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S-COHERENT PROPERTY IN TRIVIAL EXTENSION AND IN AMALGAMATED DUPLICATION

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Abstract. Bennis and El Hajoui have defined a (commutative unital) ring R to be S-coherent if each finitely generated ideal of R is a S-finitely presented R-module. Any coherent ring is an S-coherent ring. Several examples of S-coherent rings that are not coherent rings are obtained as byproducts of our study of the transfer of the S-coherent property to trivial ring extensions and amalgamated duplications.

1. Introduction

Throughout this paper, all rings are assumed to be commutative with nonzero identity and all modules are nonzero unital. Let R denote such a ring and S denote such a multiplicatively closed subset of R such that $0 \notin S$. $Reg(R)$ denotes the set of regular elements of the ring R and $Q(R) := R_{Reg(R)}$, the total quotient ring of R. For a nonnegative integer n, an R-module E is called n-presented if there is an exact sequence of R-modules:

$$F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to E \to 0,$$

where each F_i is a finitely generated free R-module. In particular, 0-presented and 1-presented R-modules are, respectively, finitely generated and finitely presented R-modules. Recall that R is an n-coherent ring if each n-presented R-module is $(n+1)$-presented. Thus, the 1-coherent rings are just the coherent rings and an n-coherent ring is $(n+1)$-coherent for any positive integer n. For instance, any coherent ring is 2-coherent and the converse is false (for example $\mathbb{Z} \propto \mathbb{Q}$ is a 2-coherent ring which is not coherent by [17, Theorem 3.1]).

A ring R is coherent if every finitely generated ideal of R is finitely presented; equivalently, if $(0 : a)$ and $I \cap J$ are finitely generated for every $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings are Noetherian rings, Boolean algebras, von Neumann regular rings, valuation domains, and Prüfer domains/semihereditary rings. The concept of coherence

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first sprang up from the study of coherent sheaves in algebraic geometry, and then developed, under the influence of Noetherian ring theory and homology, towards a full-fledged topic in algebra. During the past 30 years, several (commutative) coherent-like notions grew out of coherence such as finite conductor, quasi-coherent, v-coherent, and n-coherent. See for instance [1, 5, 13, 15, 17].

In [2], Anderson and Dumitrescu introduced the concept of S-finite modules, where S is a multiplicatively subset as follows: an R-module M is called an S-finite module if there exist a finitely generated R-submodule N of M and $s \in S$ such that $sM \subseteq N$. Also, they introduced the concept of S-Noetherian rings as follows: a ring R is called S-Noetherian if every ideal of R is S-finite. Recently, in [5], Bennis and El Hajoui investigated the S-versions of finitely presented modules and coherent modules which are called, respectively, S-finitely presented modules and S-coherent modules. An R-module M is called an S-finitely presented module for some multiplicatively closed subset S of R if there exists an exact sequence of R-modules $0 \to K \to F \to M \to 0$, where F is a finitely generated free R-module and K is an S-finite R-module. Moreover, an R-module M is said to be S-coherent if it is finitely generated and every finitely generated submodule of M is S-finitely presented. They showed that the S-coherent rings have a characterization similar to the classical one given by Chase for coherent rings (see [5, Theorem 3.8]). Any coherent ring is S-coherent and any S-Noetherian ring is S-coherent. See for instance [2, 5].

Some of our results use the $R \propto M$ construction. Let R be a ring and M be an R-module. Then $R \propto M$, the *trivial (ring) extension of R by M*, is the ring whose additive structure is that of the external direct sum $R \oplus M$ and whose multiplication is defined by $(r_1, m_1)(r_2, m_2) := (r_1 r_2, r_1 m_2 + r_2 m_1)$ for all $r_1, r_2 \in R$ and all $m_1, m_2 \in M$. The basic properties of trivial ring extensions are summarized in the books [13, 14]. Mainly, trivial ring extensions have been useful for solving many open problems and conjectures in both commutative and non-commutative ring theory. See for instance [3, 4, 10, 11, 13, 14, 16–18].

Let A be a ring and I an ideal of A. The following ring construction called the amalgamated duplication of A along I was introduced and investigated by D'Anna in [7] with the aim of applying it to curve singularities (over algebraic closed fields) where he proved that the amalgamated duplication of an algebroid curve along a regular canonical ideal yields a Gorenstein algebroid curve [7, Theorem 14 and Corollary 17]. It is the subring $A \bowtie I$ of $A \times A$ given by

$$A \bowtie I = \{(a, a+i) \mid a \in A \text{ and } i \in I\}.$$

This extension has been studied, in the general case, and from the different point of view of pullbacks, by D'anna and Fontana [9]. One main difference of this construction, with respect to the idealization, is that the ring $A \bowtie I$ can be reduced (and it is always reduced if A is an integral domain). If J is an ideal of A, then $J \bowtie I := \{(j, j+i) \mid j \in J, i \in I\}$ is an ideal of $A \bowtie I$ with $\frac{A \bowtie I}{J \bowtie I} \cong \frac{A}{J}$. Under the natural injection $A \hookrightarrow A \bowtie I$ defined by $i(a) = (a, a)$, we identify A with its respective image in $A \bowtie I$; and the natural surjection $A \bowtie I \twoheadrightarrow A$ yields the isomorphism $\frac{A \bowtie I}{(0) \bowtie I} \cong A$. See for instance [6–9, 12].

This paper investigates S-coherent condition that a trivial extension $R := A \propto E$ might inherit from the ring A for some classes of modules E. Also, we study the amalgamated duplication of a ring along an ideal to inherit the S-coherence. Our results generate new families of examples of non-coherent S-coherent rings.

2. S-coherence property in trivial ring extension

Recall that a ring R is called S-coherent if every finitely generated ideal of R is S-finitely presented. Remark that if R is S-coherent, then $S^{-1}R$ is a coherent ring. Also, any coherent ring is S-coherent for every multiplicative set.

First, we give an example of non-coherent S-coherent rings.

Example 2.1. Let R be any non-coherent domain and set $S := R - \{0\}$ be a multiplicative set of R. Then R is S-Noetherian. In particular, R is S-coherent.

Proof. Let I be a proper ideal of R and let $s \in I \setminus \{0\}$. Hence, $sI \subseteq Rs \subseteq I$ and so I is S-finite sine Rs is a finitely generated ideal of R, as desired. □

Let $R := A \propto E$ be the trivial ring extension of a ring A by an A-module E. Remark that if S is a multiplicative set of R, then $S_0 = \{a \in A \,|\, (a, e) \in S$ for some $e \in E\}$ is a multiplicative set of A. Conversely, if S_0 is a multiplicative set of A, then $S := S_0 \propto N$ is a multiplicative set of R for every submodule N of E such that $S_0 N \subseteq N$. In particular, $S_0 \propto 0$ and $S_0 \propto E$ are multiplicative sets of R.

Now, we explore the transfer of S-coherent property to the trivial ring extension of a domain A by a K-vector space E, where K is a quotient field of A.

Theorem 2.2. *Let A be an integral domain which is not a field, $K = qf(A)$, E be a K-vector space, and $R := A \propto E$ be the trivial ring extension of A by E. Then R is never S-coherent for every multiplicative set S of R.*

Proof. Let $J = R(0, f)$, where $f \in E \setminus \{0\}$, and consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R \xrightarrow{u} J \to 0,$$

where $u(a, e) = (a, e)(0, f) = (0, af)$. Hence, $Ker(u) = 0 \propto E$. We claim that $Ker(u)(= 0 \propto E)$ is not S-finite. Deny. Then there exists a finitely generated ideal $L := \sum_{i=1}^{i=n} R(a_i, e_i)$ for some positive integer n and $(a_i, e_i) \in L$, and $(s, e') \in S$, where $s \neq 0$, such that

$$(s, e')(0 \propto E) \subseteq L \subseteq 0 \propto E.$$

Hence, $a_i = 0$ for every $i = 1, \ldots, n$ since $L \subseteq 0 \propto E$. On the other hand, $(s, e')(0 \propto E) = 0 \propto sE = 0 \propto E$ since $sE = E$ for every $s \in K \setminus \{0\}$. Therefore,

$0 \propto E = L = \sum_{i=1}^{i=n} R(0, e_i) = 0 \propto \sum_{i=1}^{i=n} Ae_i$ and so $E = \sum_{i=1}^{i=n} Ae_i$. Hence, K is a finitely generated A-module and so $K = A$, a desired contradiction since A is not a field. Then $R := A \propto E$ is not S-coherent. □

Using Theorem 2.2 in the case when S consists of unit elements, we regain the result [17, Theorem 2.1(1)].

Corollary 2.3. *Let A be an integral domain which is not a field, $K = qf(A)$, E be a K-vector space, and $R := A \propto E$ be the trivial ring extension of A by E. Then R is never coherent.*

Next, we explore a different context, namely, the trivial ring extension of a local ring (A, M) by an A-module E such that $ME = 0$. If a multiplicative set S of R consists of unit elements of R, $S \subseteq U(R)$, then R is S-coherent if and only if R is coherent and it is studied by S. Kabbaj and N. Mahdou in [17, Theorem 2.6(2)]. So, we may assume that S does not consist only of unit elements of R.

Theorem 2.4. *Let (A, M) be a local ring, E an A-module with $ME = 0$ and let $R := A \propto E$ be the trivial ring extension of A by E. Let S be a multiplicative set of R and set $S_0 = \{a \in A \,|\, (a, e) \in S \text{ for some } e \in E\}$. Then*

(1) *If R is S-coherent, then A is S_0-coherent.*
(2) *Assume that $S \nsubseteq U(R)$, that is, there exists $(s_0, e) \in S$ such that $s_0 \in M \setminus 0$. Then R is S-coherent if and only if A is S_0-coherent.*

Proof. One may easily verify that R is local with maximal ideal $M \propto E$ and that each element of R is either a unit or a zero divisor.

(1) Assume that R is S-coherent and let $I = \sum_{i=1}^{i=n} Aa_i$, where $a_i \in M$ and set $J := \sum_{i=1}^{i=n} R(a_i, 0)$. Consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^n = A^n \propto E^n \xrightarrow{u} J \to 0,$$

where $u((b_i, e_i)_{i=1,\ldots,n}) = \sum_{i=1}^{i=n}(b_i, e_i)(a_i, 0) = (\sum_{i=1}^{i=n} a_i b_i, 0)$ since $a_i \in M$ for each $i = 1, \ldots, n$. On the other hand, consider the exact sequence of A-modules:

$$0 \to Ker(v) \to A^n \xrightarrow{v} I \to 0,$$

where $u((b_i)_{i=1,\ldots,n}) = \sum_{i=1}^{i=n} a_i b_i$. Then, $Ker(u) = Ker(v) \propto E^n$. But J is S-finitely presented since R is S-coherent, so $Ker(u)$ is an S-finite R-module. Then there exists a finitely generated ideal $L := \sum_{i=1}^{i=m} R(x_i, e_i) \subseteq Ker(u)$ for some $(x_i, e_i) \in L$ and a positive integer m such that

$$(s, e)Ker(u) \subseteq L \subseteq Ker(u).$$

Hence, for $L_0 = \sum_{i=1}^{i=n} Ax_i$, we have

$$sKer(v) \subseteq L_0 \subseteq Ker(v)$$

and so $Ker(v)$ is S_0-finite, as desired.

Hence, A is S_0-coherent.

(2) By (1) it remains to show that if A is S_0-coherent, then R is S-coherent, where $S \nsubseteq U(R)$.

Let $J := \sum_{i=1}^{i=n} R(a_i, e_i)$ be a finitely generated ideal of R, where $(a_i, e_i)_{i=1,\dots,n}$ is a minimal generating set of J, $a_i \in M$ and $e_i \in E$. Consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R^n \overset{u}{\to} J \to 0,$$

where $u((b_i, f_i)_{i=1,\dots,n}) = \sum_{i=1}^{i=n}(a_i, e_i)(b_i, f_i) = (\sum_{i=1}^{i=n} a_i b_i, \sum_{i=1}^{i=n} b_i e_i)$ since $a_i \in M$ for each $i = 1, \dots, n$. Further, the minimality of $(a_i, e_i)_{i=1,\dots,n}$ yields $Ker(u) = \{(b_i, f_i)_{i=1,\dots,n} \in R^n \mid \sum_{i=1}^{i=n} a_i b_i = 0\}$.

Set $I := \sum_{i=1}^{i=n} Aa_i$ and consider the surjective homomorphism v defined above. Then $Ker(v)$ is an S_0-finite A-module since A is S_0-coherent. Hence, there exists a finitely generated A-module $L_0 := \sum_{i=1}^{i=m} Ax_i$ for some $x_i \in L_0$ and a positive integer m, and $s \in S_0$ such that

$$sKer(v) \subseteq L_0 \subseteq Ker(v).$$

We may assume that s is not invertible since if $s_0 \in S_0$ is not invertible (since $S \nsubseteq U(R)$ and so $S_0 \nsubseteq U(A)$), then we have $ss_0Ker(v) \subseteq sKer(v) \subseteq L_0 \subseteq Ker(v)$ and so ss_0 is not invertible and we may replace s by ss_0.

Set $G_0 := \sum_{i=1}^{i=m} R(x_i, 0) = L_0 \propto 0$ (since $L_0 = \sum_{i=1}^{i=m} Ax_i \subseteq M^m$). Hence, $G_0 \subseteq Ker(u)$ and let $e \in E$ such that $(s, e) \in S$. Then, $(s, e)Ker(u) = (s, e)(Ker(v) \propto E^n) = sKer(v) \propto 0$ (since $Ker(v) \subseteq M^n$, $s \in M$ and $ME = 0$) $\subseteq L_0 \propto 0 = G_0$.

Therefore, $Ker(u)$ is S-finite and so R is S-coherent. □

Now, we can construct non-coherent S-coherent rings.

Example 2.5. Let (A, M) be a local coherent domain which is not a field, E be an (A/M)-vector space with infinite rank, $R := A \propto E$ be the trivial ring extension of A by E, and let S be any multiplicative set of R. Then:

(1) R is S-coherent by Theorem 2.4 since A is S_0-coherent (since A is coherent).
(2) R is not coherent by [17, Theorem 2.6(2)] since E is an (A/M)-vector space with infinite rank.

Example 2.6. Let (A, M) be a non-coherent local domain which is not a field, E be an (A/M)-vector space, $R := A \propto E$ be the trivial ring extension of A by E, and let $S = S_0 \propto \{0\}$, where $S_0 = A - \{0\}$. Then

(1) R is S-coherent by Theorem 2.4 since A is S_0-coherent (by Example 2.1).
(2) R is not coherent by [17, Theorem 2.6(2)] since A is not coherent.

Recall that any coherent ring is 2-coherent and the converse is false (for example $\mathbb{Z} \propto \mathbb{Q}$ is a 2-coherent ring which is not coherent by [17, Theorem 3.1]) (see Figure 1). Hence, we have:

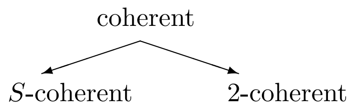

FIGURE 1.

The notions of S-coherent and 2-coherent in Figure 1 are not comparable as the following two examples show:

Example 2.7. Let (A, M) be a local coherent domain such that M is not finitely generated (for instance, take $A = K[[X_1, \dots, X_n, ...]]$ be the power series ring with countably infinite indeterminates $\{X_i, i \in \mathbb{N} - \{0\}\}$ over a field K), $E = A/M$, and $R := A \propto E$ be a trivial ring extension of A by E. Then

(1) R is S-coherent for every multiplicative set S such that $S \nsubseteq U(R)(= (A - M) \propto E)$.
(2) R is not 2-coherent.

Proof. (1) R is S-coherent by Theorem 2.4 since A is coherent and $S \nsubseteq U(R)$.

(2) Let $J = R(m, 0)$, where $m \in M - \{0\}$, and consider the exact sequence of R-modules:

$$0 \to Ker(u) \to R \stackrel{u}{\to} J \to 0,$$

where $u(a, e) = (a, e)(m, 0) = (am, 0)$. Clearly, $Ker(u) = 0 \propto A/M = R(0, \bar{1})$. Now, consider the exact sequence of R-modules:

$$0 \to Ker(v) \to R \stackrel{v}{\to} Ker(u) \to 0,$$

where $v(a, e) = (a, e)(0, \bar{1}) = (0, \bar{a})$. It is clear that $Ker(v) = M \propto E$ which is not a finitely generated ideal of R since M is not a finitely generated ideal of A. Therefore, by the exact sequence of R-modules:

$$0 \to J \to R \to R/J \to 0$$

it is clear that R/J is a 2-presented R-module which is not 3-presented. Hence, R is not 2-coherent. □

Example 2.8. Let A be a coherent integral domain which is not a field, $K = qf(A)$, $R := A \propto K$ be the trivial ring extension of A by K, and let S be any multiplicative set of R. Then

(1) R is 2-coherent by [17, Theorem 3.1(2)] since A is coherent.
(2) R is not S-coherent by Theorem 2.2.

3. Amalgamation duplication of S-coherent property

Let A be a ring, I be an ideal of A, $A \bowtie I$ be the amalgamation duplication of A along I, S be a multiplicative set of $A \bowtie I$ such that $S_0 = \{s \in A \,|\, (s, s + i) \in S \text{ for some } i \in I\}$. For instance, $S = \{(s, s) \in A \bowtie I \,|\, s \in S_0\}$ and

$S = S \bowtie I = \{(s, s+i) \mid i \in I\}$ are multiplicative sets of $A \bowtie I$ for every multiplicative set S_0 of A.

Now, the main result of this section is the following theorem.

Theorem 3.1. *Let A be a ring, I be an ideal of A, $A \bowtie I$ be the amalgamation duplication of A along I, S be a multiplicative set of $A \bowtie I$ and set $S_0 = \{s \in A \mid (s, s+i) \in S$ for some $i \in I\}$ which is a multiplicative set of A. Then*:

(1) *If $A \bowtie I$ is S-coherent, then A is S_0-coherent.*
(2) *Assume that $S = \{(s, s) \in A \bowtie I \mid s \in S_0\}$ and I is an S_0-finite ideal of A. Then $A \bowtie I$ is S-coherent if and only if A is S_0-coherent.*

Before proving Theorem 3.1, we establish the following lemma.

Lemma 3.2. *Under the hypothesis of Theorem 3.1(2), assume that A is S_0-coherent and $I \times 0$ is an S-coherent $(A \bowtie I)$-module. Then, $A \bowtie I$ is S-coherent.*

Proof. Recall that $I \times 0$ is an ideal of $A \bowtie I$ with $\frac{A \bowtie I}{I \times 0} \cong A$ by [7, Remark 1(b)]. But $(I \times 0) \cap S = \varnothing$ and $T := \{(s, s) + (I \times 0) \mid s \in S\} \cong S_0$ which is a multiplicative set of A. Therefore, $A \bowtie I$ is S-coherent by [5, Proposition 3.9 (2)] since A ($\cong \frac{A \bowtie I}{I \times 0}$) is S_0-coherent and $I \times 0$ is an S-coherent $(A \bowtie I)$-module, as desired. □

Proof of Theorem 3.1. (1) Assume that $A \bowtie I$ is S-coherent and let $J_0 := \sum_{i=1}^{n} Aa_i$ be a finitely generated proper ideal of A. Set $J := \sum_{i=1}^{n}(A \bowtie I)(a_i, a_i)$ to be an ideal of $A \bowtie I$ and consider the exact sequence of $(A \bowtie I)$-modules:

$$0 \to Ker(u) \to (A \bowtie I)^n = A^n \bowtie I^n \xrightarrow{u} J \to 0,$$

where $u((b_i, b_i + j_i)_{i=1,\dots,n}) = \sum_{i=1}^{n}(b_i, b_i + j_i)(a_i, a_i) = (\sum_{i=1}^{n} b_i a_i, \sum_{i=1}^{n}(b_i + j_i)a_i)$. On the other hand, consider the exact sequence of A-modules:

$$0 \to Ker(v) \to A^n \xrightarrow{v} J_0 \to 0,$$

where $v((b_i)_{i=1,\dots,n}) = \sum_{i=1}^{n} a_i b_i$. Hence,

$$\begin{aligned} Ker(u) &= \{((b_i, b_i + j_i)_{i=1,\dots,n}) \in A^n \bowtie I^n \mid \sum_{i=1}^{n} b_i a_i = \sum_{i=1}^{n} j_i a_i = 0\} \\ &= Ker(v) \bowtie G_0, \end{aligned}$$

where $G_0 = \{j_i \in I^n \mid \sum_{i=1}^{n} j_i a_i = 0\}$. But J is S-finitely presented since $A \bowtie I$ is S-coherent, that is, $Ker(u)$ is an S-finite $(A \bowtie I)$-module. Then, there exist $(s, s+i) \in S$ and a finitely generated $(A \bowtie I)$-module $L := \sum_{i=1}^{n}(A \bowtie I)(x_i, x_i + f_i)(\subseteq Ker(u))$ for some $(x_i, x_i + f_i) \in L$ and a positive integer m, such that

$$(s, s+i)Ker(u) \subseteq L \subseteq Ker(u).$$

Hence, for $L_0 := \sum_{i=1}^{m} Ax_i$, we have

$$sKer(v) \subseteq L_0 \subseteq Ker(v)$$

and so $Ker(v)$ is S_0-finite, as desired. Hence, A is S_0-coherent.

(2) By (1) it remains to show that if A is S_0-coherent and I is S_0-finite, then $A \bowtie I$ is S-coherent. Hence, it remains to show that $I \times 0$ is an S-coherent $(A \bowtie I)$-module by Lemma 3.2.

Let H be a finitely generated subideal of $I \times 0$ and we will show that H is S-finitely presented. Clearly, $H = \sum_{i=1}^{n}(A \bowtie I)(a_i, 0)$ for some positive integer n and $a_i \in I$. Consider the exact sequence of $(A \bowtie I)$-modules:

$$0 \to Ker(u) \to (A \bowtie I)^n = A^n \bowtie I^n \stackrel{u}{\to} H \to 0,$$

where $u(b_i, b_i + j_i)_{i=1,\dots,n} = \sum_{i=1}^{n}(b_i, b_i + j_i)(a_i, 0) = (\sum_{i=1}^{n} b_i a_i, 0)$. So that $Ker(u) = \{(b_i, b_i + j_i)_{i=1,\dots,n} \in (A \bowtie I)^n \mid \sum_{i=1}^{n} b_i a_i = 0\}$. Now, set $J := \sum_{i=1}^{n} Aa_i$ a finitely generated subideal of I, and consider the exact sequence of A-modules:

$$0 \to Ker(v) \to A^n \stackrel{v}{\to} J \to 0,$$

where $v((b_i)_{i=1,\dots,n}) = \sum_{i=1}^{n} b_i a_i$. So under the $(A \bowtie I)$-module identification $(A \bowtie I)^n = A^n \bowtie I^n$, we have $Ker(u) = Ker(v) \bowtie I^n$. But J is S_0-finitely presented since A is S_0-coherent. Hence, $Ker(v)$ is an S_0-finite A-module. Our aim is to show that $Ker(u)$ is S-finite. Since $Ker(v)$ is S_0-finite, there exist $s \in S_0$ and a finitely generated A-module $\sum_{i=1}^{m} Ae_i (\subseteq Ker(v))$ for some positive integer m and $e_i \in Ker(v)$ such that

$$(*) \qquad sKer(v) \subseteq \sum_{i=1}^{m} Ae_i \subseteq Ker(v).$$

On the other hand, since I^n is S_0-finite (since I is S_0-finite), there exist $s' \in S_0$ and a finitely generated A-module $\sum_{i=1}^{p} Af_i$ for some positive integer p and $f_i \in I^n$ such that

$$(**) \qquad s'I^n \subseteq \sum_{i=1}^{p} Af_i \subseteq I^n.$$

We may assume that $s = s'$ by replacing s and s' by ss'. Therefore, by $(*)$ and $(**)$, we have

$$\begin{aligned}(s,s)(Ker(v) \bowtie I^n) &\subseteq \sum_{i=1}^{m}(A \bowtie I)(e_i, e_i) + \sum_{i=1}^{p}(A \bowtie I)(0, f_i) \\ &\subseteq Ker(v) \bowtie I^n\end{aligned}$$

and so $Ker(u) := Ker(v) \bowtie I^n$ is S-finite. Hence, $I \times 0$ is an S-coherent $(A \bowtie I)$-module which completes the proof of Theorem 3.1. □

Using Theorem 3.1 in the case when S_0 consists of units of A and $S = \{(s, s) \mid s \in S_0\}$, we regain the result [6, Lemma 4.2].

Corollary 3.3. *Let A be a ring, I be an ideal of A, and $A \bowtie I$ be the amalgamation duplication of A along I. Then*

(1) *If $A \bowtie I$ is coherent, then so is A.*

(2) *Assume that I is a finitely generated ideal of A. Then $A \bowtie I$ is coherent if and only if so is A.*

We know that a coherent ring is an S-coherent ring for every multiplicative set. The converse is false as the following example shows.

Example 3.4. Let A be a non-coherent S_0-coherent ring (take for example $A = \mathbb{Z} + X\mathbb{R}[[X]]$ which is not coherent by [13, Theorem 5.2.3] and an S_0-Noetherian ring by Example 2.1, where $S_0 = A - \{0\}$) for some multiplicative set S_0 of A and let I be an S_0-finite ideal of A (take for example A to be an integral domain and $S_0 = A - \{0\}$). Then

(1) $A \bowtie I$ is an S-coherent ring by Theorem 3.1, where $S = \{(s, s) \in A \bowtie I \mid s \in S_0\}$.
(2) $A \bowtie I$ is not a coherent ring by [1, Corollary 2.8(1)] since A is not coherent.

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