$ contains a cycle of length 3. □

3. Relationship between the dimension graphs and zero-divisor graphs

In this section we study the relationship between the dimension graphs and the zero-divisor graphs.

Theorem 3.1. *Let $\Gamma(M) = DG(M)$. Then the following statements are true*:

(i) $Z(M) = r(\mathrm{Ann}(M))$.

(ii) *either* $Z(M)^2 \subseteq \mathrm{Ann}(M)$ *or* $Z(M)^3 \subseteq \mathrm{Ann}(M)$.

Proof. (i) Assume that $\Gamma(M) = DG(M)$ and $x \in Z(M) \setminus \mathrm{Ann}(M)$. We have to show that $x \in r(\mathrm{Ann}(M))$. If $x^2M = 0$ we are done. Otherwise, $x^2 \in Z(M) \setminus \mathrm{Ann}(M)$. If $x = x^2$, then $1 - x \in Z(M) \setminus \mathrm{Ann}(M)$, $x \neq 1 - x$ and $x(1-x)M = 0$. So $x, 1-x$ are adjacent in $\Gamma(M)$. On the other hand, we have $\dim M/(x, 1-x)M = -1 < \min\{\dim M/xM, \dim M/(1-x)M\}$ which shows that x and $1-x$ are nonadjacent vertices of $DG(M)$. Thus $x \neq x^2$. In this case, by Lemma 2.1(ii), x, x^2 are adjacent in $DG(M)$ hence by the hypotheses $x^3M = 0$. So $Z(M) = r(\mathrm{Ann}(M))$ and $DG(M)$ is a complete graph by Lemma 2.1(ii).

(ii) In the previous paragraph we show that for all $x \in Z(M) \setminus \mathrm{Ann}(M)$ either $x^2M = 0$ or $x^3M = 0$. Thus $Z(M)$ is a prime ideal of R and either $Z(M)^2 \subseteq \mathrm{Ann}(M)$ or $Z(M)^3 \subseteq \mathrm{Ann}(M)$, see [7, Theorem 2.3]. □

Corollary 3.2. *If $\Gamma(M) = DG(M)$, then $DG(M)$ is a complete graph.*

Proof. In view of Theorem 3.1 and Corollary 2.2 the result follows. □

Corollary 3.3. *If $Z(M) = r(\text{Ann}(M))$ and $Z(M)^2 \subseteq \text{Ann}(M)$, then $\Gamma(M) = DG(M)$ is a complete graph.*

Proof. In view of Corollary 2.2, $DG(M)$ is complete. Moreover, by $Z(M)^2 \subseteq \text{Ann}(M)$ it follows that $\Gamma(M)$ is complete, now the result follows. □

Example 3. Let p be a prime number and consider $M = \mathbb{Z}/p^3\mathbb{Z}$ as a $\mathbb{Z}$-module. Then $p\mathbb{Z} = Z(M) = r(\text{Ann}(M))$ and $Z(M)^3 = p^3\mathbb{Z} \subseteq \text{Ann}(M)$. By Corollary 2.2, $DG(M)$ is a complete graph but $\Gamma(M)$ is not complete.

Lemma 3.4. *Let $x, y \in Z(M) \setminus \text{Ann}(M)$ be adjacent vertices in $\Gamma(M)$. Then either x, y are adjacent in $DG(M)$ or $\dim M/yM = \dim M/\text{Ann}_M(x)$ and $\dim M/xM = \dim M/\text{Ann}_M(y)$.*

Proof. Suppose that x, y are nonadjacent vertices in $DG(M)$. We have to show that $\dim M/yM = \dim M/\text{Ann}_M(x)$ and $\dim M/xM = \dim M/\text{Ann}_M(y)$, where $\text{Ann}_M(x) = \{m \in M : xm = 0\}$. It is easy to see that $M/\text{Ann}_M(x) \cong xM$ so $\text{Ann}(M/\text{Ann}_M(x)) = \text{Ann}(xM)$. If $\text{Ann}(xM) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Spec}(R)$, then by the hypotheses $\text{Ann}(M) + Ry \subseteq \mathfrak{p}$. Therefore, we have $\text{Supp}(M/\text{Ann}_M(x)) \subseteq \text{Supp}(M/yM)$. On the other hand, if $\mathfrak{p} \in \text{Assh}(M/yM)$, then $x \notin \mathfrak{p}$ since $\dim M/(x, y)M < \dim M/xM$. Thus by $axM = 0$ it follows that $a \in \mathfrak{p}$. Hence, $\text{Ann}(xM) \subseteq \mathfrak{p}$ and therefore $\mathfrak{p} \in \text{Supp}(M/\text{Ann}_M(x))$. So $\dim M/yM = \dim M/\text{Ann}_M(x)$. By a similar argument one can show that $\dim M/\text{Ann}_M(y) = \dim M/xM$. □

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