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THE PSEUDOSPECTRA OF BOUNDED LINEAR OPERATORS ON QUASI NORMED SPACE

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Abstract. In this paper, we introduce the pseudospectra of bounded linear operators on quasi normed space and study its proprieties. Beside that, we establish the relationship between the pseudospectra of a sequence of bounded linear operators and its limit.

1. Introduction

It is well known that norm structure plays an important role in functional analysis, for example in the study of the operators and the spectral theories. But the classical definition is not satisfied for some field. For this problem, several attempts came to generalize the norm structure in the literature. Among them, we are interested in this paper of the notion of the quasi norm, denoted by $\|\cdot\|_q$, in which the third statement of the definition of the classical norm can be replaced by

$$\|x+y\|_q \leq c\Big\{\|x\|_q + \|y\|_q\Big\}$$

for some $c \geq 1$ and for any x, $y \in X$ (see Definition 1.1). Many researchers are interested to develop their works in quasi norm structure (e.g. [5, 11]). In [6], G. Rano introduced the topological concepts such as Cauchy sequence, convergent sequence, open set, closed set and established some basic theorems such as Hahn-Banach. In [6, 7], G. Rano and T. Bag, defined and developed continuity and boundedness of linear operators in quasi normed spaces and quasi normed linear spaces of bounded linear operators.

The concept of pseudospectra has long whetted the interest and drew the attention of various researchers in different spaces (e.g. [1–3]). This concept was introduced by J. M. Varah [9], and has been subsequently developed by several mathematician for example: E. B. Davies [4] and L. N. Trefethen and M. Embree [8].

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Motivated by the methods used in the study of quasi normed space, we introduce and analyze the pseudospectra. Moreover, we examine the relationship between the pseudospectra a sequence of bounded operators and its limit.

We organize our paper in the following way: Section 2 contains preliminary and auxiliary results that will be necessary in order to prove the main results. Section 3 is devoted to investigate the characterization of the pseudospectra of a bounded linear operators on a quasi normed space.

1.1. Preliminary and auxiliary results

The goal of this section consists in recalling the basic concepts of quasi norm and quasi-Banach space, and some results of the theory of linear operator in a quasi-Banach space, which are needed in the sequel. Now, we shall recall some basic properties of quasi norms.

Definition 1.1. Let X be a vector space over $\mathbb{K}$ ($=\mathbb{R}$ or $\mathbb{C}$). A quasi norm on X is a map $\|\cdot\|_q : X \to [0,+\infty[$ satisfying

(i) $\|x\|_q = 0$ if and only if $x = 0$,
(ii) $\|\lambda x\|_q = |\lambda|\ \|x\|_q$ for any $x \in X$ and any $\lambda \in \mathbb{K}$, and
(iii) there exists $c \geq 1$ such that $\|x+y\|_q \leq c\Big\{\|x\|_q + \|y\|_q\Big\}$ for any x, $y \in X$.

Then, $(X, \|\cdot\|_q)$ is called a quasi normed linear space and the least value of the constant $c \geq 1$ is called the index of the quasi norm $\|\cdot\|_q$.

Remark 1.2. (i) In a quasi normed linear space $(X, \|\cdot\|_q)$ with quasi index c,

$$\left\|\sum_{k=1}^{n} x_k\right\|_q \leq c^{n-1}\left\{\sum_{k=1}^{n}\|x_k\|_q\right\},\ \forall x_k \in X,\ \forall n \geq 1.$$

(ii) If $c = 1$, then the quasi norm $\|\cdot\|_q$ is reduced to a classical norm on X and $\Big(X, \|\cdot\|_q\Big)$ is a normed linear space.

Definition 1.3. The quasi normed linear space $(X, \|\cdot\|_q)$ is called a strong quasi normed linear space, if it satisfies the following additional condition:

$$\left\|\sum_{k=1}^{n} x_k\right\|_q \leq c\left\{\sum_{k=1}^{n}\|x_k\|_q\right\},\ \forall x_k \in X,\ \forall n \geq 1.$$

Definition 1.4. Let $(X, \|\cdot\|_q)$ be a quasi normed linear space.

(i) A sequence $(x_n) \subset X$ is said to converge to $x \in X$, denoted by $\lim_{n\to+\infty} x_n = x$, if $\|x_n - x\|_q \to 0$ as $n \to +\infty$.
(ii) A sequence $(x_n) \subset X$ is said to be a Cauchy sequence, if $\|x_n - x_m\|_q \to 0$ as $n, m \to +\infty$.
(iii) A subset $E \subseteq X$ is said to be complete if every Cauchy sequence in E converges in E.

(iv) A subset $E \subseteq X$ is said to be bounded if there exists a real number $M > 0$ such that $\|x\|_q \leq M, \ \forall x \in E$.

Lemma 1.5 ([7, Lemma 3.1]). *Let $(X, \|\cdot\|_q)$ be a quasi normed linear space and (x_n) be a sequence in X such that $\lim_{n\to+\infty} x_n = x$. Then,*

$$\|x\|_q = \|\lim_{n\to+\infty} x_n\|_q \leq c \lim_{n\to+\infty} \|x_n\|_q.$$

Definition 1.6. Let $(X, \|\cdot\|_{q_1})$ and $(Y, \|\cdot\|_{q_2})$ be quasi normed linear spaces. The linear operator $T : X \longrightarrow X$ is called bounded, if there exists $M \geq 0$ such that

$$\|Tx\|_{q_2} \leq M\|x\|_{q_1} \text{ for all } x \in X.$$

The set of all bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. If $X = Y$, then $\mathcal{L}(X, X) = \mathcal{L}(X)$.

Lemma 1.7 ([7, Theorem 3.2]). *Let $(X, \|\cdot\|_{q_1})$ and $(Y, \|\cdot\|_{q_2})$ be quasi normed linear spaces. For $T \in \mathcal{L}(X, Y)$, we define*

$$\|T\|_q = \sup_{x\in X\setminus\{0\}} \frac{\|Tx\|_{q_2}}{\|x\|_{q_1}}. \tag{1.1}$$

Then, $(\mathcal{L}(X, Y), \|\cdot\|_q)$ is a quasi normed linear space.

Remark 1.8. Let $(X, \|\cdot\|_{q_1})$ and $(Y, \|\cdot\|_{q_2})$ be quasi normed linear spaces. If $T \in \mathcal{L}(X, Y)$, then

$$\|T\|_q = \sup_{\substack{x\in X \\ \|x\|_{q_1=1}}} \|Tx\|_{q_2}.$$

If $Y = \mathbb{K}$ and $\|\cdot\|_{q_2} = |\cdot|$, then $(\mathcal{L}(X, \mathbb{K}), \|\cdot\|_q)$ which is called the set of all bounded linear functional defined on $(X, \|\cdot\|_{q_1})$, is called the dual space of $(X, \|\cdot\|_{q_1})$.

For simplicity, we use $\|\cdot\|_q$ for both the quasi norm on X and the quasi operator norm on $\mathcal{L}(X)$.

Definition 1.9. Let $(X, \|\cdot\|_q)$ be a quasi normed linear space. If $T \in \mathcal{L}(X)$, then

(i) T is said to be one-to-one if $N(T) = \{0\}$.
(ii) T is said to be onto if $R(T) = X$.
(iii) T is said to be invertible if it is both one-to-one and onto.

Remark 1.10. Let $(X, \|\cdot\|_q)$ be a quasi normed linear space and let $T \in \mathcal{L}(X)$. If T is invertible, then there exists a unique bounded linear operator denoted $T^{-1} : X \longrightarrow X$ called the inverse of T such that $T^{-1}T = TT^{-1} = I_X$, where $I_X : X \to X$ is the identity operator.

Proposition 1.11 ([7, Theorems 3.1 and 3.3]). *Let $(X, \|\cdot\|_q)$ be a quasi normed linear space.*

(i) *If T, $S \in \mathcal{L}(X)$ and $\lambda \in \mathbb{K}$, then $T + S$, λT belong to $\mathcal{L}(X)$.*

(ii) *If $(X, \|\cdot\|_q)$ is complete, then space $(\mathcal{L}(X), \|\cdot\|_q)$ is a complete quasi normed linear space.*

Lemma 1.12. *Let $(X, \|\cdot\|_q)$ be a strong quasi Banach space and let $T \in \mathcal{L}(X)$. If $\|T\|_q < 1$, then $(I_X - T)$ is invertible and $(I_X - T)^{-1} = \sum_{k=0}^{\infty} T^k$.*

Proof. Putting $S_n = \sum_{k=0}^{n} T^k$. Then, we have

$$\begin{aligned} S_n(I_X - T) &= \sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1} \\ &= I_X - T^{n+1}, \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} (I_X - T)S_n &= \sum_{k=0}^{n} T^k - \sum_{k=0}^{n} T^{k+1} \\ &= I_X - T^{n+1}. \end{aligned} \tag{1.3}$$

Since $T \in \mathcal{L}(X)$, then for all $x \in X$, we have

$$\begin{aligned} \|T^n x\|_q &= \|T(T^{n-1})x\|_q \\ &\leq \|T\|_q \, \|T^{n-1}x\|_q \\ &\vdots \\ &\leq \|T\|_q^n \, \|x\|_q. \end{aligned}$$

This leads to

$$\|T^n\|_q = \sup_{x \in X \setminus \{0\}} \frac{\|T^n x\|_q}{\|x\|_q} \leq \|T\|_q^n. \tag{1.4}$$

Based on the assumption $\|T\|_q < 1$, we infer that $\|T\|_q^n \to 0$ as $n \to +\infty$. Hence, it follows from (1.4) that $\|T^n\|_q \longrightarrow 0$ as $n \to +\infty$ which yields

$$T^n \longrightarrow 0 \text{ as } n \to +\infty.$$

Letting $n \to +\infty$ in (1.2) and (1.3), we obtain

$$(I_X - T)\sum_{k=0}^{\infty} T^k = \left(\sum_{k=0}^{\infty} T^k\right)(I_X - T) = I_X.$$

Moreover, we have

$$\begin{aligned} \left\|\sum_{k=0}^{\infty} T^k\right\|_q &= \left\|\lim_{n\to+\infty} \sum_{k=0}^{n} T^k\right\|_q \\ &\leq c \lim_{n\to+\infty} \left\|\sum_{k=0}^{n} T^k\right\|_q \quad \text{(by Lemma 1.5)} \end{aligned}$$

$$\begin{aligned}
&\le c \lim_{n\to+\infty} \left\| T^0 + \sum_{k=1}^{n} T^k \right\|_q \\
&\le c^2 \lim_{n\to+\infty} \left(\left\|T^0\right\|_q + \left\| \sum_{k=1}^{n} T^k \right\|_q \right) \text{ (by Definition 1.1(iii))} \\
&\le c^2 \lim_{n\to+\infty} \left(\left\|T^0\right\|_q + c \sum_{k=1}^{n} \left\|T^k\right\|_q \right) \text{ (by Definition 1.3)} \\
&\le c^3 \lim_{n\to+\infty} \left(\left\|T^0\right\|_q + \sum_{k=1}^{n} \left\|T^k\right\|_q \right) \text{ (as } c \ge 1) \\
&\le c^3 \lim_{n\to+\infty} \sum_{k=0}^{n} \left\|T^k\right\|_q
\end{aligned}$$

$$\le c^3 \sum_{k=0}^{\infty} \|T\|_q^k . \tag{1.5}$$

The fact that $\lim_{n\to+\infty} \sqrt[n]{\|T\|_q^n} < 1$ implies that $\sum_{k=0}^{\infty} \|T\|_q^k$ is convergent. Hence, we deduce that $\sum_{k=0}^{\infty} T^k \in \mathcal{L}(X)$. □

Proposition 1.13 ([6, Theorem 3.4])**.** *Let $(X, \|\cdot\|_q)$ be a quasi normed linear space and f be a bounded linear functional which is defined on a subspace Z of X. Then, f has a linear extension $\hat{f}$ from Z to Z_1 which is a higher dimensional subspace of X and bounded on Z_1 satisfying $\|f\|_q \le \|\hat{f}\|_q \le c\ \|f\|_q$.*

Lemma 1.14. *Let $(X, \|\cdot\|_q)$ be a quasi normed linear space and $x_0 \in X\backslash\{0\}$. Then, there is a bounded linear functional f which is defined on X such that $f(x_0) = \|x_0\|_q$ and $1 \le \|f\|_q \le c$.*

Proof. Let E be the subspace spanned by x_0 and let us define $g : E \longrightarrow \mathbb{K}$ such that $g(\alpha x_0) = \alpha \|x_0\|_q$. Then, $\|g\|_q = 1$.

This implies that g is a bounded linear functional which is defined on a subspace E of X. Then, we infer from Proposition 1.13 that g has a linear extension f from E to X and bounded on X satisfying $\|g\|_q \le \|f\|_q \le c\ \|g\|_q$. Hence, $f(x_0) = g(x_0) = \|x_0\|_q$ and $1 \le \|f\|_q \le c$. □

2. Main results

The goal of this section is to introduce and study some aspects of spectral theory of bounded linear operators on a quasi normed space.

2.1. The spectrum of bounded linear operators

The aim of this subsection is to introduce and study the spectra and the pseudospectra of bounded linear operators on a quasi normed space.

Definition 2.1. Let $(X, \|\cdot\|_q)$ be a quasi normed linear space and $T \in \mathcal{L}(X)$.
(i) The resolvent set of T is defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is invertible in } \mathcal{L}(X)\}.$$

(ii) The spectrum of T is defined by $\sigma(T) = \mathbb{C}\backslash\rho(T)$.

Proposition 2.2. *Let $(X, \|\cdot\|_q)$ be a strong quasi Banach space and $T \in \mathcal{L}(X)$.*
(i) $\rho(T)$ is open.
(ii) $\sigma(T) \subset \overline{\mathcal{B}}(0, \|T\|_q)$, where $\overline{\mathcal{B}}(0, \|T\|_q) = \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|_q\}$.

Proof. (i) If $\rho(T) = \emptyset$, then $\rho(T)$ is open. If $\rho(T) \neq \emptyset$, then there exists $\lambda_0 \in \rho(T)$. It sufficient to find $r > 0$ and $\mathbb{B}(\lambda_0, r) = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < r\}$ such that

$$\mathbb{B}(\lambda_0, r) \subset \rho(T). \tag{2.1}$$

For $\lambda \in \mathbb{C}$, we have

$$\begin{aligned}\lambda - T &= \lambda - \lambda_0 + \lambda_0 - T \\ &= (\lambda_0 - T)[(\lambda - \lambda_0)\ (\lambda_0 - T)^{-1} + I].\end{aligned}$$

If $|\lambda - \lambda_0| < \|(\lambda_0 - T)^{-1}\|_q$, then by using Lemma 1.12, we obtain

$$[(\lambda - \lambda_0)\ (\lambda_0 - T)^{-1} + I]^{-1} \in \mathcal{L}(X).$$

The fact that $\lambda_0 \in \rho(T)$ implies that $(\lambda - T)^{-1} \in \mathcal{L}(X)$. Consequently, $\lambda \in \rho(T)$. Hence, we conclude that $\rho(T)$ is open.

(ii) Let $\lambda \notin \overline{\mathcal{B}}(0, \|T\|_q)$. Then, $\lambda \in \mathbb{C}$ such that $0 \leq \|T\|_q < |\lambda|$. Hence, we can write $\lambda - T = \lambda(I - \lambda^{-1}T)$. Moreover, we have

$$\|\lambda^{-1}T\|_q = |\lambda^{-1}|\ \|T\|_q < |\lambda^{-1}|\ |\lambda| = 1.$$

This implies from Lemma 1.12 that $(\lambda - T)^{-1} \in \mathcal{L}(X)$ which yields $\lambda \in \rho(T)$. Hence, we deduce that $\sigma(T) \subset \overline{\mathcal{B}}(0, \|T\|_q)$. □

Remark 2.3. It follows from Proposition 2.2 that $\sigma(T)$ is compact.

Remark 2.4. Let $(X, \|\cdot\|_q)$ be a quasi normed linear space and $T \in \mathcal{L}(X)$. Then, $\lambda \in \sigma(T)$ if, and only if, $\lambda + \mu \in \sigma(T + \mu)$ for all $\mu \in \mathbb{C}$. Indeed, for $\mu \in \mathbb{C}$, we have

$$\lambda - T = \lambda + \mu - (T + \mu). \tag{2.2}$$

Let $\lambda \in \sigma(T)$. We assume that $\lambda + \mu \in \rho(T + \mu)$, then $\lambda + \mu - (T + \mu)$ is invertible and $\Big(\lambda + \mu - (T + \mu)\Big)^{-1} \in \mathcal{L}(X)$. Hence, by referring to (2.2), we infer that $\lambda \in \rho(T)$. This contradiction implies that $\lambda + \mu \in \sigma(T + \mu)$. Conversely, a same reasoning as before leads to the result.

2.2. The pseudospectra of bounded linear operators

The aim of this subsection is to introduce the pseudospectra of bounded linear operators on a quasi normed space and study its proprieties.

We start by discussing the stability of solutions of the following equation

$$(\lambda - T)x = y, \tag{2.3}$$

where T is a bounded linear operator on quasi normed linear space X and $x, y \in X$. For $\lambda \in \rho(T)$, the equation (2.3) admits the unique solution which expressed in the form $x = (\lambda - T)^{-1}y$. We perturb (2.3) by $\tau \in X$ such that $\|\tau\|_q \leq \varepsilon$ and we obtain

$$(\lambda - T)x_1 = y + \tau, \ x_1 \in X. \tag{2.4}$$

As above, $x_1 = (\lambda - T)^{-1}(y + \tau)$ is a unique solution of (2.4) for $\lambda \in \rho(T)$. It follows that

$$\|x - x_1\|_q = \|(\lambda - T)^{-1}\tau\|_q.$$

Since $\lambda \in \rho(T)$, then by the boundedness of the operator $(\lambda - T)^{-1}$, we have

$$\begin{aligned} \|x - x_1\|_q &\leq \|(\lambda - T)^{-1}\|_q \ \|\tau\|_q \\ &\leq \varepsilon \ \|(\lambda - T)^{-1}\|_q. \end{aligned}$$

For x is very close to x_1, we need to know that $\|(\lambda - T)^{-1}\|_q$ is small. This implies that $\|(\lambda - T)^{-1}\|_q \leq \frac{1}{\varepsilon}$. Hence,

$$\lambda \in \rho(T) \bigcap \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\|_q \leq \frac{1}{\varepsilon} \right\}.$$

Definition 2.5. Let $(X, \|\cdot\|_q)$ be a quasi normed linear space, $T \in \mathcal{L}(X)$ and let $\varepsilon > 0$. The pseudospectrum of T is defined by

$$\sigma_\varepsilon(T) = \sigma(T) \bigcup \left\{ \lambda \in \mathbb{C} : \|(\lambda - T)^{-1}\|_q > \frac{1}{\varepsilon} \right\},$$

by convention $\|(\lambda - T)^{-1}\|_q = +\infty$ if and only if $\lambda \in \sigma(T)$.

Theorem 2.6. *Let $(X, \|\cdot\|_q)$ be a strong quasi normed linear space, $\varepsilon > 0$ and let $T \in \mathcal{L}(X)$. If there exists a bounded operator $S \in \mathcal{L}(X)$ such that $\|S\|_q < \varepsilon$ and $\lambda \in \sigma(T + S)$, then $\lambda \in \sigma_\varepsilon(T)$.*

Proof. Let us assume that there exists a bounded operator $S \in \mathcal{L}(X)$ such that $\|S\| < \varepsilon$ and $\lambda \in \sigma(T+S)$. We argue by contradiction. Suppose that $\lambda \notin \sigma_\varepsilon(T)$ which yields $\lambda \in \rho(T)$ and $\|(\lambda - T)^{-1}\|_q \leq \frac{1}{\varepsilon}$. Let us consider the operator C defined on X by

$$C = (\lambda - T)^{-1} \sum_{n=0}^{+\infty} \left(S(\lambda - T)^{-1} \right)^n.$$

The fact that

$$\|S(\lambda - T)^{-1}\|_q \leq \|S\|_q \ \|(\lambda - T)^{-1}\|_q < 1,$$

implies from Lemma 1.12 that $\sum_{n=0}^{+\infty}\Big(S(\lambda-T)^{-1}\Big)^n \in \mathcal{L}(X)$. Moreover, we have $(\lambda-T)^{-1}\in\mathcal{L}(X)$. By referring to Proposition 1.11(i), we infer that $C\in\mathcal{L}(X)$. Again by Lemma 1.12, we can write

$$\begin{aligned} C &= (\lambda-T)^{-1}\Big(I-S(\lambda-T)^{-1}\Big)^{-1} \\ &= \Big[\big(I-S(\lambda-T)^{-1}\big)(\lambda-T)\Big]^{-1} \\ &= (\lambda-T-S)^{-1}. \end{aligned}$$

This implies that $\lambda-T-S$ is invertible with $C=(\lambda-T-S)^{-1}$. □

Remark 2.7. The proof of Theorem 2.8 is same as [4, Theorem 9.2.13].

Theorem 2.8. *Let $(X,\|\cdot\|_q)$ be a quasi normed linear space with index c, $\varepsilon>0$ and let $T\in\mathcal{L}(X)$. If $\lambda\in\sigma_\varepsilon(T)$, then there exists a bounded operator $S\in\mathcal{L}(X)$ such that $\|S\|<c\,\varepsilon$ and $\lambda\in\sigma(T+S)$.*

Proof. Let us assume that $\lambda\in\sigma_\varepsilon(T)$. We discuss two cases.

First case. If $\lambda\in\sigma(T)$, then we may put $S=0$.

Second case. Suppose that $\lambda\in\sigma_\varepsilon(T)$ and $\lambda\notin\sigma(T)$. Then, there exists $y\in X\backslash\{0\}$ such that

$$\|(\lambda-T)^{-1}y\|_q>\frac{1}{\varepsilon}\|y\|_q. \tag{2.5}$$

Putting $x=(\lambda-T)^{-1}y$. Then, $y=(\lambda-T)x$. By referring to (2.5), we infer that $\|(\lambda-T)x\|_q<\varepsilon\|x\|_q$. Consequently,

$$\|(\lambda-T)x_0\|_q<\varepsilon,\quad x_0=\frac{x}{\|x\|_q}.$$

At this level, by using Lemma 1.14, there exists a linear functional f defined on X satisfying $f(x_0)=1$ and $1\le\|f\|\le c$. We consider the following linear operator

$$Sy=f(y)(\lambda-T)x_0.$$

Let us observe that

$$\|Sy\|_q\le|f(y)|\;\|(\lambda-T)x_0\|_q<c\,\varepsilon\|y\|_q.$$

Then we infer that $\|S\|_q\le c\varepsilon$ and $\mathcal{D}(S)=X$. This implies that S is bounded. Moreover, we have $(\lambda-T-S)x_0=0$, $\|x_0\|_q=1$. Hence, $\lambda-T-S$ is not invertible. This enables us to conclude that $\lambda\in\sigma(T+S)$. □

Remark 2.9. The only difference between the proof of Theorem 2.6 and its Banach counterpart [4, Thoerem 9.2.13] is the index c.

As a direct consequence of Theorems 2.6 and 2.8, we infer the following result:

Corollary 2.10. *Let $(X, \|\cdot\|_q)$ be a strong quasi normed linear space with index c, $\varepsilon > 0$ and let $T \in \mathcal{L}(X)$. Then*

$$\bigcup_{\|S\|<\varepsilon} \sigma(T+S) \subset \sigma_\varepsilon(T) \subset \bigcup_{\|S\|<c\varepsilon} \sigma(T+S).$$

Remark 2.11. We should notice that

$$\sigma_\varepsilon(T) = \bigcup_{\|S\|<\varepsilon} \sigma(T+S),$$

only for $c = 1$ (i.e., $(X, \|\cdot\|_q)$ will be a normed linear space).

Lemma 2.12. *Let $(X, \|\cdot\|_q)$ be a strong quasi normed linear space with index c, $\varepsilon, \delta > 0$ and let $T \in \mathcal{L}(X)$.*

(i) $\sigma(T) + \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\} \subseteq \sigma_\varepsilon(T)$.

(ii) $\sigma_\delta(T) + \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\} \subseteq \sigma_{c^2\ \varepsilon + c\delta}(T)$.

Proof. (i) Suppose that $\lambda \in \sigma(T) + \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\}$. Then, there exist $\lambda_1 \in \sigma(T)$ and $|\lambda_2| < \varepsilon$ such that $\lambda = \lambda_1 + \lambda_2$. This implies from Remark 2.4 that

$$\lambda = \lambda_1 + \lambda_2 \in \sigma(T + \lambda_2).$$

The use of Theorem 2.6 allows us to conclude that $\lambda \in \sigma_\varepsilon(T)$.

(ii) Let us assume that $\lambda \in \sigma_\varepsilon(T) + \{\lambda \in \mathbb{C} : |\lambda| < \delta\}$. Then, there exist $\lambda_1 \in \sigma_\varepsilon(T)$ and $|\lambda_2| < \delta$ such that $\lambda = \lambda_1 + \lambda_2$. This implies from Theorem 2.8 that there exists $S \in \mathcal{L}(X)$ such that $\|S\|_q < c\varepsilon$ and $\lambda_1 \in \sigma(T + S)$. From Remark 2.4, we infer that

$$\lambda = \lambda_1 + \lambda_2 \in \sigma(T + S + \lambda_2).$$

Moreover, $S + \lambda_2 \in \mathcal{L}(X)$ and

$$\begin{aligned} \|S + \lambda_2\|_q &\le c(\|S\|_q + |\lambda_2|) \\ &< c\ (c\ \varepsilon + |\lambda_2|) \\ &< c^2\ \varepsilon + c\delta. \end{aligned}$$

Hence, by using Theorem 2.6, we deduce that $\lambda \in \sigma_{c^2\ \varepsilon + c\delta}(T)$. □

Proposition 2.13. *Let $(X, \|\cdot\|_q)$ be a quasi normed linear space, $\varepsilon > 0$ and let $T \in \mathcal{L}(X)$. Then*

(i) $\sigma_\varepsilon(T) = \sigma(T) \bigcup \{\lambda \in \mathbb{C} : \exists\ x \in X \text{ and } \|(\lambda - T)x\|_q < \varepsilon \|x\|_q\}$.

(ii) $\sigma_\varepsilon(T) = \sigma(T) \bigcup \{\lambda \in \mathbb{C} : \exists\ x_n \in X, \|x_n\|_q = 1 \text{ and } \lim\limits_{n\to+\infty} \|(\lambda - T)x_n\|_q < \varepsilon\}$.

Proof. (i) Let $\lambda \notin \sigma(T)$ and $\lambda \in \{\lambda \in \mathbb{C} : \exists\ x \in X \text{ and } \|(\lambda - T)x\|_q < \varepsilon\|x\|_q\}$. Then, we have $\lambda \in \rho(T)$ and

$$\|(\lambda - T)x\|_q < \varepsilon\|x\|_q. \tag{2.6}$$

Putting $y = (\lambda - T)x$. Then, $x = (\lambda - T)^{-1}y$ and $x \in X$. It follows from (2.6) that

$$\|(\lambda - T)^{-1}y\|_q > \frac{1}{\varepsilon}\|y\|_q. \tag{2.7}$$

At this level, we are going to prove that (2.7) holds for $y \neq 0$. We assume that $y = 0$. Then, $(\lambda - T)x = 0$ which yields $x \in N(\lambda - T)$. The fact that $\lambda - T$ is invertible implies that $x = 0$. By referring to (2.6), we obtain $0 < 0$. This contradiction implies that $y \neq 0$. Since

$$\|(\lambda - T)^{-1}\|_q \geq \frac{\|(\lambda - T)^{-1}y\|_q}{\|y\|_q}, \quad y \neq 0,$$

then by (2.7), we infer that

$$\|(\lambda - T)^{-1}\|_q > \frac{1}{\varepsilon}.$$

Hence, $\lambda \in \sigma_\varepsilon(T)$. Thus, we conclude

$$\sigma(T) \bigcup \{\lambda \in \mathbb{C} : \exists\, x \in X \text{ and } \|(\lambda - T)x\|_q < \varepsilon\|x\|_q\} \subseteq \sigma_\varepsilon(T).$$

Conversely, let $\lambda \in \sigma_\varepsilon(T)\backslash\sigma(T)$. Then, $\lambda \in \rho(T)$ and $\|(\lambda - T)^{-1}\|_q > \frac{1}{\varepsilon}$. This implies from (1.1) that there exist $y \neq 0$ such that

$$\frac{\|(\lambda - T)^{-1}y\|_q}{\|y\|_q} > \frac{1}{\varepsilon}. \tag{2.8}$$

Putting $x = (\lambda - T)^{-1}y$. Then, $x \in X$ and $y = (\lambda - T)x$. This implies from (2.8) that $\|(\lambda - T)x\|_q < \varepsilon\|x\|_q$. Hence,

$$\sigma_\varepsilon(T) = \sigma(T) \bigcup \{\lambda \in \mathbb{C} : \exists\, x \in X \text{ and } \|(\lambda - T)x\|_q < \varepsilon\|x\|_q\}.$$

(ii) Let us assume that $\lambda \in \sigma_\varepsilon(T)\backslash\sigma(T)$. Then, $\lambda \in \rho(T)$ and $\|(\lambda - T)^{-1}\|_q > \frac{1}{\varepsilon}$. This implies from Remark 1.8 that

$$\|(\lambda - T)^{-1}\|_q = \sup_{\substack{y \in X \\ \|y\|_{q=1}}} \|(\lambda - T)^{-1}y\|_q.$$

Using the characterization of the upper bound, we obtain that for all $\delta \in \mathbb{R}$ there exists $(y_n)_n$ such that $\|y_n\|_q = 1$ and

$$\|(\lambda - T)^{-1}\|_q - \delta < \|(\lambda - T)^{-1}y_n\|_q < \|(\lambda - T)^{-1}\|_q \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

For $\delta = \frac{1}{n}$, we have

$$\|(\lambda - T)^{-1}\|_q - \frac{1}{n} < \|(\lambda - T)^{-1}y_n\|_q < \|(\lambda - T)^{-1}\|_q \quad \text{for all } n \in \mathbb{N}^*.$$

Now, letting $n \to +\infty$, we obtain

$$\lim_{n \to +\infty} \|(\lambda - T)^{-1}y_n\|_q = \|(\lambda - T)^{-1}\|_q. \tag{2.10}$$

At this level, putting $x_n = \|(\lambda - T)^{-1}y_n\|_q^{-1}(\lambda - T)^{-1}y_n$. Then, $\|x_n\|_q = 1$ and

$$\begin{aligned}\|(\lambda - T)x_n\|_q &= \|(\lambda - T)\|(\lambda - T)^{-1}y_n\|_q^{-1}(\lambda - T)^{-1}y_n\|_q \\ &= \|(\lambda - T)^{-1}y_n\|_q \ \|(\lambda - T)(\lambda - T)^{-1}y_n\|_q \\ &= \|(\lambda - T)^{-1}y_n\|_q^{-1} \ \|y_n\|_q \\ &= \|(\lambda - T)^{-1}y_n\|_q^{-1}.\end{aligned}$$

The use of (2.10) makes us conclude that

$$\lim_{n\to+\infty} \|(\lambda - T)x_n\|_q = \|(\lambda - T)^{-1}\|_q^{-1} < \varepsilon.$$

It is show that

$$\sigma_\varepsilon(T) \subseteq \sigma(T)\bigcup\{\lambda \in \mathbb{C} : \exists\ x_n \in X, \|x_n\|_q = 1 \text{ and } \lim_{n\to+\infty} \|(\lambda - T)x_n\|_q < \varepsilon\}.$$

Conversely, let $\lambda \in \rho(T)$ such that there exists $x_n \in X, \|x_n\|_q = 1$ and

$$\lim_{n\to+\infty} \|(\lambda - T)x_n\|_q < \varepsilon. \tag{2.11}$$

Putting $y_n = \|(\lambda - T)x_n\|^{-1}(\lambda - T)x_n$. Then, $\|y_n\|_q = 1$ and

$$\begin{aligned}\|(\lambda - T)^{-1}y_n\|_q &= \|(\lambda - T)x_n\|_q^{-1} \ \|x_n\|_q \\ &= \|(\lambda - T)x_n\|_q^{-1}. \qquad (2.12)\end{aligned}$$

It follows from Remark 1.8 that $\|(\lambda - T)^{-1}y_n\|_q \leq \|(\lambda - T)^{-1}\|_q$. Hence, by using (2.12), we obtain

$$\|(\lambda - T)x_n\|_q^{-1} \leq \|(\lambda - T)^{-1}\|_q. \tag{2.13}$$

Now, letting $n \to +\infty$, we infer from (2.11) and (2.13) that

$$\|(\lambda - T)^{-1}\|_q > \frac{1}{\varepsilon}.$$

Thus, $\sigma(T)\bigcup\{\lambda \in \mathbb{C} : \exists\ x_n \in X, \|x_n\|_q = 1 \text{ and } \lim_{n\to+\infty} \|(\lambda - T)x_n\|_q < \varepsilon\} \subseteq \sigma_\varepsilon(T)$. ☐

In the following result, we discussed the pseudospectra of a sequence of bounded linear operators.

Theorem 2.14. *Let $(X, \|\cdot\|_q)$ be a strong quasi normed linear space with index $c > 1$, $\varepsilon > 0$, (T_n) be a sequence of bounded linear operators on X and $T \in \mathcal{L}(X)$ such that $\|T_n - T\|_q \longrightarrow 0$ when $n \to +\infty$. Then, there exists $n_0 \in \mathbb{N}$ such that $n_2 \geq n_1$ such that*

$$\sigma_{c^{-7}\varepsilon}(T) \subset \sigma_{c^{-2}\varepsilon}(T_n) \subset \sigma_{c\varepsilon}(T) \text{ for all } n \geq n_2.$$

Proof. Let $\lambda \notin \sigma_{c\varepsilon}(T)$. Then, $\lambda \in \rho(T)$ and $\|(\lambda - T)^{-1}\|_q < \frac{1}{c\varepsilon}$. The fact that $\|T_n - T\|_q \longrightarrow 0$ as $n \to 0$ implies that there exists $n_0 \in \mathbb{N}$ such that

$$\|T_n - T\|_q < c\varepsilon - \varepsilon \text{ for all } n \geq n_0.$$

This implies that

$$\begin{aligned}\left\|(T_n-T)(\lambda-T)^{-1}\right\|_q &\leq \|T_n-T\|_q \left\|(\lambda-T)^{-1}\right\|_q\\ &< \frac{c\varepsilon-\varepsilon}{c\varepsilon}\\ &< 1-\frac{1}{c}\\ &< 1.\end{aligned}$$

It follows from Lemma 1.12 that

$$\left(I-(T_n-T)(\lambda-T)^{-1}\right)^{-1}=\sum_{k=0}^{\infty}\left((T_n-T)(\lambda-T)^{-1}\right)^k. \tag{2.14}$$

Now, we can write

$$\begin{aligned}\lambda-T_n &= \lambda-T+T-T_n\\ &= \left(I-(T_n-T)(\lambda-T)^{-1}\right)(\lambda-T). \end{aligned}\tag{2.15}$$

Based on the assumption $\lambda\in\rho(T)$, we infer from (2.14) and (2.15) that $\lambda\in\rho(T_n)$ and

$$\begin{aligned}\left\|(\lambda-T_n)^{-1}\right\|_q &= \left\|\left(\left(I-(T_n-T)(\lambda-T)^{-1}\right)(\lambda-T)\right)^{-1}\right\|_q\\ &\leq \left\|(\lambda-T)^{-1}\right\|_q\left\|\sum_{k=0}^{\infty}\left((T_n-T)(\lambda-T)^{-1}\right)^k\right\|_q.\end{aligned}$$

Substituting T for $(T_n-T)(\lambda-T)^{-1}$ in (1.5) yields

$$\begin{aligned}\left\|(\lambda-T_n)^{-1}\right\|_q &\leq \left\|(\lambda-T)^{-1}\right\|_q c^3\sum_{k=0}^{\infty}\left\|\left((T_n-T)(\lambda-T)^{-1}\right)^k\right\|_q\\ &\leq c^3\left\|(\lambda-T)^{-1}\right\|_q\sum_{k=0}^{\infty}\left\|(T_n-T)(\lambda-T)^{-1}\right\|_q^k\\ &\leq \frac{c^2}{\varepsilon}\sum_{k=0}^{\infty}\left\|(T_n-T)(\lambda-T)^{-1}\right\|_q^k\\ &\leq \frac{c^2}{\varepsilon}\quad\frac{1}{1-\left(1-\frac{1}{c}\right)}\\ &\leq \frac{1}{c^{-3}\varepsilon}. \end{aligned}\tag{2.16}$$

This shows that $\lambda\in\sigma_{c^{-3}\varepsilon}(T_n)$. Let us assume $\lambda\notin\sigma_{c^{-3}\varepsilon}(T_n)$. Then, $\lambda\in\rho(T_n)$ and $\|(\lambda-T_n)^{-1}\|_q<\frac{1}{c^{-3}\ \varepsilon}$. Using the fact that $\|T_n-T\|_q\longrightarrow 0$ as $n\to 0$

implies for all $\eta > 0$ that there exists $n_0 \in \mathbb{N}$ such that

$$\|T_n - T\|_q < \eta \quad \text{for all } n \geq n_0.$$

Now, we write

$$c^{-4}\varepsilon - c^{-3}\varepsilon = c^{-3}\varepsilon(c^{-1} - 1). \tag{2.17}$$

Since $c > 1$ and $\varepsilon > 0$, then by (2.17), we infer that $c^{-4}\varepsilon < c^{-3}\varepsilon$. Hence, for $\eta = c^{-3}\varepsilon - c^{-4}\varepsilon$, we obtain

$$\|T_n - T\|_q < c^{-3}\varepsilon - c^{-4}\varepsilon \quad \text{for all } n \geq n_0.$$

Consequently,

$$\begin{aligned} \left\|(T_n - T)(\lambda - T_n)^{-1}\right\|_q &\leq \|T_n - T\|_q \left\|(\lambda - T_n)^{-1}\right\|_q \\ &< \frac{c^{-3}\ \varepsilon - c^{-4}\ \varepsilon}{c^{-3}\ \varepsilon} \\ &< 1 - \frac{c^{-4}}{c^{-3}} \\ &< 1. \end{aligned}$$

This implies from Lemma 1.12 that

$$\left(I - (T_n - T)(\lambda - T)^{-1}\right)^{-1} = \sum_{k=0}^{n} \left((T_n - T)(\lambda - T)^{-1}\right)^k. \tag{2.18}$$

Now, we can write

$$\begin{aligned} \lambda - T_n &= \lambda - T + T - T_n \\ &= \left(I - (T_n - T)(\lambda - T)^{-1}\right)(\lambda - T). \end{aligned} \tag{2.19}$$

Based on the assumption $\lambda \in \rho(T_n)$, we infer from (2.18) and (2.19) that $\lambda \in \rho(T)$. Substituting T_n for T and T for T_n in (2.16) yields

$$\begin{aligned} \left\|(\lambda - T)^{-1}\right\|_q &\leq c^3 \left\|(\lambda - T_n)^{-1}\right\|_q \sum_{k=0}^{n} \left\|(T_n - T)(\lambda - T_n)^{-1}\right\|_q^k \\ &\leq \frac{c^3}{c^{-3}\ \varepsilon} \quad \frac{1}{1 - \left(1 - \dfrac{c^{-4}}{c^{-3}}\right)} \\ &\leq \frac{1}{c^{-7}\varepsilon}. \end{aligned}$$

□

Remark 2.15. Theorem 2.14 is a strong quasi normed with index $c > 1$ counterpart of [10, Proposition 1.1].

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