Commun. Korean Math. Soc. **38** (2023), No. 3, pp. 649–661
https://doi.org/10.4134/CKMS.c220160
pISSN: 1225-1763 / eISSN: 2234-3024

LOCAL-GLOBAL PRINCIPLE AND GENERALIZED LOCAL COHOMOLOGY MODULES

Bui Thi Hong Cam, Nguyen Minh Tri, and Do Ngoc Yen

Abstract. Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. We introduce the concept of $\mathcal{M}$-minimax R-modules and investigate the local-global principle for generalized local cohomology modules that concerns to the $\mathcal{M}$-minimaxness. We also provide the $\mathcal{M}$-finiteness dimension $f_I^{\mathcal{M}}(M,N)$ of M,N relative to I which is an extension the finiteness dimension $f_I(N)$ of a finitely generated R-module N relative to I.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring and I is an ideal of R. Let M,N be two finitely generated R-modules. The i-th local cohomology module of an R-module X with respect to I is denoted by $H_I^i(X)$. Local cohomology was first defined and studied by Grothendieck. The readers may refer [4,8] for more details about local cohomology. Since the local cohomology theory has a lot of useful applications, there are some extensions of this theory. The following generalization is given by J. Herzog in [10]. Let j be a non-negative integer, M a finitely generated R-module and X an R-module. The j-th generalized local cohomology module of M and X with respect to I is defined by

$$H_I^j(M,X) \cong \varinjlim_n \operatorname{Ext}_R^j(M/I^nM, X).$$

If $M=R$, then $H_I^i(M,X)=H_I^i(X)$ the usual local cohomology module.

An important theorem in local cohomology is Faltings' local-global principle for the finiteness dimension of local cohomology modules [6, Satz 1]. The Faltings' theorem was stated that for a given finitely generated R-module and a positive integer n, the $R_{\mathfrak{p}}$-module $H^i_{IR_{\mathfrak{p}}}(N_{\mathfrak{p}})$ is finitely generated for all $0 \leq i \leq n$ and for all $\mathfrak{p} \in \operatorname{Spec}R$ if and only if the R-module $H_I^i(N)$ is finitely

Received May 31, 2022; Revised December 23, 2022; Accepted February 10, 2023.
2020 *Mathematics Subject Classification.* Primary 13D45, 13C60.
Key words and phrases. Finiteness dimension, generalized local cohomology, local-global principle, Serre subcategories.

generated for all $0 \leq i \leq n$. The Faltings' local-global principle for the finiteness dimension of local cohomology modules has been improved in [2,5,9,14,15,17].

The Faltings' local-global principle induces the concept of the finiteness dimension $f_I(M)$ which is the least integer i such that a local cohomology module $H^i_I(M)$ is not a finitely generated R-module.

Recently, Faltings' local-global principle has been applied to the generalized local cohomology modules. Some results relating to this problem can be seen in [7,11].

In this paper, we will introduce the concept of $\mathcal{M}$-minimax R-modules, where $\mathcal{M}$ is a stable Serre subcategory of the category of R-modules. This notion is based on the concept of $\mathcal{S}$-minimax R-modules [13] and some results in [17]. Recall that a stable Serre subcategory of the category of R-modules is a Serre subcategory that is closed under taking injective hulls. An R-module K is said to be $\mathcal{M}$-minimax if there is a finitely generated R-module T of K such that $K/T \in \mathcal{M}$. We investigate the local-global principle for generalized local cohomology modules that concerns to the $\mathcal{M}$-minimaxness. One of the our tools for proving the main results in Section 2, is the following theorem.

Theorem 1.1 (Theorem 2.5). *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. Assume that M, N are two finitely generated R-modules and t is a non-negative integer such that $H^i_I(M,N)$ is $\mathcal{M}$-minimax for all $i < t$. Then $\operatorname{Hom}_R(R/I, H^t_I(M,N))$ is $\mathcal{M}$-minimax.*

As the first main result of this paper, we prove the following.

Theorem 1.2. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules, t a non-negative integer, I an ideal of R and M, N two finitely generated R-modules. Then the following statements are equivalent:*

(i) *The module $H^i_I(M,N)$ is an $\mathcal{M}$-minimax R-module for all $i < t$;*
(ii) *The module $H^i_I(M,N)_\mathfrak{p}$ is an $(\mathcal{M} \otimes_R R_\mathfrak{p})$-minimax $R_\mathfrak{p}$-module for all $i < t$ and for all $\mathfrak{p} \in \operatorname{Spec}R$;*
(iii) *The module $H^i_I(M,N)_\mathfrak{m}$ is an $(\mathcal{M} \otimes_R R_\mathfrak{m})$-minimax $R_\mathfrak{m}$-module for all $i < t$ and for all $\mathfrak{m} \in \operatorname{Max}R$.*

This result is a generalization of Faltings' local-global principle, which includes the local-global principles for the Artinianness and the modules in dimension $< n$ of local cohomology modules as well as of generalized local cohomology modules. Another main result of this paper is Theorem 2.13 which shows some equivalent conditions such that the module $H^i_I(M,N)$ is $\mathcal{M}$-minimax for all $i < t$. This result inspires us to provide the concept $\mathcal{M}$-finiteness dimension $f^{\mathcal{M}}_I(M,N)$ of M, N with respect to I. The paper is closed by some consequents relating to some certain finiteness dimensions in [2,3,11].

Throughout this article, $\mathcal{M}$ is a stable Serre subcategory of the category of R-modules. We shall use $\operatorname{Max}R$ to denote the set of all maximal ideals of R. Also, for any ideal I of R, we denote $\{\mathfrak{p} \in \operatorname{Spec}R \,|\, I \subseteq \mathfrak{p}\}$ by $V(I)$. For any ideal J of R, the radical of J, denoted by $\sqrt{J}$, is defined to be the set

$\{x \in R \,|\, x^n \in J$ for some $n \in \mathbb{N}\}$. We denote by $E_R(M)$ the injective hull of an R-module M. Let $\mathcal{S}$ be a subcategory of the category of R-modules and $\mathfrak{p}$ be a prime ideal of R, we denote by $\mathcal{S} \otimes_R R_\mathfrak{p}$ the set [17]

$$\mathcal{S} \otimes_R R_\mathfrak{p} = \{M \otimes_R R_\mathfrak{p} \,|\, M \in \mathcal{S}\}.$$

Moreover, the set $\cup_{M \in \mathcal{S}} \mathrm{Supp}_R M$ is denoted by $\mathrm{Supp}_R \mathcal{S}$.

2. Main results

In [18], H. Zöschinger introduced the class of minimax modules. An R-module K is said to be a minimax module if K has a finitely generated submodule T such that K/T is Artinian.

Next, we recall that a Serre subcategory $\mathcal{S}$ of the category of R-modules is a subcategory of the category of R-modules if it is closed under taking submodules, quotients and extensions. A Serre subcategory of the category of R-modules is called *stable* if it is closed under taking injective hulls.

Definition. Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. An R-module M is called $\mathcal{M}$-minimax if there exists a finitely generated submodule N of M such that $M/N \in \mathcal{M}$.

Example 2.1.

(i) Note that the class of Artinian R-modules is a stable Serre subcategory of the category of R-modules. Hence all Artinian R-modules are $\mathcal{M}$-minimax.

(ii) It is clear that finitely generated R-modules are $\mathcal{M}$-minimax.

(iii) The class of minimax R-modules, which was introduced by Zöchinger in [18], is $\mathcal{M}$-minimax.

(iv) Since $\mathrm{Ass}_R X = \mathrm{Ass}_R E(X)$, the subcategory $D_{\leq n-1}$ is a stable subcategory. So, the concept of $\mathrm{FD}_{\leq n-1}$ modules in [1] and the modules in dimension $< n$ in [2] are $\mathcal{M}$-minimax.

Lemma 2.2. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. The class of $\mathcal{M}$-minimax R-modules is a Serre subcategory of the category of R-modules.*

Proof. It follows from [16, Corollary 3.5]. □

Lemma 2.3. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. Let M be a finitely generated R and N an $\mathcal{M}$-minimax R-module. Then $\mathrm{Ext}^i_R(M, N)$ and $\mathrm{Tor}^R_i(M, N)$ are $\mathcal{M}$-minimax for all $i \geq 0$.*

Proof. Since M is a finitely generated R-module and R is a Noetherian ring, there exists a free resolution of M

$$\mathbf{F}: \quad \cdots F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

where F_i is finitely generated free for all $i \geq 0$. For each non-negative integer i, one has that $\mathrm{Hom}_R(F_i, N) = \oplus^t N$ for some positive integer t. Since

$\mathrm{Ext}^i_R(M,N) = H^i(\mathrm{Hom}_R(\mathbf{F},N))$ which is a subquotient of the $\mathcal{M}$-minimax R-module $\oplus^t N$, it follows from Lemma 2.2 that $\mathrm{Ext}^i_R(M,N)$ is $\mathcal{M}$-minimax for all $i \geq 0$. The proof of Tor modules is similar. □

Next, we summarize some basic properties of generalized local cohomology modules which follow easily from the definition of generalized local cohomology modules.

Lemma 2.4. *Let M be a finitely generated R-module and X an R-module. The following statements are true.*

(i) $\Gamma_I(M,X) \cong \mathrm{Hom}_R(M,\Gamma_I(X)) \cong \Gamma_I(\mathrm{Hom}_R(M,X))$.

(ii) *If $\Gamma_I(X) = X$, then $H^i_I(M,X) \cong \mathrm{Ext}^i_R(M,X)$ for all $i \geq 0$.*

Theorem 2.5. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. Assume that M,N are two finitely generated R-modules and t is a non-negative integer such that $H^i_I(M,N)$ is $\mathcal{M}$-minimax for all $i < t$. Then $\mathrm{Hom}_R(R/I,H^t_I(M,N))$ is $\mathcal{M}$-minimax.*

Proof. The proof is by induction on t. Let $t = 0$. We see that

$$\mathrm{Hom}_R(R/I,H^0_I(M,N)) \subseteq H^0_I(M,N) = \Gamma_I(\mathrm{Hom}_R(M,N)).$$

Since M,N are two finitely generated R-modules, so is $H^0_I(M,N)$ and then $\mathrm{Hom}_R(R/I,H^0_I(M,N))$ is $\mathcal{M}$-minimax.

Now, let $t > 0$. The short exact sequence

$$0 \to \Gamma_I(N) \to N \to N/\Gamma_I(N) \to 0$$

induces a long exact sequence

$$\cdots \to H^t_I(M,\Gamma_I(N)) \xrightarrow{\alpha} H^t_I(M,N) \xrightarrow{\beta} H^t_I(M,N/\Gamma_I(N)) \xrightarrow{\gamma} \cdots.$$

Lemma 2.4(ii) shows that $H^i_I(M,\Gamma_I(N)) \cong \mathrm{Ext}^i_R(M,\Gamma_I(N))$ for all $i \geq 0$. It follows from the assumption that $\mathrm{Ext}^i_R(M,\Gamma_I(N))$ is finitely generated for all $i \geq 0$. Hence, $H^i_I(M,\Gamma_I(N))$ is $\mathcal{M}$-minimax for all $i \geq 0$. Let $\overline{N} = N/\Gamma_I(N)$. The hypothesis induces that $H^i_I(M,\overline{N})$ is $\mathcal{M}$-minimax for all $i < t$. There are short exact sequences

$$0 \to \mathrm{Im}\alpha \to H^t_I(M,N) \to \mathrm{Im}\beta \to 0$$

and

$$0 \to \mathrm{Im}\beta \to H^t_I(M,N/\Gamma_I(N)) \to \mathrm{Im}\gamma \to 0.$$

Applying the functor $\mathrm{Hom}_R(R/I,-)$ to these above short exact sequences, we obtain the following exact sequences

$$\begin{aligned} 0 \to \mathrm{Hom}_R(R/I,\mathrm{Im}\alpha) &\to \mathrm{Hom}_R(R/I,H^t_I(M,N)) \\ &\to \mathrm{Hom}_R(R/I,\mathrm{Im}\beta) \to \mathrm{Ext}^1_R(R/I,\mathrm{Im}\alpha) \end{aligned}$$

and

$$0 \to \mathrm{Hom}_R(R/I,\mathrm{Im}\beta) \to \mathrm{Hom}_R(R/I,H^t_I(M,N/\Gamma_I(N))) \to \mathrm{Hom}_R(R/I,\mathrm{Im}\gamma).$$

By Lemma 2.2, $\mathrm{Im}\alpha$ and $\mathrm{Im}\gamma$ are $\mathcal{M}$-minimax. Lemma 2.3 induces that $\mathrm{Hom}_R(R/I, \mathrm{Im}\alpha)$, $\mathrm{Ext}^1_R(R/I, \mathrm{Im}\alpha)$ and $\mathrm{Hom}_R(R/I, \mathrm{Im}\gamma)$ are $\mathcal{M}$-minimax R-modules. Hence, the proof is complete by showing that

$$\mathrm{Hom}_R(R/I, H^t_I(M, N/\Gamma_I(N)))$$

is $\mathcal{M}$-minimax. It is clear that $\overline{N}$ is I-torsion free. Consequently, there is an element $x \in I$ which is $\overline{N}$-regular. The short exact sequence

$$0 \to \overline{N} \xrightarrow{x} \overline{N} \to \overline{N}/x\overline{N} \to 0$$

yields the following exact sequence

$$\cdots \xrightarrow{f} H^{t-1}_I(M, \overline{N}/x\overline{N}) \xrightarrow{g} H^t_I(M, \overline{N}) \xrightarrow{x} H^t_I(M, \overline{N}) \to \cdots .$$

This implies that $H^i_I(M, \overline{N}/x\overline{N})$ is $\mathcal{M}$-minimax for all $i < t-1$. Therefore, we can claim by the inductive hypothesis that $\mathrm{Hom}_R(R/I, H^{t-1}_I(M, \overline{N}/x\overline{N}))$ is $\mathcal{M}$-minimax. Now, the short exact sequence

$$0 \to \mathrm{Im}f \to H^{t-1}_I(M, \overline{N}/x\overline{N}) \to (0 :_{H^t_I(M,\overline{N})} x) \to 0$$

induces a long exact sequence

$$0 \to \mathrm{Hom}_R(R/I, \mathrm{Im}f) \to \mathrm{Hom}_R(R/I, H^{t-1}_I(M, \overline{N}/x\overline{N}))$$
$$\to \mathrm{Hom}_R(R/I, (0 :_{H^t_I(M,\overline{N})} x)) \to \mathrm{Ext}^1_R(R/I, \mathrm{Im}f) \to \cdots .$$

Since $\mathrm{Im}f$ is an $\mathcal{M}$-minimax R-module, combining Lemma 2.2 with Lemma 2.3, we see that $\mathrm{Hom}_R(R/I, (0 :_{H^t_I(M,\overline{N})} x))$ is $\mathcal{M}$-minimax. Moreover, since $x \in I$, there is an isomorphism

$$\mathrm{Hom}_R(R/I, (0 :_{H^t_I(M,\overline{N})} x)) \cong \mathrm{Hom}_R(R/I, H^t_I(M, \overline{N})),$$

which completes the proof. □

Proposition 2.6. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules. Let M, N be two finitely generated R-modules and t a non-negative integer such that $H^i_I(M, N)$ is $\mathcal{M}$-minimax for all $i < t$. Then the set $\{\mathfrak{p} \in \mathrm{Ass}_R H^t_I(M, N) \mid R/\mathfrak{p}$ is not in $\mathcal{M}\}$ is finite.*

Proof. It follows from Theorem 2.5 that there is a finitely generated R-modules X and an R-module Y in $\mathcal{M}$ such that

$$0 \to X \to \mathrm{Hom}_R(R/I, H^t_I(M, N)) \to Y \to 0$$

is a short exact sequence. It should be noted that

$$\mathrm{Ass}_R \mathrm{Hom}_R(R/I, H^t_I(M, N)) = \mathrm{Ass}_R H^t_I(M, N)$$

and

$$\mathrm{Ass}_R H^t_I(M, N) \subseteq \mathrm{Ass}_R X \cup \mathrm{Ass}_R Y.$$

Let $\mathfrak{p} \in \mathrm{Ass}_R H^t_I(M, N)$ and $R/\mathfrak{p}$ be not in $\mathcal{M}$. We show $\mathfrak{p} \notin \mathrm{Ass}_R Y$. If $\mathfrak{p} \in \mathrm{Ass}_R Y$, then $R/\mathfrak{p}$ is an isomorphism to a submodule of Y. Since $\mathcal{M}$ is a Serre subcategory of the category of R-modules, we can conclude that

$R/\mathfrak{p} \in \mathcal{M}$, a contradiction. Consequently, we have $\mathfrak{p} \in \mathrm{Ass}_R X$. Moreover, since X is a finitely generated R-module, the set $\mathrm{Ass}_R X$ is finite. Thus, we get the assertion. □

Let $\mathfrak{p} \in \mathrm{Spec} R$, we denote the set

$$\mathcal{M} \otimes_R R_\mathfrak{p} = \{M \otimes_R R_\mathfrak{p} \mid M \text{ is an } R\text{-module in } \mathcal{M}\}.$$

It follows from [17, Proposition 3.2] that

$$\mathcal{M} \otimes_R R_\mathfrak{p} = \{M \text{ is an } R_\mathfrak{p}\text{-module} \mid M \text{ is in } \mathcal{M} \text{ as an } R\text{-module}\}$$

and $\mathcal{M} \otimes_R R_\mathfrak{p}$ is a stable Serre subcategory of the category of $R_\mathfrak{p}$-modules.

Lemma 2.7. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules and M an $\mathcal{M}$-minimax R-module. Then $\mathcal{M}_\mathfrak{p}$ is an $(\mathcal{M} \otimes_R R_\mathfrak{p})$-minimax $R_\mathfrak{p}$-module for all $\mathfrak{p} \in \mathrm{Spec} R$.*

Proof. Since M is an $\mathcal{M}$-minimax R-module, there is a short exact sequence

$$0 \to A \to M \to B \to 0,$$

where A is a finitely generated R-module and $B \in \mathcal{M}$. Let $\mathfrak{p} \in \mathrm{Spec} R$. Applying the functor $- \otimes_R R_\mathfrak{p}$ to the above exact sequence, we obtain the short exact sequence

$$0 \to A_\mathfrak{p} \to M_\mathfrak{p} \to B_\mathfrak{p} \to 0.$$

Note that $B_\mathfrak{p} \cong B \otimes_R R_\mathfrak{p} \in \mathcal{M} \otimes_R R_\mathfrak{p}$ and $A_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module. It follows from [17, Proposition 3.2] that $\mathcal{M} \otimes_R R_\mathfrak{p}$ is a stable Serre subcategory of the category of $R_\mathfrak{p}$-modules. Hence, $M_\mathfrak{p}$ is an $(\mathcal{M} \otimes_R R_\mathfrak{p})$-minimax $R_\mathfrak{p}$-module. □

We are going to state and prove the first main result of this paper.

Theorem 2.8. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules, t a non-negative integer, I an ideal of R and M, N two finitely generated R-modules. Then the following statements are equivalent*:

(i) *The module $H^i_I(M, N)$ is an $\mathcal{M}$-minimax R-module for all $i < t$*;
(ii) *The module $H^i_I(M, N)_\mathfrak{p}$ is an $(\mathcal{M} \otimes_R R_\mathfrak{p})$-minimax $R_\mathfrak{p}$-module for all $i < t$ and for all $\mathfrak{p} \in \mathrm{Spec} R$*;
(iii) *The module $H^i_I(M, N)_\mathfrak{m}$ is an $(\mathcal{M} \otimes_R R_\mathfrak{m})$-minimax $R_\mathfrak{m}$-module for all $i < t$ and for all $\mathfrak{m} \in \mathrm{Max} R$.*

Proof. The implications (i) ⇒ (ii) ⇒ (iii) hold by Lemma 2.7.

(iii) ⇒ (i) The proof is by induction on t. Let $t = 0$. It follows Lemma 2.4 that $H^0_I(M, N)$ is an R-submodule of the finitely generated R-module $\mathrm{Hom}_R(M, N)$. Then we get the conclusion in this case.

We assume that $t > 0$ and the theorem is true for $t - 1$. The inductive hypothesis shows that $H^i_I(M, N)$ is $\mathcal{M}$-minimax for all $i \le t - 2$. Now, we prove that $H^{t-1}_I(M,N)$ is also $\mathcal{M}$-minimax. Proposition 2.6 indicates that the set

$$\{\mathfrak{p}\in \mathrm{Ass}_R H^{t-1}_I(M,N)\,|\,R/\mathfrak{p} \text{ is not in } \mathcal{M}\}=\{\mathfrak{p}_1,\mathfrak{p}_2,\ldots,\mathfrak{p}_k\}$$

is finite. Fix $1\leq i\leq k$. There is an ideal $\mathfrak{m}_i\in \mathrm{Max}R$ such that $\mathfrak{p}_i\subseteq\mathfrak{m}_i$. By the hypothesis (iii), $H^{t-1}_I(M,N)_{\mathfrak{m}_i}$ is an $(\mathcal{M}\otimes_R R_{\mathfrak{m}_i})$-minimax $R_{\mathfrak{m}_i}$-module. Now, there is a finitely generated R-module X_i and an R-module Y_i in $\mathcal{M}$ such that

$$0\to X_i\otimes_R R_{\mathfrak{m}_i}\to H^{t-1}_I(M,N)_{\mathfrak{m}_i}\to Y_i\otimes_R R_{\mathfrak{m}_i}\to 0$$

is a short exact sequence of $R_{\mathfrak{m}_i}$-modules. Applying the functor $-\otimes_{R_{\mathfrak{m}_i}}(R_{\mathfrak{m}_i})_{\mathfrak{p}_iR_{\mathfrak{m}_i}}$ to the above exact sequence, we get, by [12, Corollary 4, p. 24], the following short exact sequence of $R_{\mathfrak{p}_i}$-modules

$$0\to X_i\otimes_R R_{\mathfrak{p}_i}\to H^{t-1}_I(M,N)_{\mathfrak{p}_i}\to Y_i\otimes_R R_{\mathfrak{p}_i}\to 0.$$

Since $R/\mathfrak{p}_i$ is not in $\mathcal{M}$, we see that $Y_i\otimes_R R_{\mathfrak{p}_i}=0$. This implies that

$$X_i\otimes_R R_{\mathfrak{p}_i}\cong H^{t-1}_I(M,N)_{\mathfrak{p}_i}.$$

Hence $H^{t-1}_I(M,N)_{\mathfrak{p}_i}$ is a finitely generated $R_{\mathfrak{p}_i}$-module. There exists a positive integer m_i such that $(IR_{\mathfrak{p}_i})^{m_i}H^{t-1}_I(M,N)_{\mathfrak{p}_i}=0$. Let $m=\max\{m_1,m_2,\ldots,m_k\}$. Then we get

$$\{\mathfrak{p}_1,\mathfrak{p}_2,\ldots,\mathfrak{p}_k\}\cap \mathrm{Supp}_R I^mH^{t-1}_I(M,N)=\emptyset.$$

Let $\mathfrak{q}\in \mathrm{Ass}_R I^mH^{t-1}_I(M,N)$. Then $\mathfrak{q}\in \mathrm{Ass}_R H^{t-1}_I(M,N)\setminus\{\mathfrak{p}_1,\mathfrak{p}_2,\ldots,\mathfrak{p}_k\}$. Therefore, $R/\mathfrak{q}$ is in $\mathcal{M}$ and then $\mathrm{Ass}_R I^mH^{t-1}_I(M,N)\subseteq \mathrm{Supp}_R\mathcal{M}$. This induces that

$$\mathrm{Ass}_R\mathrm{Hom}_R(R/I,I^mH^{t-1}_I(M,N))\subseteq \mathrm{Supp}_R\mathcal{M}.$$

On the other hand, by the inductive hypothesis and Theorem 2.5, we can claim that $\mathrm{Hom}_R(R/I,H^{t-1}_I(M,N))$ is $\mathcal{M}$-minimax. Since

$$\mathrm{Hom}_R(R/I,I^mH^{t-1}_I(M,N))$$

is a submodule of $\mathrm{Hom}_R(R/I,H^{t-1}_I(M,N))$, we get the $\mathcal{M}$-minimaxness of $\mathrm{Hom}_R(R/I,I^mH^{t-1}_I(M,N))$. Thus, there are a finitely generated R-module A and an R-module $B\in\mathcal{M}$ such that

$$0\to A\to \mathrm{Hom}_R(R/I,I^mH^{t-1}_I(M,N))\to B\to 0$$

is a short exact sequence. We also have

$$\mathrm{Ass}_RA\subseteq \mathrm{Ass}_R\mathrm{Hom}_R(R/I,I^mH^{t-1}_I(M,N))\subseteq \mathrm{Ass}_RA\cup\mathrm{Ass}_RB.$$

Since $\mathrm{Ass}_RA\subseteq \mathrm{Supp}_R\mathcal{M}$ and A is a finitely generated R-module, the module $E_R(A)$ is the zero module or a finite direct sum of copies of indecomposable injective R-modules $E_R(R/\mathfrak{p})$ with $\mathfrak{p}\in \mathrm{Ass}_RA\subseteq \mathrm{Supp}_R(\mathcal{M})$. It follows from [17, Lemma 4.1] that $R/\mathfrak{p}\in\mathcal{M}$ for all $\mathfrak{p}\in\mathrm{Ass}_RA$. Since $\mathcal{M}$ is stable, this implies that $E_R(R/\mathfrak{p})\in\mathcal{M}$. Consequently, we claim that $E_R(A)\in\mathcal{M}$. Furthermore, the injective homomorphism $A\to \mathrm{Hom}_R(R/I,I^mH^{t-1}_I(M,N))$ induces a homomorphism $\mathrm{Hom}_R(R/I, I^m H_I^{t-1}(M,N)) \to E_R(A)$. Then there is an injective homomorphism $E_R(\mathrm{Hom}_R(R/I, I^m H_I^{t-1}(M,N))) \to E_R(A)$. This shows that $E_R(\mathrm{Hom}_R(R/I, I^m H_I^{t-1}(M,N)))$ is a direct summand of $E_R(A) \in \mathcal{M}$. On the other hand, there is an inclusion

$$I^m H_I^{t-1}(M,N) \subseteq E_R(I^m H_I^{t-1}(M,N)) = E_R(\mathrm{Hom}_R(R/I, I^m H_I^{t-1}(M,N))).$$

Hence, we can claim that $I^m H_I^{t-1}(M,N) \in \mathcal{M}$ and then it is an $\mathcal{M}$-minimax R-module. By [17, Lemma 5.1(2)] and Lemma 2.2, one gets that

$$H_I^{t-1}(M,N)/(0 :_{H_I^{t-1}(M,N)} I^m)$$

is $\mathcal{M}$-minimax.

Now, combining Theorem 2.5 with the inductive hypothesis, we assert that $(0 :_{H_I^{t-1}(M,N)} I)$ is $\mathcal{M}$-minimax. Again, using [17, Lemma 5.1(3)], we have that $(0 :_{H_I^{t-1}(M,N)} I^m)$ is $\mathcal{M}$-minimax. Finally, the short exact sequence

$$0 \to (0 :_{H_I^{t-1}(M,N)} I^m) \to H_I^{t-1}(M,N) \to H_I^{t-1}(M,N)/(0 :_{H_I^{t-1}(M,N)} I^m) \to 0$$

and Lemma 2.2 show that $H_I^{t-1}(M,N)$ is an $\mathcal{M}$-minimax R-module, which complete the proof. □

Corollary 2.9 (See [6, Satz 1])**.** *Let N be a finitely generated R-module and t a positive integer. Then the following statements are equivalent*:

(i) *The module $H_I^i(N)$ is a finitely generated R-module for all $i < t$;*
(ii) *The module $H_I^i(N)_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for all $i < t$ and for all $\mathfrak{p} \in \mathrm{Spec}R$;*
(iii) *The module $H_I^i(N)_\mathfrak{m}$ is a finitely generated $R_\mathfrak{m}$-module for all $i < t$ and for all $\mathfrak{m} \in \mathrm{Max}R$.*

Proof. The assertion follows from Theorem 2.8 when $M = R$ and $\mathcal{M} = \{0\}$ the zero subcategory of the category of R-modules. □

Corollary 2.10 (See [7, Theorem 5.3])**.** *Let M, N be two finitely generated R-modules and t a positive integer. Then the following statements are equivalent*:

(i) *The module $H_I^i(M,N)$ is an Artinian R-module for all $i < t$;*
(ii) *The module $H_I^i(M,N)_\mathfrak{p}$ is an Artinian $R_\mathfrak{p}$-module for all $i < t$ and for all $\mathfrak{p} \in \mathrm{Spec}R$;*
(iii) *The module $H_I^i(M,N)_\mathfrak{m}$ is an Artinian $R_\mathfrak{m}$-module for all $i < t$ and for all $\mathfrak{m} \in \mathrm{Max}R$.*

Proof. Applying Theorem 2.8 which $\mathcal{M}$ is the class of Artinian R-modules. □

Corollary 2.11 (See [11, Theorem 2.2])**.** *Let M, N be finitely generated R-modules and n, t two non-negative integers. Then the following statements are equivalent*:

(i) *The module $H_I^i(M,N)$ is in dimension $< n$ for all $i < t$;*

(ii) *The module* $H^i_I(M,N)_\mathfrak{p}$ *is in dimension* $< n$ *as* $R_\mathfrak{p}$*-module for all* $i < t$ *and for all* $\mathfrak{p} \in \mathrm{Spec}R$;

(iii) *The module* $H^i_I(M,N)_\mathfrak{m}$ *is in dimension* $< n$ *as* $R_\mathfrak{m}$*-module for all* $i < t$ *and for all* $\mathfrak{m} \in \mathrm{Max}R$.

Proof. The assertion follows from Theorem 2.8 by applying $\mathcal{M}$ to be the stable Serre subcategory of R-modules in dimension $< n$. □

Corollary 2.12 ([5, Theorem 2.2])**.** *Let* N *be a finitely generated* R*-module and* n, t *two non-negative integers. Then the following statements are equivalent*:

(i) *The module* $H^i_I(N)$ *is in dimension* $< n$ *for all* $i < t$;

(ii) *The module* $H^i_I(N)_\mathfrak{p}$ *is in dimension* $< n$ *as* $R_\mathfrak{p}$*-module for all* $i < t$ *and for all* $\mathfrak{p} \in \mathrm{Spec}R$;

(iii) *The module* $H^i_I(N)_\mathfrak{m}$ *is in dimension* $< n$ *as* $R_\mathfrak{m}$*-module for all* $i < t$ *and for all* $\mathfrak{m} \in \mathrm{Max}R$.

Proof. The assertion follows from Theorem 2.8 by applying $M = R$ and $\mathcal{M}$ to be the stable Serre subcategory of R-modules in dimension $< n$. □

The following theorem is the second main result of this paper, which provides some equivalent conditions for the $\mathcal{M}$-minimaxness of the generalized local cohomology modules.

Theorem 2.13. *Let* $\mathcal{M}$ *be a stable Serre subcategory of the category of* R*-modules,* t *a non-negative integer,* I *an ideal of* R *and* M, N *two finitely generated* R*-modules. Then the following statements are equivalent*:

(i) *The module* $H^i_I(M,N)$ *is* $\mathcal{M}$*-minimax for all* $i < t$;

(ii) *There exists a positive integer* m *such that* $I^m H^i_I(M,N)$ *is in* $\mathcal{M}$ *for all* $i < t$.

(iii) *The module* $H^i_I(M,N)_\mathfrak{p}$ *is a finitely generated* $R_\mathfrak{p}$*-module for all* $i < t$ *and for all* $\mathfrak{p} \in \mathrm{Supp}_R M \cap \mathrm{Supp}_R N \cap V(I)$ *with* $R/\mathfrak{p} \notin \mathcal{M}$.

Proof. (i) ⇒ (ii) Let $i < t$ be an integer. Since $H^i_I(M,N)$ is $\mathcal{M}$-minimax and $\mathrm{Supp}_R H^i_I(M,N) \subseteq V(I)$, by [13, Theorem 2.8] there exists an integer m such that $I^m H^i_I(M,N) \in \mathcal{M}$.

(ii) ⇒ (i) We proceed by induction on t. There is nothing to do in the case $t = 0$. Let $t = 1$. Since M, N are two finitely generated R-modules and the module $H^0_I(M,N)$ is a submodule of $\mathrm{Hom}_R(M,N)$, we see that $H^0_I(M,N)$ is finitely generated and then it is also $\mathcal{M}$-minimax.

Now, consider the case where $t > 1$. Let $m \geq 1$ be an integer such that $I^m H^s_I(M,N) \in \mathcal{M}$ for all $s < t$. It is obvious that $I^m H^s_I(M,N)$ is $\mathcal{M}$-minimax. One has that $H^s_I(M,N)/(0 :_{H^s_I(M,N)} I^m)$ is $\mathcal{M}$-minimax by [17, Lemma 5.1 (2)]. The inductive assumption induces that $H^i_I(M,N)$ is $\mathcal{M}$-minimax for $i < s$. Also, in view of Theorem 2.5, the module $\mathrm{Hom}_R(R/I, H^s_I(M,N))$ is $\mathcal{M}$-minimax. We have by [17, Lemma 5.1(3)] that $(0 :_{H^s_I(M,N)} I^m)$ is $\mathcal{M}$-minimax.

The short exact sequence

$$0 \to (0 :_{H^s_I(M,N)} I^m) \to H^s_I(M,N) \to H^s_I(M,N)/(0 :_{H^s_I(M,N)} I^m) \to 0$$

and Lemma 2.2 induce that $H^s_I(M,N)$ is an $\mathcal{M}$-minimax R-module, which complete the proof.

(i) ⇒ (iii) For each non-negative integer $i < t$, there is the short exact sequence

$$0 \to X \to H^i_I(M,N) \to Y \to 0,$$

where X is a finitely generated R-module and $Y \in \mathcal{M}$. Let $\mathfrak{p} \in \operatorname{Spec} R$ such that $R/\mathfrak{p} \notin \mathcal{M}$, it is clear that $Y_{\mathfrak{p}} = 0$. Therefore, the isomorphism

$$X_{\mathfrak{p}} \cong H^i_I(M,N)_{\mathfrak{p}}$$

shows that $H^i_I(M,N)_{\mathfrak{p}}$ is a finitely generated $R_{\mathfrak{p}}$-module.

(iii) ⇒ (i) We prove the implication by induction on t. The case where $t = 0$ is trivial. Suppose that the result has been proved for smaller than $t-1$. By the inductive hypothesis $H^i_I(M,N)$ is $\mathcal{M}$-minimax for all $i < t-1$, now we show that $H^{t-1}_I(M,N)$ is $\mathcal{M}$-minimax. It follows from Proposition 2.6 that

$$\{\mathfrak{p} \in \operatorname{Ass}_R H^{t-1}_I(M,N) \mid R/\mathfrak{p} \notin \mathcal{M}\} = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k\}$$

is a finite set. The assumption shows that $H^{t-1}_I(M,N)_{\mathfrak{p}_i}$ is a finitely generated $R_{\mathfrak{p}_i}$-module for all $1 \le i \le k$. Then, there is an integer m_i such that $(I^{m_i} H^{t-1}_I(M,N))_{\mathfrak{p}_i} = 0$. Let $m = \max\{m_1, \ldots, m_k\}$. Then

$$\{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\} \cap \operatorname{Supp}_R I^m H^{t-1}_I(M,N) = \emptyset.$$

Therefore, one has

$$\begin{aligned} \operatorname{Ass}_R I^m H^{t-1}_I(M,N) &\subseteq \{\mathfrak{p} \in \operatorname{Supp}_R M \cap \operatorname{Supp}_R N \cap V(I) \mid R/\mathfrak{p} \in \mathcal{M}\} \\ &\subseteq \operatorname{Supp}_R \mathcal{M}. \end{aligned}$$

By using the same arguments in the proof of Theorem 2.8 (iii) ⇒ (i), we get the claim. □

Corollary 2.14 ([7, Proposition 3.1])**.** *Let M, N be finitely generated R-modules and t a non-negative integer. Then the following statements are equivalent*:

(i) *The module $H^i_I(M,N)$ is finitely generated for all $i < t$*;

(ii) *There exists a positive integer m such that $I^m H^i_I(M,N) = 0$ for all $i < t$.*

Let M be a finitely generated R-module. Following [4, Definition 9.1.3], the finiteness dimension $f_I(M)$ of M relative to I is defined as follows:

$$f_I(M) = \inf\{i \in \mathbb{N}_0 \mid H^i_I(M) \text{ is not finitely generated}\},$$

with the usual convention that the infimum of the empty set of integers is interpreted as ∞. It is well-known that

$$f_I(M) = \inf\{i \in \mathbb{N} \mid I \not\subseteq \sqrt{H^i_I(M)}\}$$

$$= \inf\{f_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Spec} R\}.$$

We introduce the concept of an $\mathcal{M}$-finiteness dimension of M, N with respect to I which is an extension of the concept of an nth finiteness dimension $f_I^n(M)$ in [3] as well as the concept of an n-th finiteness dimension $f_I^n(M, N)$ in [11]. This is also a generalization of the concept of an $\mathcal{S}$-finiteness dimension of M with respect to I in [13].

Definition. Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules, I an ideal of R and M, N two finitely generated R-modules. The $\mathcal{M}$-finiteness dimension of M, N with respect to I is defined as follows

$$f_I^{\mathcal{M}}(M, N) = \inf\{i \mid H_I^i(M, N) \text{ is not } \mathcal{M}\text{-minimax}\}.$$

Corollary 2.15. *Let $\mathcal{M}$ be a stable Serre subcategory of the category of R-modules, I an ideal of R and M, N two finitely generated R-modules. Then*

$$\begin{aligned} f_I^{\mathcal{M}}(M, N) &= \inf\{i \mid I^n H_I^i(M, N) \notin \mathcal{M} \text{ for all } n \in \mathbb{N}\} \\ &= \inf\{f_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N) \cap V(I) \text{ and } \\ &\qquad R/\mathfrak{p} \notin \mathcal{M}\}, \end{aligned}$$

where $f_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = \inf\{i \in \mathbb{N}_0 \mid H^i_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \text{ is not finitely generated}\}$.

Proof. It follows from Theorem 2.13. □

Corollary 2.16 ([11, Theorem 2.4])**.** *Let I be an ideal of R and M, N two finitely generated R-modules. Then*

$$\begin{aligned} f_I^n(M, N) &:= \inf\{f_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N) \cap V(I) \text{ and } \\ &\qquad \dim R/\mathfrak{p} \geq n\} \\ &= \inf\{i \in \mathbb{N}_0 \mid H_I^i(M, N) \text{ is not in dimension } < n\}. \end{aligned}$$

Corollary 2.17 ([2, Theorem 2.5], [3, Theorem 2.10])**.** *Let I be an ideal of R and N a finitely generated R-module. Then*

$$\begin{aligned} f_I^n(M, N) &:= \inf\{f_{IR_{\mathfrak{p}}}(N_{\mathfrak{p}}) \mid \mathfrak{p} \in \operatorname{Supp}_R(N/IN) \text{ and } \dim R/\mathfrak{p} \geq n\} \\ &= \inf\{i \in \mathbb{N}_0 \mid H_I^i(N) \text{ is not in dimension } < n\}. \end{aligned}$$

Acknowledgments. The author is deeply grateful to the referee for careful reading of the manuscript and for the helpful suggestions. This research is funded by Vietnam National University HoChiMinh City (VNU-HCM) under grant number C2023-26-15.

References

[1] M. Aghapournahr and K. Bahmanpour, *Cofiniteness of weakly Laskerian local cohomology modules*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **57(105)** (2014), no. 4, 347–356.

[2] D. Asadollahi and R. Naghipour, *Faltings' local-global principle for the finiteness of local cohomology modules*, Comm. Algebra **43** (2015), no. 3, 953–958. https://doi.org/10.1080/00927872.2013.849261

[3] K. Bahmanpour, R. Naghipour, and M. Sedghi, *Minimaxness and cofiniteness properties of local cohomology modules*, Comm. Algebra **41** (2013), no. 8, 2799–2814. https://doi.org/10.1080/00927872.2012.662709

[4] M. P. Brodmann and R. Y. Sharp, *Local cohomology*, second edition, Cambridge Studies in Advanced Mathematics, 136, Cambridge Univ. Press, Cambridge, 2013.

[5] M. R. Doustimehr, *Faltings' local-global principle and annihilator theorem for the finiteness dimensions*, Comm. Algebra **47** (2019), no. 5, 1853–1861. https://doi.org/10.1080/00927872.2018.1523423

[6] G. Faltings, *Der Endlichkeitssatz in der lokalen Kohomologie*, Math. Ann. **255** (1981), no. 1, 45–56. https://doi.org/10.1007/BF01450555

[7] A. Fathi, *Local-global principle for the finiteness and Artinianness of generalised local cohomology modules*, Bull. Aust. Math. Soc. **87** (2013), no. 3, 480–492. https://doi.org/10.1017/S0004972712000640

[8] A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), North-Holland, Amsterdam, 1968.

[9] M. Hatamkhani and H. Roshan-Shekalgourabi, *On the local-global principles for the* cd *local cohomology modules*, Algebraic Structures and Their Applications **8** (2021), no. 2, 1–15.

[10] J. Herzog, *Komplexe, Auflösungen und dualität in der localen Algebra*, Habilitationsschrift, Universität Regensburg, 1970.

[11] N. V. Hoang, *On Faltings' local-global principle of generalized local cohomology modules*, Kodai Math. J. **40** (2017), no. 1, 58–62. https://doi.org/10.2996/kmj/1490083223

[12] H. Matsumura, *Commutative ring theory*, translated from the Japanese by M. Reid, Cambridge Studies in Advanced Mathematics, 8, Cambridge Univ. Press, Cambridge, 1986.

[13] S. Rezaei, $\mathcal{S}$-minimaxness and local cohomology modules, Arch. Math. (Basel) **110** (2018), no. 6, 563–572. https://doi.org/10.1007/s00013-018-1157-1

[14] H. Roshan-Shekalgourabi, *On the local-global principle for the local cohomology modules*, Math. Rep. (Bucur.) **22(72)** (2020), no. 2, 107–120.

[15] Z. M. Tang, *Local-global principle for the Artinianness of local cohomology modules*, Comm. Algebra **40** (2012), no. 1, 58–63. https://doi.org/10.1080/00927872.2010.511640

[16] T. Yoshizawa, *Subcategories of extension modules by Serre subcategories*, Proc. Amer. Math. Soc. **140** (2012), no. 7, 2293–2305. https://doi.org/10.1090/S0002-9939-2011-11108-0

[17] T. Yoshizawa, *Remarks on the local-global principle for a subcategory consisting extension modules*, J. Algebra Appl. **18** (2019), no. 12, 1950236 (18 pages)

[18] H. Zöschinger, *Minimax-moduln*, J. Algebra **102** (1986), no. 1, 1–32. https://doi.org/10.1016/0021-8693(86)90125-0

Bui Thi Hong Cam
Department of Natural Science Education
Dong Nai University
Dong Nai Province, Vietnam
Email address: hongcamb@gmail.com

Nguyen Minh Tri
Vietnam National University
Ho Chi Minh City, Vietnam
and
Department of Mathematics and Physics
University of Information Technology
Ho Chi Minh City, Vietnam
Email address: trinm@uit.edu.vn, triminhng@gmail.com

Do Ngoc Yen
Faculty of Fundamental Science 2
Posts and Telecommunications Institute of Technology
Ho Chi Minh City, Vietnam
Email address: dongocyen.dhsp@gmail.com