Commun. Korean Math. Soc. **38** (2023), No. 3, pp. 755–772
https://doi.org/10.4134/CKMS.c220132
pISSN: 1225-1763 / eISSN: 2234-3024

FRACTIONAL INTEGRATION AND DIFFERENTIATION OF THE (p,q)–EXTENDED MODIFIED BESSEL FUNCTION OF THE SECOND KIND AND INTEGRAL TRANSFORMS

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Abstract. Our aim is to establish certain image formulas of the (p,q)–extended modified Bessel function of the second kind $M_{\nu,p,q}(z)$ by employing the Marichev-Saigo-Maeda fractional calculus (integral and differential) operators including their composition formulas and using certain integral transforms involving (p,q)–extended modified Bessel function of the second kind $M_{\nu,p,q}(z)$. Corresponding assertions for the Saigo's, Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the (p,q)–extended modified Bessel function of the second kind $M_{\nu,p,q}(z)$ and Fox-Wright function ${}_r\Psi_s(z)$.

1. Introduction and preliminaries

The general pair of fractional integral operators so-called Marichev-Saigo-Maeda (M-S-M) involving third Appell's function of two-variables $F_3(\cdot)$ are defined by: (see, for details, [1, 11, 20–22].)

Definition 1. Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$. Then for $\Re(\gamma) > 0$,

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\, t^{-\alpha'} \times F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, \mathrm{d}t, \tag{1.1}$$

and

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1}\, t^{-\alpha}$$

Received May 13, 2022; Revised October 17, 2022; Accepted November 22, 2022.
2020 *Mathematics Subject Classification.* Primary 26A33, 33B20, 33C20; Secondary 26A09, 33B15, 33C05.
Key words and phrases. (p,q)–extended Gauss hypergeometric function, $F_{p,q}(a,b;c;z)$, extended beta function, (p,q)–extended modified Bessel function of the second kind $M_{\nu,p,q}(z)$, fractional calculus operators.

$$\times F_3\left(\alpha,\alpha',\beta,\beta';\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,\mathrm{d}t. \tag{1.2}$$

Here $F_3(\cdot)$ denotes the Appell's hypergeometric function of two variables [25].

Definition 2. Let $\alpha,\alpha',\beta,\beta',\gamma\in\mathbb{C}$ and $x>0$. Then for $\Re(\eta)>0$,

$$\begin{aligned}&\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)\\ =&\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x)\\ =&\left(\frac{d}{dx}\right)^n\left(I_{0+}^{-\alpha',-\alpha,-\beta'+n,-\beta,-\gamma+n}f\right)(x)\quad(n=[Re(\gamma)]+1)\\ =&\frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dx}\right)^n x^{\alpha'}\int_0^x(x-t)^{n-\eta-1}\,t^{\sigma}\\ &\times F_3\left(-\alpha',-\alpha,n-\beta',-\beta;n-\gamma;1-\frac{t}{x},1-\frac{x}{t}\right)f(t)\,\mathrm{d}t,\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}&\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}f\right)(x)\\ =&\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma}f\right)(x)\\ =&\left(-\frac{d}{dx}\right)^n\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta'+n,-\gamma+n}f\right)(x)\quad(n=[Re(\eta)]+1)\\ =&\frac{1}{\Gamma(n-\gamma)}\left(-\frac{d}{dx}\right)^n x^{\alpha'}\int_x^\infty(t-x)^{n-\gamma-1}\,t^{\sigma'}\\ &\times F_3\left(-\alpha',-\alpha,\beta',n-\beta;n-\gamma;1-\frac{x}{t},1-\frac{t}{x}\right)f(t)\,\mathrm{d}t.\end{aligned} \tag{1.4}$$

For various choice of parameters operators (1.1), (1.2), (1.3), and (1.4) involves Saigo, Riemann-Liouville and Erdélyi-Kober fractional calculus operators as particular cases. If we take $\alpha=\alpha+\beta$, $\alpha'=\beta'=0$, $\beta=-\eta$, $\gamma=\alpha$, we immediately obtain *Saigo* fractional integral and differential operators involving the hypergeometric function ${}_2F_1$ [20]:

$$\begin{aligned}&\left(I_{0+}^{\alpha,\beta,\eta}f\right)(x)\\ =&\frac{x^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{t}{x}\right)f(t)\,\mathrm{d}t,\end{aligned} \tag{1.5}$$

$$\begin{aligned}&\left(I_{-}^{\alpha,\beta,\eta}f\right)(x)\\ =&\frac{1}{\Gamma(\alpha)}\int_x^\infty(t-x)^{\alpha-1}t^{-\alpha-\beta}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{x}{t}\right)f(t)\,\mathrm{d}t,\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha,\beta,\eta}f\right)(x) &= \left(I_{0,x}^{-\alpha,-\beta,\alpha+\eta}f\right)(x)\\ &= \left(\frac{d}{dx}\right)^{n}\left(I_{0,x}^{-\alpha+n,-\beta-n,\alpha+\eta-n}f\right)(x) \quad (n=[\Re(\alpha)]+1),\end{aligned} \tag{1.7}$$

$$\begin{aligned}\left(D_{-}^{\alpha,\beta,\eta}f\right)(x) &= \left(I_{x,\infty}^{-\alpha,-\beta,\alpha+\eta}f\right)(x)\\ &= (-1)^{n}\left(\frac{d}{dx}\right)^{n}\left(I_{x,\infty}^{-\alpha+n,-\beta-n,\alpha+\eta}f\right)(x) \quad (n=[\Re(\alpha)]+1).\end{aligned} \tag{1.8}$$

Here and in what follows, $[x]$ denotes the greatest integer less than or equal to the real number x. Setting $\beta=-\alpha$ in (1.5), (1.6), (1.7), and (1.8) yields the familiar *Riemann-Liouville* fractional integrals and derivatives of order $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ and $x\in\mathbb{R}^{+}$ (see, e.g., [8, 9, 23, 26]):

$$\left(I_{0+}^{\alpha,-\alpha,\eta}f\right)(x)=\left(I_{0,x}^{\alpha}f\right)(x)\equiv\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}f(t)\,\mathrm{d}t, \tag{1.9}$$

$$\left(I_{-}^{\alpha,-\alpha,\eta}f\right)(x)=\left(I_{x,\infty}^{\alpha}f\right)(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)\,\mathrm{d}t, \tag{1.10}$$

and

$$\begin{aligned}\left(D_{0+}^{\alpha,-\alpha,\eta}f\right)(x) &= \left(D_{0,x}^{\alpha}f\right)(x)=\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-t)^{n-\alpha-1}f(t)\,\mathrm{d}t\\ &= \left(\frac{d}{dx}\right)^{n}\left(I_{0,x}^{n-\alpha}f\right)(x) \qquad (n=[\Re(\alpha)]+1),\end{aligned} \tag{1.11}$$

$$\begin{aligned}\left(D_{-}^{\alpha,-\alpha,\eta}f\right)(x) &= \left(D_{x,\infty}^{\alpha}f\right)(x)\\ &= (-1)^{n}\left(\frac{d}{dx}\right)^{n}\frac{1}{\Gamma(n-\alpha)}\int_{x}^{\infty}(t-y)^{n-\alpha-1}f(t)\,\mathrm{d}t\\ &= (-1)^{n}\left(\frac{d}{dx}\right)^{n}\left(I_{x,\infty}^{n-\alpha}f\right)(x) \qquad (n=[\Re(\alpha)]+1).\end{aligned} \tag{1.12}$$

Setting $\beta=0$ in (1.5), (1.6), (1.7), and (1.8) yields the so-called *Erdélyi-Kober* fractional integrals and derivatives of order $\alpha\in\mathbb{C}$ with $\Re(\alpha)>0$ and $x\in\mathbb{R}^{+}$ (see, e.g., [8, 9, 23, 26]):

$$\left(I_{0+}^{\alpha,0,\eta}f\right)(x)=\left(I_{\eta,\alpha}^{+}f\right)(x)=\frac{x^{-\alpha-\eta}}{\Gamma(\alpha)}\int_{0}^{x}(x-t)^{\alpha-1}t^{\eta}f(t)\,\mathrm{d}t, \tag{1.13}$$

$$\left(I_{-}^{\alpha,0,\eta}f\right)(x)=\left(K_{\eta,\alpha}^{-}f\right)(x)\equiv\frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)\,\mathrm{d}t, \tag{1.14}$$

and

$$\left(D_{0+}^{\alpha,0,\eta}f\right)(x)=\left(D_{\eta,\alpha}^{+}f\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0,x}^{-\alpha+n,-\alpha,\alpha+\eta-n} f\right)(x) \quad (n = [\Re(\alpha)] + 1), \tag{1.15}$$

$$\left(D_{-}^{\alpha,0,\eta} f\right)(x) = \left(D_{\eta,\alpha}^{-} f\right)(x)$$
$$= (-1)^n \left(\frac{d}{dx}\right)^n \left(I_{x,\infty}^{-\alpha+n,-\alpha,\alpha+\eta} f\right)(x) \quad (n = [\Re(\alpha)] + 1), \tag{1.16}$$

$$\left(D_{\eta,\alpha}^{+} f\right)(x) = x^{-\eta} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)}$$
$$\times \int_0^x t^{\alpha+\eta}(x-t)^{n-\alpha-1} f(t)\, \mathrm{d}t \quad (n = [\Re(\alpha)] + 1), \tag{1.17}$$

$$\left(D_{\eta,\alpha}^{-} f\right)(x) = x^{\eta+\alpha} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)}$$
$$\times \int_x^\infty t^{-\eta}(t-x)^{n-\alpha-1} f(t)\, \mathrm{d}t \quad (n = [\Re(\alpha)] + 1), \tag{1.18}$$

respectively. In recent years, various authors studied and investigated the (p, q)–variant, and in turn, when $p = q$ the p–variant together with the set of related higher transcendental hypergeometric type special functions (see, for details, [2–7, 10, 12, 14, 15, 17–19]). In particular, Parmar and Pogány [16] introduced and studied the (p, q)–extended modified Bessel function of the second kind $M_{\nu,p,q}(z)$ in the form:

$$\mathbf{M}_{\nu,p,q}(x) = -\frac{2\left(\frac{x}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\, \mathrm{e}^{-xt-\frac{p}{t^2}-\frac{q}{1-t^2}}\, \mathrm{d}t\,,$$

with $\min\{p, q\} \geq 0$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$. The routine calculation gives the series representation of the (p, q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$ with $\min\{p, q\} \geq 0$, $|z| < 1$ and $\Re(\nu) > -\frac{1}{2}$ when $p = q = 0$ in the form:

$$\mathbf{M}_{\nu,p,q}(z) = -\frac{\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})} \sum_{n\geq 0} \mathrm{B}\left(\frac{n}{2}+\frac{1}{2},\, \nu+\frac{1}{2}; p, q\right) \frac{(-z)^n}{n!}, \tag{1.19}$$

where $\mathrm{B}(x, y; p, q)$ is the (p, q)–extended Beta function introduced by Choi *et al.* [7]

$$\mathrm{B}(x, y; p, q) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{e}^{-\frac{p}{t}-\frac{q}{1-t}}\, \mathrm{d}t\,, \tag{1.20}$$

when $\min\{\Re(x), \Re(y)\} > 0$; $\min\{\Re(p), \Re(q)\} \geq 0$. They developed and studied its several properties such as integral representations, Mellin transforms, summation formulas, transformation formulas, and so on. The concept of the Hadamard product (or convolution) of two analytic functions is required in our current investigation. It can aid in the decomposition of a newly emerged function into two known functions. If one of the power series, in particular, describes an entire function, then the Hadamard product series also defines an entire function. If we assume

$$g(z) := \sum_{n=0}^{\infty} c_n z^n \ (|z| < R_f) \ \text{and} \ h(z) := \sum_{n=0}^{\infty} d_n z^n \ (|z| < R_g)$$

two given power series and whose radii of convergence are given by R_f and R_g, respectively. Then their Hadamard product (or convolution) is the power series defined by

$$(g * h)(z) := \sum_{n=0}^{\infty} c_n\, d_n z^n = (h * g)(z) \ (|z| < R), \tag{1.21}$$

where

$$R = \lim_{n\to\infty} \left| \frac{c_n\, d_n}{c_{n+1}\, d_{n+1}} \right| = \left(\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| \right) \cdot \left(\lim_{n\to\infty} \left| \frac{d_n}{d_{n+1}} \right| \right) = R_f \cdot R_g,$$

so that, in general, we have $R \geqq R_f \cdot R_g$.

The Fox-Wright function${}_r\Psi_s(z)$ $(r, s \in \mathbb{N}_0)$, which is a generalization of hypergeometric function, is defined as follows: (see, for details, [8, 13]; see also [23, 25].)

$$ {}_r\Psi_s \left[\begin{array}{l} (a_1, A_1), \ldots, (a_s, A_s); \\ (b_1, B_1), \ldots, (b_s, B_s); \end{array} z \right] = \sum_{n=0}^{\infty} \frac{\Gamma(a_1 + A_1 n) \cdots \Gamma(a_r + A_r n)}{\Gamma(b_1 + B_1 n) \cdots \Gamma(b_s + B_s n)} \, \frac{z^n}{n!} \tag{1.22}$$

$$\left(A_\ell \in \mathbb{R}^+ \ (\ell = 1, \ldots, r); \ B_\ell \in \mathbb{R}^+ \ (\ell = 1, \ldots, s); \ 1 + \sum_{\ell=1}^{s} B_\ell - \sum_{\ell=1}^{r} A_\ell \geqq 0 \right),$$

where the equality in the convergence condition holds true for

$$|z| < \nabla := \left(\prod_{\ell=1}^{r} A_\ell^{-A_\ell} \right) \cdot \left(\prod_{\ell=1}^{s} B_\jmath^{B_\ell} \right).$$

The paper contains the certain image formulas of the Marichev-Saigo-Maeda fractional integration and differentiation operators (1.1), (1.2), (1.3) and (1.4) including their composition formulas and using certain integral transforms involving (p,q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$. Corresponding assertions for the Saigo's, Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators are deduced. All the results are represented in terms of the Hadamard product of the (p,q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$ and Fox-Wright function ${}_r\Psi_s(z)$.

2. Fractional integration of the $\mathbf{M}_{\nu,p,q}(z)$

We begin the main results exposition with presenting a composition formulas of generalized fractional integrals (1.1) and (1.2) involving (p,q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$. We prove that such compositions are expressed in terms of the Hadamard product (1.21) of (p,q)–extended modified Bessel function of the second kind (1.19) and Fox-Wright function ${}_r\Psi_s(z)$ (1.22).

Lemma 1. *Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma \in \mathbb{C}$ and $x>0$. Then the following relation exists*

(a) *If $\Re(\gamma)>0$ and $\Re(\sigma)>\max\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re(\alpha'-\beta')\}$, then*

$$\begin{aligned}&\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}\right)(x)\\ &=\frac{\Gamma(\sigma)\Gamma(\sigma+\gamma-\alpha-\alpha'-\beta)\Gamma(\sigma+\beta'-\alpha')}{\Gamma(\sigma+\beta')\Gamma(\sigma+\gamma-\alpha-\alpha')\Gamma(\sigma+\gamma-\alpha'-\beta)}x^{\sigma+\gamma-\alpha-\alpha'-1}.\end{aligned}\tag{2.1}$$

(b) *If $\Re(\gamma)>0$ and $\Re(\sigma)<1+\min\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re(\alpha+\beta'-\gamma)\}$, then*

$$\begin{aligned}&\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\sigma-1}\right)(x)\\ &=\frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma-\gamma+\alpha+\alpha')\Gamma(1-\sigma-\gamma+\alpha+\beta')}{\Gamma(1-\sigma)\Gamma(1-\sigma-\gamma+\alpha+\alpha'+\beta')\Gamma(1-\sigma+\alpha-\beta)}x^{\sigma+\gamma-\alpha-\alpha'-1}.\end{aligned}\tag{2.2}$$

Lemma 2. *Let α, β, $\eta \in \mathbb{C}$.*

(a) *If $\Re(\alpha)>0$ and $\Re(\sigma)>\max\{0,\Re(\beta-\eta)\}$, then*

$$(I_{0+}^{\alpha,\beta,\eta}t^{\sigma-1})(x)=\frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}x^{\sigma-\beta-1}.\tag{2.3}$$

In particular, for $x>0$, we have

$$(I_{0+}^{\alpha}t^{\sigma-1})(x)=\frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)}x^{\sigma+\alpha-1}\quad(\min\{\Re(\alpha),\ \Re(\sigma)\}>0),\tag{2.4}$$

and

$$(I_{\eta,\alpha}^{+}t^{\sigma-1})(x)=\frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)}x^{\sigma-1}\quad(\Re(\alpha)>0,\ \Re(\sigma)>-\Re(\eta)).\tag{2.5}$$

(b) *If $\Re(\alpha)>0$ and $\Re(\sigma)<1+\min\{\Re(\beta),\Re(\eta)\}$, then*

$$(I_{-}^{\alpha,\beta,\eta}t^{\sigma-1})(x)=\frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)}x^{\sigma-\beta-1}.\tag{2.6}$$

In particular, for $x>0$, we have

$$(I_{-}^{\alpha}t^{\sigma-1})(x)=\frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)}x^{\sigma+\alpha-1}\quad(0<\Re(\alpha)<1-\Re(\sigma)),\tag{2.7}$$

and

$$(K_{\eta,\alpha}^{-}t^{\sigma-1})(x)=\frac{\Gamma(\eta-\sigma+1)}{\Gamma(\alpha+\eta-\sigma+1)}x^{\sigma-1}\quad(\Re(\sigma)<1+\Re(\sigma)).\tag{2.8}$$

Theorem 1. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0, \Re(\alpha+\alpha'+b-c), \Re(\alpha'-b')]$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned fractional integration formula holds true*:

$$\begin{aligned}&\left(I_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\ &= x^{\varrho+c-\alpha-\alpha'-1}\mathbf{M}_{\nu,p,q}(x^{\gamma})\\ &* {}_4\Psi_3\left[\begin{matrix}(1,1),(\varrho+\gamma\nu,\gamma),(\varrho+\gamma\nu+c-\alpha-\alpha'-b\gamma),(\varrho+b'-\alpha'+\gamma\nu,\gamma);\\ (\varrho+\gamma\nu+b',\gamma),(\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma)(\varrho+c-\alpha'-b+\gamma\nu,\gamma);\end{matrix}\ (-x^{\gamma})\right].\end{aligned} \tag{2.9}$$

Proof. Applying definition (1.19), using (1.1) and (2.1) and changing the orders of integration and summation, we find for $x > 0$

$$\begin{aligned}&\left(I_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\ &= -\frac{\left(\frac{t^{\gamma}}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\frac{(-t^{\gamma})^k}{k!}\ \left(I_{0+}^{\alpha,\alpha',b,b',c}t^{\varrho+\gamma\nu+\gamma k-1}\right)(x)\\ &= -x^{\varrho+c-\alpha-\alpha'-1}\ \frac{\left(\frac{x^{\gamma}}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\\ &\times\frac{\Gamma(1+k)\Gamma(\varrho+\gamma\nu+\gamma k)\Gamma(\varrho+\gamma\nu+c+\alpha-\alpha'-b+\gamma k)\Gamma(\varrho+b'-\alpha'+\gamma\nu+\gamma k)}{\Gamma(\varrho+b'+\gamma\nu+\gamma k)\Gamma(\varrho+c-\alpha-\alpha'+\gamma\nu+\gamma k)\Gamma(\varrho+c--\alpha'-b+\gamma\nu+\gamma k)\,k!}\frac{(-x^{\gamma})^k}{k!}.\end{aligned} \tag{2.10}$$

By applying the Hadamard product (1.21) in (2.10), which in view of (1.19) and (1.22), yields the desired formula (2.9). ☐

Theorem 2. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) < 1 + \min\{\Re(-b), \Re(\alpha+\alpha'-c), \Re(\alpha+b'-c)\}$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned fractional integration holds true*:

$$\begin{aligned}&\left(I_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)\\ &= x^{\varrho+c-\alpha-\alpha'-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right)\\ &* {}_4\Psi_3\left[\begin{matrix}(1,1),(1-\varrho-b+\gamma\nu,\gamma),(1-\varrho-c+\alpha+\alpha'+\gamma\nu,\gamma),(1-\varrho-c+\alpha+b'+\gamma\nu,\gamma);\\ (1-\varrho+\gamma\nu,\gamma),(1-\varrho-c+\alpha+\alpha'+b'+\gamma\nu,\gamma)(1-\varrho+\alpha-b+\gamma\nu,\gamma);\end{matrix}\ -\frac{1}{x^{\gamma}}\right].\end{aligned} \tag{2.11}$$

Proof. Applying definition (1.19), using (1.6) and (2.2) and changing the orders of integration and summation, we find for $x > 0$

$$\begin{aligned}&\left(I_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)\\ &= -\frac{\left(\frac{1}{2t^{\gamma}}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\frac{\left(\frac{-1}{t^{\gamma}}\right)^k}{k!}\ \left(I_{-}^{\alpha,\alpha',b,b',c}t^{\varrho-\gamma\nu-\gamma k-1}\right)(x)\\ &= -x^{\varrho+c-\alpha-\alpha'-1}\frac{\left(\frac{1}{2x^{\gamma}}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\\ &\times\frac{\Gamma(1+k)\Gamma(1-\varrho-b+\gamma\nu+\gamma k)\Gamma(1-\varrho-c+\gamma\nu+c+\alpha+\alpha'-b+\gamma k)}{\Gamma(1-\varrho+\gamma\nu+\gamma k)\Gamma(1-\varrho-c+\alpha+\alpha'+b'+\gamma\nu+\gamma k)}\end{aligned}$$

$$\times\frac{\Gamma(1-\varrho-c+b'+\alpha+\gamma\nu+\gamma k)}{\Gamma(1-\varrho-\alpha-b+\gamma\nu+\gamma k)\,k!}\frac{\left(\frac{-1}{x^{\gamma}}\right)^{k}}{k!}. \tag{2.12}$$

By applying the Hadamard product (1.21) in (2.12), which in view of (1.19) and (1.22), yields the desired formula (2.11). □

Next by specializing the parameters in Theorem 1 and Theorem 2, we immediately obtain the results for the *Saigo's*, Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral operators given by below corollaries.

Corollary 2.1. *Let* $\alpha, b, c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(\alpha) > 0$ *and* $\Re(\varrho) > \max[0, \Re(b-c)]$ *and* $\min\{\Re(p), \Re(q)\} \geq 0$. *Then the below mentioned Saigo hypergeometric fractional integral* $I_{0,+}^{\alpha,b,c}$ *of* $\mathbf{M}_{\nu,p,q}(t^{\gamma})$ *holds true*:

$$\begin{aligned}&\left(I_{0+}^{\alpha,b,c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\ &= x^{\varrho-b-1}\mathbf{M}_{\nu,p,q}(x^{\gamma}) * {}_3\Psi_2\left[\begin{array}{c}(1,1),(\varrho+\gamma\nu,\gamma),(\varrho+c--b+\gamma\nu,\gamma);\\ (\varrho-b+\gamma\nu,\gamma),(\varrho+c+\alpha+\gamma\nu,\gamma);\end{array}(-x^{\gamma})\right].\end{aligned} \tag{2.13}$$

Corollary 2.2. *Let* $\alpha, b, c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(\alpha) > 0$, $\Re(\varrho) < 1 + \min\{\Re(b), \Re(c)\}$ *and* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned Saigo hypergeometric fractional integral* $I_{-}^{\alpha,b,c}$ *of* $\mathbf{M}_{\nu,p,q}(\frac{1}{t^{\gamma}})$ *holds true*:

$$\begin{aligned}&\left(I_{-}^{\alpha,b,c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)\\ &= x^{\varrho-b-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right)\\ &\quad * {}_3\Psi_2\left[\begin{array}{c}(1,1),(1-\varrho+b+\gamma\nu,\gamma),(1-\varrho+c+\gamma\nu,\gamma);\\ (1-\varrho+\gamma\nu,\gamma),(1-\varrho-c+\alpha+b+c+\gamma\nu,\gamma);\end{array} -\frac{1}{x^{\gamma}}\right].\end{aligned} \tag{2.14}$$

Corollary 2.3. *Let* $\alpha, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(\alpha) > 0$ *and* $\Re(\varrho) > 0$ *and* $\min\{\Re(p), \Re(q)\} \geq 0$. *Then the below mentioned Riemann-Liouville fractional integral* $I_{0,+}^{\alpha}$ *of* $\mathbf{M}_{\nu,p,q}(t^{\gamma})$ *holds true*:

$$\begin{aligned}&\left(I_{0+}^{\alpha}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\ &= x^{\varrho+\alpha-1}\mathbf{M}_{\nu,p,q}(x^{\gamma}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\varrho+\gamma\nu,\gamma);\\ (\varrho+\gamma\nu+\alpha,\gamma);\end{array}(-x^{\gamma})\right].\end{aligned} \tag{2.15}$$

Corollary 2.4. *Let* $\alpha, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $0 < \Re(\alpha) < 1 - \Re(\varrho)$ *and* $\Re(\varrho) > 0$ *and* $\min\{\Re(p), \Re(q)\} \geq 0$. *Then the below mentioned Riemann-Liouville fractional integral* I_{-}^{α} *of* $\mathbf{M}_{\nu,p,q}(\frac{1}{t^{\gamma}})$ *holds true*:

$$\begin{aligned}&\left(I_{-}^{\alpha}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)\\ &= x^{\varrho+\alpha-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1-\alpha-\varrho+\gamma\nu,\gamma);\\ (1-\varrho+\gamma\nu,\gamma);\end{array} -\frac{1}{x^{\gamma}}\right].\end{aligned} \tag{2.16}$$

Corollary 2.5. *Let $\alpha, c, \varrho, p, q \in \mathbb{C}$ and $\Re(\nu) > -\frac{1}{2}$ with $\gamma \in \mathbb{R}^+$ such that $\Re(\alpha) > 0$ and $\Re(\varrho) > -\Re(c)$ and $\min\{\Re(p), \Re(q)\} \geq 0$. Then the below mentioned Erdélyi-Kober fractional integral $I^+_{c,\alpha}$ of $\mathbf{M}_{\nu,p,q}(t^\gamma)$ holds true:*

$$\begin{aligned}&\left(I^+_{c,\alpha}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^\gamma)\right\}\right)(x)\\ &= x^{\varrho-1}\mathbf{M}_{\nu,p,q}(x^\gamma) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\varrho+c+\gamma\nu,\gamma);\\ (\varrho+\alpha+c+\gamma\nu,\gamma);\end{array}(-x^\gamma)\right].\end{aligned} \tag{2.17}$$

Corollary 2.6. *Let $\alpha, c, \varrho, p, q \in \mathbb{C}$ and $\Re(\nu) > -\frac{1}{2}$ with $\gamma \in \mathbb{R}^+$ such that $\Re(\alpha) > 0$ and $\Re(\varrho) > 1 + \Re(c)$ and $\min\{\Re(p), \Re(q)\} \geq 0$. Then the below mentioned Erdélyi-Kober fractional integral $K^-_{c,\alpha}$ of $\mathbf{M}_{\nu,p,q}(\frac{1}{t^\gamma})$ holds true:*

$$\begin{aligned}&\left(K^-_{c,\alpha}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^\gamma}\right)\right\}\right)(x)\\ &= x^{\varrho+\alpha-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^\gamma}\right) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1-\varrho+c+\gamma\nu,\gamma);\\ (1-\varrho+\alpha+c+\gamma\nu,\gamma);\end{array} -\frac{1}{x^\gamma}\right].\end{aligned} \tag{2.18}$$

3. Fractional differentiation of the $\mathbf{M}_{\nu,p,q}(z)$

In this section, we obtain a composition formulas of generalized fractional differentiation (1.3) and (1.4) involving (p,q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$. We prove that such compositions are expressed in terms of the Hadamard product (1.21) of (p,q)–extended modified Bessel function of second kind (1.19) and Fox-Wright function ${}_p\Psi_q(z)$ (1.22).

Lemma 3. *Let $\alpha, \alpha', \beta, \beta', \gamma, \sigma \in \mathbb{C}$ and $x > 0$. Then the following relation exists:*

(a) *If $\Re(\gamma) > 0$ and $\Re(\sigma) > \max\{0, \Re(\gamma-\alpha-\alpha'+\beta'), \Re(\beta-\alpha)\}$, then*

$$\begin{aligned}&\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{0+}t^{\sigma-1}\right)(x)\\ &= \frac{\Gamma(\sigma)\Gamma(\sigma-\gamma+\alpha+\alpha'+\beta')\Gamma(\sigma-\beta+\alpha)}{\Gamma(\sigma-\beta)\Gamma(\sigma-\gamma+\alpha+\alpha')\Gamma(\sigma-\gamma+\alpha+\beta')}x^{\sigma-\gamma+\alpha+\alpha'-1}.\end{aligned} \tag{3.1}$$

(b) *If $\Re(\gamma) > 0$ and $\Re(\sigma) < 1 + \min\{\Re(\beta'), \Re(\gamma-\alpha-\alpha', \Re(\eta-\alpha'-\beta)\}$, then*

$$\begin{aligned}&\left(D^{\alpha,\alpha',\beta,\beta',\gamma}_{-}t^{\sigma-1}\right)(x)\\ &= \frac{\Gamma(1-\sigma-\beta')\Gamma(1-\sigma+\gamma-\alpha-\alpha')\Gamma(1-\sigma+\gamma-\alpha'-\beta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\gamma-\alpha-\alpha'-\nu)\Gamma(1-\sigma-\alpha'-\beta')}x^{\sigma-\gamma+\alpha+\alpha'-1}.\end{aligned} \tag{3.2}$$

Lemma 4. *Let α, β, $\eta \in \mathbb{C}$. Then*

(a) *If $\Re(\alpha) > 0$ and $\Re(\sigma) > -\min\{0, \Re(\alpha+\beta+\eta)\}$, then*

$$(D^{\alpha,\beta,\eta}_{0+}t^{\sigma-1})(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\alpha+\beta+\eta)}{\Gamma(\sigma+\beta)\Gamma(\sigma+\eta)}x^{\sigma+\beta-1}. \tag{3.3}$$

In particular, for $x > 0$, we have

$$(D^{\alpha}_{0+}t^{\sigma-1})(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma-\alpha)}x^{\sigma-\alpha-1} \quad (\Re(\alpha) > 0,\ \Re(\sigma) > 0), \tag{3.4}$$

and

$$(D^{+}_{\eta,\alpha}t^{\sigma-1})(x)=\frac{\Gamma(\sigma+\alpha+\eta)}{\Gamma(\sigma+\eta)}x^{\sigma-1}\ (\Re(\alpha)>0,\ \Re(\sigma)>-\Re(\alpha+\eta)). \tag{3.5}$$

(b) *If* $\Re(\alpha)>0,\Re(\sigma)<1+\min\{\Re(-\beta-n),\Re(\alpha+\eta)\}$ *and* $n=[\Re(\alpha)]+1$, *then*

$$(D^{\alpha,\beta,\eta}_{-}t^{\sigma-1})(x)=\frac{\Gamma(1-\sigma-\beta)\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\eta-\beta)}x^{\sigma+\beta-1}. \tag{3.6}$$

In particular, for $x>0$, *we have*

$$\begin{aligned}&(D^{\alpha}_{-}t^{\sigma-1})(x)\\&=\frac{\Gamma(1-\sigma+\alpha)}{\Gamma(1-\sigma)}x^{\sigma-\alpha-1}\quad(\Re(\alpha)>0,\ \Re(\sigma)<1+\Re(\alpha)-n),\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}&(D^{-}_{\eta,\alpha}t^{\sigma-1})(x)\\&=\frac{\Gamma(1-\sigma+\alpha+\eta)}{\Gamma(1-\sigma-\eta)}x^{\sigma-1}\quad(\Re(\alpha)>0,\ \Re(\sigma)<1+\Re(\alpha+\eta)-n).\end{aligned} \tag{3.8}$$

Theorem 3. *Let* $\alpha,\alpha',b,b',c,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(c)>0$ *and* $\Re(\varrho)>\max[0,\Re(c-\alpha+\alpha'-b'),\Re(b-\alpha)]$ *with* $\min\{\Re(p),\Re(q)\}\geq 0$ *and* $|t|<1$. *Then the below mentioned fractional differentiation formula holds true*:

$$\begin{aligned}&\left(D^{\alpha,\alpha',b,b',c}_{0+}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\&=x^{\varrho+c-\alpha-\alpha'-1}\mathbf{M}_{\nu,p,q}(x^{\gamma})\\&*\ {}_4\Psi_3\left[\begin{matrix}(1,1),(\varrho+\gamma\nu,\gamma),(\varrho+\gamma\nu-c-\alpha+\alpha+\alpha'-b',\gamma),(\varrho-b-\alpha+\gamma\nu,\gamma);\\(\varrho-b,\gamma),(\varrho-c-\alpha-\alpha'+\gamma\nu,\gamma)(\varrho-c-\alpha+b'+\gamma\nu,\gamma);\end{matrix}\ (-x^{\gamma})\right].\end{aligned} \tag{3.9}$$

Proof. By virtue of the formulas (1.3) and (1.19), the term-by-term fractional differentiation and the application of the relation (3.3), yields for $x>0$

$$\begin{aligned}&\left(D^{\alpha,\alpha',b,b',c}_{0+}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\&=-\frac{\left(\frac{t^{\gamma}}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\nu+\frac{1}{2};p,q\right)\frac{(-t^{\gamma})^{k}}{k!}\ \left(D^{\alpha,\alpha',b,b',c}_{0+}t^{\varrho+\gamma\nu+\gamma k-1}\right)(x)\\&=-x^{\varrho+c-\alpha-\alpha'-1}\ \frac{\left(\frac{x^{\gamma}}{2}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\nu+\frac{1}{2};p,q\right)\\&\times\frac{\Gamma(1+k)\Gamma(\varrho+\gamma\nu+\gamma k)\Gamma(\varrho+\gamma\nu-c+\alpha+\alpha'+b'+\gamma k)\Gamma(\varrho-b+\alpha+\gamma\nu+\gamma k)}{\Gamma(\varrho-b+\gamma k)\Gamma(\varrho-c+\alpha+\alpha'+\gamma\nu+\gamma k)\Gamma(\varrho-c+\alpha+b'+\gamma\nu+\gamma k)\,k!}\frac{(-x^{\gamma})^{k}}{k!}.\end{aligned} \tag{3.10}$$

By applying the Hadamard product (1.21) in (3.10), which in view of (1.19) and (1.22), yields the desired formula (3.9). □

Theorem 4. *Let* $\alpha,\alpha',b,b',c,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(c)>0$ *and* $\Re(\varrho)<1+\min\{\Re(b'),\Re(c-\alpha-\alpha'),\Re(c-\alpha'-b)\}$ *with*

$\min\{\Re(p),\Re(q)\}\geq 0$ *and* $|\frac{1}{t}|<1$. *Then the below mentioned fractional differentiation holds true*:

$$\begin{aligned}&\left(D_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)\\&=x^{\varrho-c+\alpha+\alpha'-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right)\\&\quad*\ {}_4\Psi_3\left[\begin{matrix}(1,1),(1-\varrho-b'+\gamma\nu,\gamma),(1-\varrho-c-\alpha-\alpha'+\gamma\nu,\gamma),(1-\varrho+c-\alpha'-b+\gamma\nu,\gamma);\\(1-\varrho+\gamma\nu,\gamma),(1-\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma)(1-\varrho-\alpha'-b_1'+\gamma\nu,\gamma);\end{matrix}\ -\frac{1}{x^{\gamma}}\right].\end{aligned}\tag{3.11}$$

Proof. By virtue of the formulas (1.8) and (1.19), the term-by-term fractional differentiation and the application of the relation (3.6), yields for $x>0$

$$\begin{aligned}&\left(D_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)\\&=-\frac{\left(\frac{1}{2t^{\gamma}}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\frac{\left(\frac{-1}{t^{\gamma}}\right)^{k}}{k!}\ \left(D_{-}^{\alpha,\alpha',b,b',c}t^{\varrho-\gamma\nu-\gamma k-1}\right)(x)\\&=-x^{\varrho+c+\alpha+\alpha'-1}\frac{\left(\frac{1}{2x^{\gamma}}\right)^{\nu}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\sum_{k\geq 0}\mathrm{B}\left(\frac{k}{2}+\frac{1}{2},\,\nu+\frac{1}{2};p,q\right)\\&\quad\times\frac{\Gamma(1+k)\Gamma(1-\varrho-b'+\gamma\nu+\gamma k)\Gamma(1-\varrho+c-\alpha-\alpha'+\gamma\nu+\gamma k)}{\Gamma(1-\varrho+\gamma\nu+\gamma k)\Gamma(1-\varrho+c-\alpha-\alpha'-b+\gamma\nu+\gamma k)}\\&\quad\times\frac{\Gamma(1-\varrho+c-b-\alpha'+\gamma\nu+\gamma k)}{\Gamma(1-\varrho-\alpha'-b_1'+\gamma\nu+\gamma k)\,k!}\frac{\left(\frac{-1}{x^{\gamma}}\right)^{k}}{k!}.\end{aligned}\tag{3.12}$$

By applying the Hadamard product (1.21) in (3.12), which in view of (1.19) and (1.22), yields the desired formula (3.11). □

Next by specializing the parameters in Theorem 3 and Theorem 4, we immediately obtain the results for the *Saigo's*, Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional differential operators given by below corollaries.

Corollary 4.1. *Let* $\alpha,b,c,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(\alpha)\geq 0$ *and* $\Re(\varrho)>-\min[0,\Re(a+b+c)]$ *and* $\min\{\Re(p),\Re(q)\}\geq 0$. *Then the below mentioned Saigo hypergeometric fractional derivative* $D_{0,+}^{\alpha,b,c}$ *of* $\mathbf{M}_{\nu,p,q}(t^{\gamma})$ *holds true*:

$$\begin{aligned}&\left(D_{0+}^{\alpha,b,c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)\\&=x^{\varrho+b-1}\mathbf{M}_{\nu,p,q}(x^{\gamma})\\&\quad*\ {}_3\Psi_2\left[\begin{matrix}(1,1),(\varrho+\gamma\nu,\gamma),(\varrho+\alpha+c+b+\gamma\nu,\gamma);\\(\varrho+b+\gamma\nu,\gamma),(\varrho+c+\gamma\nu,\gamma);\end{matrix}\ (-x^{\gamma})\right].\end{aligned}\tag{3.13}$$

Corollary 4.2. *Let* $\alpha,b,c,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(\alpha)\geq 0$, $\Re(\varrho)<1+\min\{\Re(-b-n),\Re(\alpha+c),n=[\Re(\alpha)]+1\}$ *and* $\min\{\Re(p),\Re(q)\}\geq 0$. *Then the below mentioned Saigo hypergeometric fractional derivative* $D_{-}^{\alpha,b,c}$ *of* $\mathbf{M}_{\nu,p,q}(\frac{1}{t^{\gamma}})$ *holds true*:

$$\left(D_{-}^{\alpha,b,c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)$$

$$= x^{\varrho+b-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right)$$

$$* \ {}_3\Psi_2\left[\begin{array}{c}(1,1),(1-\varrho-b+\gamma\nu,\gamma),(1-\varrho+c+\alpha+\gamma\nu,\gamma);\\ (1-\varrho+\gamma\nu,\gamma),(1-\varrho-b+c+\gamma\nu,\gamma);\end{array} -\frac{1}{x^{\gamma}}\right]. \tag{3.14}$$

Corollary 4.3. *Let* $\alpha,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(\alpha)\geq 0$ *and* $\Re(\varrho)>0$ *and* $\min\{\Re(p),\Re(q)\}\geq 0$. *Then the below mentioned Riemann-Liouville fractional differentiation* $D^{\alpha}_{0,+}$ *of* $\mathbf{M}_{\nu,p,q}(t^{\gamma})$ *holds true*:

$$\left(D^{\alpha}_{0+}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)$$

$$= x^{\varrho-\alpha-1}\mathbf{M}_{\nu,p,q}(x^{\gamma}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\varrho+\gamma\nu,\gamma);\\ (\varrho+\gamma\nu-\alpha,\gamma);\end{array} (-x^{\gamma})\right]. \tag{3.15}$$

Corollary 4.4. *Let* $\alpha,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(\alpha)\geq 0,\Re(\varrho)<\Re(\alpha)-[Re(\alpha)]$ *and* $\min\{\Re(p),\Re(q)\}\geq 0$. *Then the below mentioned Riemann-Liouville fractional integral* D^{α}_{-} *of* $\mathbf{M}_{\nu,p,q}(\frac{1}{t^{\gamma}})$ *holds true*:

$$\left(D^{\alpha}_{-}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)$$

$$= x^{\varrho-\alpha-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1+\alpha-\varrho+\gamma\nu,\gamma);\\ (1-\varrho+\gamma\nu,\gamma);\end{array} -\frac{1}{x^{\gamma}}\right]. \tag{3.16}$$

Corollary 4.5. *Let* $\alpha,c,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(\alpha)\geq 0$ *and* $\Re(\varrho)>-\Re(\alpha+c)$ *and* $\min\{\Re(p),\Re(q)\}\geq 0$. *Then the below mentioned Erdélyi-Kober fractional integral* $D^{+}_{c,\alpha}$ *of* $\mathbf{M}_{\nu,p,q}(t^{\gamma})$ *holds true*:

$$\left(D^{+}_{c,\alpha}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}(t^{\gamma})\right\}\right)(x)$$

$$= x^{\varrho-1}\mathbf{M}_{\nu,p,q}(x^{\gamma}) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(\varrho+c+\alpha+\gamma\nu,\gamma);\\ (\varrho+c+\gamma\nu,\gamma);\end{array} (-x^{\gamma})\right]. \tag{3.17}$$

Corollary 4.6. *Let* $\alpha,c,\varrho,p,q\in\mathbb{C}$ *and* $\Re(\nu)>-\frac{1}{2}$ *with* $\gamma\in\mathbb{R}^{+}$ *such that* $\Re(\alpha)\geq 0$ *and* $\Re(\varrho)<\Re(\alpha+c)-[\Re(\alpha)]$ *and* $\min\{\Re(p),\Re(q)\}\geq 0$. *Then the below mentioned Erdélyi-Kober fractional differentiation* $D^{-}_{c,\alpha}$ *of* $\mathbf{M}_{\nu,p,q}(\frac{1}{t^{\gamma}})$ *holds true*:

$$\left(D^{-}_{c,\alpha}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{t^{\gamma}}\right)\right\}\right)(x)$$

$$= x^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^{\gamma}}\right) * {}_2\Psi_1\left[\begin{array}{c}(1,1),(1-\varrho+c+\alpha+\gamma\nu,\gamma);\\ (1-\varrho+c+\gamma\nu,\gamma);\end{array} -\frac{1}{x^{\gamma}}\right]. \tag{3.18}$$

4. Certain integral transforms of the $\mathbf{M}_{\nu,p,q}(z)$

Here, in this section, first we give definition of the certain integral transforms as Euler-Beta, Laplace and Whittaker transforms. Then we apply to the composition formulas for generalized fractional integrals and differential operators.

Definition 3. The Euler-Beta transformation [24] of the function $f(z)$ is defined, as usual, by

$$\mathrm{B}(f(z);a,b)=\int_0^1 z^{\alpha-1}(1-z)^{b-1}f(z)\mathrm{d}t\,. \tag{4.1}$$

Definition 4. The Laplace transformation (see, e.g., [24]) of the function $f(z)$ is defined, as usual, by

$$\mathrm{L}(f(z);t) = \int_0^\infty \mathrm{e}^{-tz} f(z)\mathrm{d}z \quad (\Re(t) > 0). \tag{4.2}$$

The below integral containing Whittaker function $W_{\kappa,\nu}$

$$\begin{aligned} &\int_0^\infty t^{x-1}\mathrm{e}^{-\frac{1}{2}at} W_{\kappa,\nu}(at)\mathrm{d}z \\ &= a^{-\rho}\frac{\Gamma(\frac{1}{2}\pm\nu+\rho)}{\Gamma(1-\kappa+\rho)} \quad (\Re(\alpha) > 0, \Re(\rho\pm\nu) > -\frac{1}{2}) \end{aligned} \tag{4.3}$$

is well known.

Now we establish certain theorems involving the results obtained in previous section associated with the integral transforms such as Euler-Beta transform, Laplace transform and Whittaker transform. The results are simply based on the definitions of Euler-Beta, Laplace and Whittaker transforms given by (4.1), (4.2) and (4.3), respectively. So, we omit the proofs of the theorems in this section.

Theorem 5. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0, \Re(\alpha+\alpha'+b-c), \Re(\alpha'-b')]$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned Euler-Beta transformation holds true*:

$$\begin{aligned} &(\mathrm{B}\left(I_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((tz)^\gamma)\right\}\right)(x) : l, m) \\ &= x^{\varrho+c-\alpha-\alpha'-1}\Gamma(m)\mathbf{M}_{\nu,p,q}(x^\gamma) \\ &\quad * {}_5\Psi_4\left[\begin{matrix} (1,1), (l+\gamma\nu,\gamma), (\varrho+\gamma\nu,\gamma), (\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma)(\varrho+b'-\alpha'+\gamma\nu,\gamma); \\ (l+m+\gamma\nu,\gamma), (\varrho+b'+\gamma\nu,\gamma)(\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma)(\varrho+c-\alpha'-b+\gamma\nu,\gamma); \end{matrix} (-x^\gamma)\right]. \end{aligned} \tag{4.4}$$

Theorem 6. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) < 1 + \min\{\Re(-b), \Re(\alpha+\alpha'-c), \Re(\alpha+b'-c)\}$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned Euler-Beta transformation holds true*:

$$\begin{aligned} &(\mathrm{B}\left(I_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{z}{t}\right)^\gamma\right\}\right)(x) : l, m) \\ &= x^{\varrho+c-\alpha-\alpha'-1}\Gamma(m)\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^\gamma}\right) \\ &\quad * {}_5\Psi_4\left[\begin{matrix} (1,1), (l+\gamma\nu,\gamma), (1-\varrho-b+\gamma\nu,\gamma), (1-\varrho-c+\alpha+\alpha'+\gamma\nu,\gamma), \\ (1-\varrho-c+\alpha+b'+\gamma\nu,\gamma); \\ (l+m+\gamma\nu,\gamma), (1-\varrho+\gamma\nu,\gamma)(1-\varrho-c+\alpha+\alpha'+b'+\gamma\nu,\gamma)(1-\varrho+\alpha-b+\gamma\nu,\gamma); \end{matrix} -\frac{1}{x^\gamma}\right]. \end{aligned} \tag{4.5}$$

Theorem 7. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0, \Re(c-\alpha-\alpha'-b'), \Re(b-\alpha)]$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned Euler–Beta transformation holds true*:

$$\begin{aligned} &(\mathrm{B}\left(D_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((tz)^\gamma)\right\}\right)(x) : l, m) \\ &= x^{\varrho-c-\alpha-\alpha'-1}\Gamma(m)\mathbf{M}_{\nu,p,q}(x^\gamma) \end{aligned}$$

$$(4.6) \quad * \, {}_5\Psi_4\left[\begin{matrix} (1,1),(l+\gamma\nu,\gamma),(\varrho+\gamma\nu,\gamma),(\varrho+c+\alpha+\alpha'+b'+\gamma\nu,\gamma),(\varrho-b+\alpha+\gamma\nu,\gamma); \\ (l+m+\gamma\nu,\gamma),(\varrho-b+\gamma\nu,\gamma)(\varrho-c+\alpha+\alpha'+\gamma\nu,\gamma)(\varrho-c+\alpha+b'+\gamma\nu,\gamma); \end{matrix} (-x^\gamma)\right].$$

Theorem 8. *Let* $\alpha,\alpha',b,b',c,\varrho,p,q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) < 1 + \min\{\Re(b'),\Re(c-\alpha-\alpha'),\Re(c-\alpha'-b)\}$ *with* $\min\{\Re(p),\Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned Euler-Beta transformation holds true*:

$$(\mathrm{B}\left(D_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{z}{t}\right)^\gamma\right\}\right)(x):l,m)$$
$$= x^{\varrho-c+\alpha+\alpha'-1}\Gamma(m)\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{x^\gamma}\right)$$
$$(4.7) \quad * \, {}_5\Psi_4\left[\begin{matrix} (1,1),(l+\gamma\nu,\gamma),(1-\varrho-b'+\gamma\nu,\gamma),(1-\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma), \\ (1-\varrho+c-\alpha'-b+\gamma\nu,\gamma); \\ (l+m+\gamma\nu,\gamma),(1-\varrho+\gamma\nu,\gamma)(1-\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma)(1-\varrho-\alpha'-b'+\gamma\nu,\gamma); \end{matrix} -\frac{1}{x^\gamma}\right].$$

Theorem 9. *Let* $\alpha,\alpha',b,b',c,\varrho,p,q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0,\Re(\alpha+\alpha'+b-c),\Re(\alpha'-b')]$ *with* $\min\{\Re(p),\Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned Laplace-transformation holds true*:

$$\mathrm{L}\left(z^{l-1}(I_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((tz)^\gamma)\right\})\right)(x)$$
$$= \frac{x^{\varrho+c-\alpha-\alpha'-1}}{s^{l+\gamma\nu}}\mathbf{M}_{\nu,p,q}\left(\tfrac{x}{s}\right)^\gamma$$
$$(4.8) \quad * \, {}_5\Psi_3\left[\begin{matrix} (1,1),(l+\gamma\nu,\gamma),(\varrho+\gamma\nu,\gamma),(\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma) \\ (\varrho+b'+\alpha'+\gamma\nu,\gamma); \\ (\varrho+b'+\gamma\nu,\gamma),(\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma)(\varrho+c-\alpha'-b+\gamma\nu,\gamma); \end{matrix} \left(-\frac{x}{s}\right)^\gamma\right].$$

Theorem 10. *Let* $\alpha,\alpha',b,b',c,\varrho,p,q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) < 1 + \min\{\Re(-b),\Re(\alpha+\alpha'-c),\Re(\alpha+b'-c)\}$ *with* $\min\{\Re(p),\Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned Laplace-transformation holds true*:

$$\mathrm{L}\left(z^{l-1}(I_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((\tfrac{z}{t})^\gamma)\right\})\right)(x)$$
$$= \frac{x^{\varrho+c-\alpha-\alpha'-1}}{s^{l+\gamma\nu}}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{xs}\right)^\gamma$$
$$(4.9) \quad * \, {}_5\Psi_3\left[\begin{matrix} (1,1),(l+\gamma\nu,\gamma),(1-\varrho-b+\gamma\nu,\gamma),(1-\varrho-c+\alpha+\alpha'-b+\gamma\nu,\gamma) \\ (1-\varrho-c+\alpha+b'+\gamma\nu,\gamma); \\ (1-\varrho+\gamma\nu,\gamma),(1-\varrho-c+\alpha+\alpha'+b'+\gamma\nu,\gamma)(1-\varrho+\alpha-b+\gamma\nu,\gamma); \end{matrix} \left(-\frac{1}{xs}\right)^\gamma\right].$$

Theorem 11. *Let* $\alpha,\alpha',b,b',c,\varrho,p,q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0,\Re(c-\alpha-\alpha'-b'),\Re(b-\alpha)]$ *with* $\min\{\Re(p),\Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned Laplace-transformation holds true*:

$$\mathrm{L}\left(z^{l-1}(D_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((tz)^\gamma)\right\})\right)(x)$$
$$= \frac{x^{\varrho-c+\alpha+\alpha'-1}}{s^{l+\gamma\nu}}\mathbf{M}_{\nu,p,q}\left(\tfrac{x}{s}\right)^\gamma$$
$$(4.10) \quad * \, {}_5\Psi_3\left[\begin{matrix} (1,1),(l+\gamma\nu,\gamma),(\varrho+\gamma\nu,\gamma),(\varrho-c+\alpha+\alpha'+b'+\gamma\nu,\gamma) \\ (\varrho-b+\alpha+\gamma\nu,\gamma); \\ (\varrho-b+\gamma\nu,\gamma),(\varrho-c+\alpha+\alpha'+\gamma\nu,\gamma)(\varrho-c+\alpha+b'+\gamma\nu,\gamma); \end{matrix} \left(-\frac{x}{s}\right)^\gamma\right].$$

Theorem 12. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) < 1 + \min\{\Re(b'), \Re(c-\alpha-\alpha'), \Re(c-\alpha'-b)\}$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned Laplace-transformation holds true*:

$$\begin{aligned}
&\mathrm{L}\left(z^{l-1}(D_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((\tfrac{z}{t})^{\gamma})\right\})\right)(x) \\
&= \frac{x^{\varrho-c+\alpha+\alpha'-1}}{s^{l+\gamma\nu}}\mathbf{M}_{\nu,p,q}\left(\tfrac{1}{xs}\right)^{\gamma} \\
&* {}_5\Psi_3\left[\begin{matrix}(1,1),(l+\gamma\nu,\gamma),(1-\varrho-b'+\gamma\nu,\gamma),(1-\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma)\\ (1-\varrho+c-\alpha'-b+\gamma\nu,\gamma);\\ (1-\varrho+\gamma\nu,\gamma),(1-\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma)(1-\varrho-\alpha'-b'+\gamma\nu,\gamma);\end{matrix}\ \left(-\frac{1}{xs}\right)^{\gamma}\right].
\end{aligned} \tag{4.11}$$

Theorem 13. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0, \Re(\alpha+\alpha'+b-c), \Re(\alpha'-b')]$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned integral holds true*:

$$\begin{aligned}
&\int_0^\infty z^{l-1}\mathrm{e}^{-\frac{1}{2}\delta z}\,W_{\tau,\varsigma}(\delta z)\left(I_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((wtz)^{\gamma})\right\}\right)(x)\mathrm{d}z \\
&= \frac{x^{\varrho+c-\alpha-\alpha'-1}}{\delta^{l}}\mathbf{M}_{\nu,p,q}\left(\tfrac{wx}{\delta}\right)^{\gamma} \\
&* {}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\varsigma+l+\gamma\nu,\gamma),(\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma),(\varrho+\gamma\nu,\gamma)(\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma)\\ (\varrho+b'-\alpha'+\gamma\nu,\gamma);\\ (\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma)(\varrho+b'+\gamma\nu,\gamma),(\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma)(\varrho+c-\alpha'-b+\gamma\nu,\gamma);\end{matrix}\ \left(-\tfrac{wx}{\delta}\right)^{\gamma}\right].
\end{aligned} \tag{4.12}$$

Theorem 14. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) < 1 + \min\{\Re(-b), \Re(\alpha+\alpha'-c), \Re(\alpha+b'-c)\}$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|\frac{1}{t}| < 1$. *Then the below mentioned integral holds true*:

$$\begin{aligned}
&\int_0^\infty z^{l-1}\mathrm{e}^{-\frac{1}{2}\delta z}\,W_{\tau,\varsigma}(\delta z)\left(I_{-}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}\left(\tfrac{wz}{t}\right)^{\gamma}\right\}\right)(x)\mathrm{d}z \\
&= \frac{x^{\varrho+c-\alpha-\alpha'-1}}{\delta^{l}}\mathbf{M}_{\nu,p,q}\left(\tfrac{w}{\delta x}\right)^{\gamma} \\
&* {}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\varsigma+l+\gamma\nu,\gamma),(\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma),(1-\varrho-b+\gamma\nu,\gamma)(1-\varrho-c+\alpha+\alpha'+\gamma\nu,\gamma)\\ (1-\varrho-c+b'+\alpha+\gamma\nu,\gamma);\\ (\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma)(1-\varrho+\gamma\nu,\gamma),(1-\varrho-c+\alpha+\alpha'+b'+\gamma\nu,\gamma)(1-\varrho+\alpha-b+\gamma\nu,\gamma);\end{matrix}\ \left(-\tfrac{w}{\delta x}\right)^{\gamma}\right].
\end{aligned} \tag{4.13}$$

Theorem 15. *Let* $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ *and* $\Re(\nu) > -\frac{1}{2}$ *with* $\gamma \in \mathbb{R}^+$ *such that* $\Re(c) > 0$ *and* $\Re(\varrho) > \max[0, \Re(c-\alpha-\alpha'-b'), \Re(b-\alpha)]$ *with* $\min\{\Re(p), \Re(q)\} \geq 0$ *and* $|t| < 1$. *Then the below mentioned integral holds true*:

$$\begin{aligned}
&\int_0^\infty z^{l-1}\mathrm{e}^{-\frac{1}{2}\delta z}\,W_{\tau,\varsigma}(\delta z)\left(D_{0+}^{\alpha,\alpha',b,b',c}\left\{t^{\varrho-1}\mathbf{M}_{\nu,p,q}((wtz)^{\gamma})\right\}\right)(x)\mathrm{d}z \\
&= \frac{x^{\varrho+c-\alpha-\alpha'-1}}{\delta^{l}}\mathbf{M}_{\nu,p,q}\left(\tfrac{wx}{\delta}\right)^{\gamma} \\
&* {}_6\Psi_4\left[\begin{matrix}(1,1),(\frac{1}{2}+\varsigma+l+\gamma\nu,\gamma),(\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma),(1-\varrho-b+\gamma\nu,\gamma)(1-\varrho-c+\alpha+\alpha'+\gamma\nu,\gamma)\\ (1-\varrho-c+\alpha+b'+\gamma\nu,\gamma);\\ (\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma)(1-\varrho+\gamma\nu,\gamma),(1-\varrho-c+\alpha+\alpha'+b'+\gamma\nu,\gamma)(1-\varrho+\alpha-b+\gamma\nu,\gamma);\end{matrix}\ \left(-\tfrac{wx}{\delta}\right)^{\gamma}\right].
\end{aligned} \tag{4.14}$$

Theorem 16. *Let $\alpha, \alpha', b, b', c, \varrho, p, q \in \mathbb{C}$ and $\Re(\nu) > -\frac{1}{2}$ with $\gamma \in \mathbb{R}^+$ such that $\Re(c) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(b'), \Re(c-\alpha-\alpha'), \Re(c-\alpha'-b)\}$ with $\min\{\Re(p), \Re(q)\} \geq 0$ and $|\frac{1}{t}| < 1$. Then the below mentioned integral holds true*:

$$\int_0^\infty z^{l-1} \mathrm{e}^{-\frac{1}{2}\delta z}\, W_{\tau,\varsigma}(\delta z) \left(D_-^{\alpha,\alpha',b,b',c} \left\{t^{\varrho-1} \mathbf{M}_{\nu,p,q}\left(\tfrac{wz}{t}\right)^\gamma\right\}\right)(x)\mathrm{d}z$$

$$= \frac{x^{\varrho-c+\alpha+\alpha'-1}}{\delta^l} \mathbf{M}_{\nu,p,q}\left(\tfrac{w}{\delta x}\right)^\gamma$$

$$* \;{}_6\Psi_4\left[\begin{matrix} (1,1), (\frac{1}{2}+\varsigma+l+\gamma\nu,\gamma), (\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma), (1-\varrho-b'+\gamma\nu,\gamma)(1-\varrho+c-\alpha-\alpha'+\gamma\nu,\gamma) \\ (1-\varrho+c-b-\alpha'+\gamma\nu,\gamma); \\ (\frac{1}{2}-\varsigma+l+\gamma\nu,\gamma)(1-\varrho+\gamma\nu,\gamma), (1-\varrho+c-\alpha-\alpha'-b+\gamma\nu,\gamma)(1-\varrho-\alpha'-b'+\gamma\nu,\gamma); \end{matrix} \left(-\tfrac{w}{\delta x}\right)^\gamma\right]. \tag{4.15}$$

5. Concluding remarks and observations

In our present investigation, with the help of the concept of the Hadamard product (or the convolution) of two analytic functions, we have obtained the composition formulas of the generalized Marichev-Saigo-Maeda fractional integrals and differential operators (1.1), (1.2), (1.3) and (1.4) involving the (p,q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$ in terms of the Hadamard product (1.21) of the (p,q)–extended modified Bessel function of the second kind $\mathbf{M}_{\nu,p,q}(z)$ and the Fox–Wright function ${}_r\Psi_s(z)$.Nex, we have also deduced the certain image formulas for the Saigo's, Riemann-Liouville (R-L) and Erdélyi-Kober (E-K) fractional integral and differential operators as particular cases. Further, we establish certain theorems involving the results obtained in Section 2 and Section 3 associated with the integral transforms such as Euler-Beta transform, Laplace transform and Whittaker transform.

Acknowledgments. The authors thanks to the anonymous referee for valuable comments and useful suggestions to improving the quality of the article.

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