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GENERIC LIGHTLIKE SUBMANIFOLDS OF SEMI-RIEMANNIAN PRODUCT MANIFOLDS

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Abstract. We introduce the study of generic lightlike submanifolds of a semi-Riemannian product manifold. We establish a characterization theorem for the induced connection on a generic lightlike submanifold to be a metric connection. We also find some conditions for the integrability of the distributions associated with generic lightlike submanifolds and discuss the geometry of foliations. Then we search for some results enabling a generic lightlike submanifold of a semi-Riemannian product manifold to be a generic lightlike product manifold. Finally, we examine minimal generic lightlike submanifolds of a semi-Riemannian product manifold.

1. Introduction

The concept of CR-submanifolds of a Kaehler manifold was firstly introduced and developed by Bejancu [2] in 1978. He studied totally real as well as complex submanifolds as the sub-cases of a CR-submanifold. After that different geometric aspects of CR-submanifolds of a Kaehler manifold were examined by other geometers ([3–7]). Then, Deshmukh et al. [8] initiated the study of CR-submanifolds of nearly Kaehler manifolds. Husain and Deshmukh [15] investigated several fundamental results on CR-submanifolds of a nearly Kaehler manifold. They also proved the non-existence of complex hypersurfaces in nearly Kaehler manifolds with constant holomorphic sectional curvature. Moreover, Duggal [10] studied the interaction of CR-structures with Lorentzian geometry which has outstanding applications in relativity. On a similar note, the class of generic submanifolds emerged as an important class of submanifolds of almost Hermitian manifolds as in this case the normal bundle is mapped to the tangent bundle under the action of an almost complex structure $\bar{J}$. The geometry of generic submanifolds was dealt in details by Yano and Kon in [22] and [23].

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It is well known that a submanifold of a semi-Riemannian manifold is called a lightlike submanifold, if the induced metric is degenerate. Due to the degenerate metric, in the case of a lightlike submanifold the normal vector bundle intersects with the tangent vector bundle. This unique feature complicates the study of lightlike submanifolds. In recent studies, several significant applications of lightlike submanifolds have been observed in mathematical physics and relativity. For example, lightlike submanifolds are useful to study black holes, four-dimensional electromagnetic space times, Einstein Field Equations, different types of horizons (Cauchy's horizons, event horizons and Kruskal's horizons) (for details, see [11]). Thus, Duggal and Bejancu [11] established a new class of lightlike submanifolds, namely CR-lightlike submanifolds of indefinite Kaehler manifolds. Then they observed that CR-lightlike submanifolds exclude invariant and totally real cases. Thereafter, Duggal and Sahin [13] introduced SCR-lightlike submanifolds of indefinite Kaehler manifolds containing invariant and totally real sub-cases. They concluded that SCR and CR-lightlike submanifolds are entirely different from each other. Therefore, Duggal and Sahin [14] initiated the study of GCR-lightlike submanifolds of indefinite Kaehler manifolds which acts as an umbrella for CR and SCR-lightlike submanifolds. On a similar note, Kumar et al. [19] studied GCR-lightlike submanifolds of indefinite nearly Kaehler manifolds. In [12], Duggal and Jin introduced the general notion of generic lightlike submanifolds of indefinite Sasakian manifolds. Since then, numerous studies have been devoted to this class of lightlike submanifolds, such as ([16–18, 21]). In [9], Dogan et al. investigated screen generic lightlike submanifolds of indefinite Kaehler manifolds.

It may be noted that the semi-Riemannian product manifolds are generalization of Riemannian product manifolds in semi-Riemannian case and they have rich geometric properties. In [20], Kumar et al. considered geometry of GCR-lightlike submanifolds of a semi-Riemannian product manifolds and proved several geometric characterization for this class of submanifolds. However, the concept of generic lightlike submanifolds is yet to be explored in semi-Riemannian product manifolds.

Therefore, in this paper, we study generic lightlike submanifolds of a semi-Riemannian product manifold. At first, we define a generic lightlike submanifold of a semi-Riemannian product manifold followed by a non-trivial example for such lightlike submanifolds. Then we prove a necessary and sufficient condition for the induced connection on a generic lightlike submanifold to be a metric connection. We also find some conditions for the integrability of distributions associated with generic lightlike submanifolds and examine the geometry of foliations. Further, we obtain some necessary and sufficient conditions for a generic lightlike submanifold to be a generic lightlike product manifold. At last, we investigate minimal generic lightlike submanifolds in a semi-Riemannian product manifold.

2. Preliminaries

2.1. Lightlike submanifolds

Let (M, g) be an m-dimensional submanifold of an $(m+n)$-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ with constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then T_pM and $T_pM^{\perp}$ both are degenerate and there exists a radical (null) subspace $Rad(T_pM)$ such that $Rad(T_pM) = T_pM \cap T_pM^{\perp}$. If $Rad(TM) : p \in M \to Rad(T_pM)$ is a smooth distribution on M with rank $r > 0$ and $1 \leq r \leq m$, then M is called an r-lightlike submanifold of $\bar{M}$. While the radical distribution $Rad(TM)$ of TM is defined as:

$$Rad(TM) = \cup_{p \in M} \{\xi \in T_pM \mid g(u, \xi) = 0, \ \forall\, u \in T_pM, \xi \neq 0\}.$$

Let $S(TM)$ be the screen distribution in TM such that

$$TM = Rad(TM) \perp S(TM) \tag{1}$$

and $S(TM^{\perp})$ is a complementary vector sub-bundle to $Rad(TM)$ in $TM^{\perp}$.

Moreover, there exists a local null frame $\{N_i\}$ of null sections with values in the orthogonal complement of $S(TM^{\perp})$ in $S(TM^{\perp})^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0 \ \text{ for any } i, j \in \{1, 2, \ldots, r\},$$

where $\{\xi_i\}$ is any local basis of $\Gamma(Rad(TM))$.

Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{3}$$

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (3), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h(Y_1, Y_2), \quad \forall\, Y_1, Y_2 \in \Gamma(TM), \tag{4}$$

$$\bar{\nabla}_{Y_1} U = -A_U Y_1 + \nabla^{\perp}_{Y_1} U, \quad \forall\, Y_1 \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{5}$$

where $\{\nabla_{Y_1} Y_2, A_U Y_1\}$ and $\{h(Y_1, Y_2), \nabla^{\perp}_{Y_1} U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called the second fundamental form and A_U is a linear operator on M known as shape operator.

According to Eq. (2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, Eqs. (4) and (5) become

$$\bar{\nabla}_{Y_1} Y_2 = \nabla_{Y_1} Y_2 + h^l(Y_1, Y_2) + h^s(Y_1, Y_2), \tag{6}$$

$$\bar{\nabla}_{Y_1} U = -A_U Y_1 + D^l_{Y_1} U + D^s_{Y_1} U, \tag{7}$$

where we put $h^l(Y_1, Y_2) = L(h(Y_1, Y_2))$, $h^s(Y_1, Y_2) = S(h(Y_1, Y_2))$, $D^l_{Y_1}U = L(\nabla^{\perp}_{Y_1}U)$, $D^s_{Y_1}U = S(\nabla^{\perp}_{Y_1}U)$. As h^l and h^s are $ltr(TM)$-valued and $S(TM^{\perp})$-valued bilinear forms, respectively, known as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_{Y_1}N = -A_N Y_1 + \nabla^l_{Y_1}N + D^s(Y_1, N), \tag{8}$$

$$\bar{\nabla}_{Y_1}V = -A_V Y_1 + \nabla^s_{Y_1}V + D^l(Y_1, V), \tag{9}$$

where $Y_1 \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $V \in \Gamma(S(TM^{\perp}))$. Then using Eqs. (6)-(9), we obtain

$$\bar{g}(h^s(Y_1, Y_2), V) + \bar{g}(Y_2, D^l(Y_1, V)) = g(A_V Y_1, Y_2), \tag{10}$$

$$\bar{g}(h^l(Y_1, Y_2), \xi) + \bar{g}(Y_2, h^l(Y_1, \xi)) + \bar{g}(Y_2, \nabla_{Y_1}\xi) = 0 \tag{11}$$

for $\xi \in \Gamma(Rad(TM))$, $V \in \Gamma(S(TM^{\perp}))$ and $Y_1, Y_2 \in \Gamma(TM)$.

Let P denote the projection morphism of TM on $S(TM)$. Then using Eq. (1) we can induce some new geometric objects on $S(TM)$ of M as

$$\nabla_{Y_1}PY_2 = \nabla^*_{Y_1}PY_2 + h^*(Y_1, Y_2), \tag{12}$$

$$\nabla_{Y_1}\xi = -A^*_{\xi}Y_1 + \nabla^{*t}_{Y_1}\xi \tag{13}$$

for $Y_1, Y_2 \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_{Y_1}PY_2, A^*_{\xi}Y_1\}$ and $\{h^*(Y_1, Y_2), \nabla^{*t}_{Y_1}\xi\}$ belongs to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Further, ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. Moreover, h^* and A^* are $Rad(TM)$-valued and $S(TM)$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $Rad(TM)$, respectively.

Using Eqs. (6), (7), (12) and (13), we obtain

$$\bar{g}(h^l(Y_1, PY_2), \xi) = g(A^*_{\xi}Y_1, PY_2), \tag{14}$$

$$\bar{g}(h^*(Y_1, PY_2), N) = \bar{g}(A_N Y_1, PY_2) \tag{15}$$

for $Y_1, Y_2 \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

From the geometry of non-degenerate submanifolds, it is well known that the induced connection ∇ on a non-degenerate submanifold is always a metric connection. However, this is not true for a lightlike submanifold. Since $\bar{\nabla}$ is a metric connection on $\bar{M}$, thus we have

$$(\nabla_{Y_1}g)(Y_2, Y_3) = \bar{g}(h^l(Y_1, Y_2), Y_3) + \bar{g}(h^l(Y_1, Y_3), Y_2) \tag{16}$$

for $Y_1, Y_2, Y_3 \in \Gamma(TM)$. By direct calculations, the equation of Codazzi is given by

$$\begin{aligned}(\bar{R}(Y_1, Y_2)Y_3)^{\perp} = {} & (\nabla_{Y_1}h^l)(Y_2, Y_3) - (\nabla_{Y_2}h^l)(Y_1, Y_3) + D^l(Y_1, h^s(Y_2, Y_3)) \\ & - D^l(Y_2, h^s(Y_1, Y_3)) + (\nabla_{Y_1}h^s)(Y_2, Y_3) - (\nabla_{Y_2}h^s)(Y_1, Y_3) \\ & + D^s(Y_1, h^l(Y_2, Y_3)) - D^s(Y_2, h^l(Y_1, Y_3)),\end{aligned} \tag{17}$$

where

$$(\nabla_{Y_1} h^l)(Y_2, Y_3) = \nabla^l_{Y_1}(h^l(Y_2, Y_3)) - h^l(\nabla_{Y_1} Y_2, Y_3) - h^l(Y_2, \nabla_{Y_1} Y_3), \tag{18}$$

$$(\nabla_{Y_1} h^s)(Y_2, Y_3) = \nabla^s_{Y_1}(h^s(Y_2, Y_3)) - h^s(\nabla_{Y_1} Y_2, Y_3) - h^s(Y_2, \nabla_{Y_1} Y_3). \tag{19}$$

Definition 1 ([5])**.** A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there exist a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M such that

$$h(X, Y) = Hg(X, Y) \tag{20}$$

for any $X, Y \in \Gamma(TM)$. Using Eqs. (6), (8) and (9), it is clear that M is totally umbilical if and only if on each coordinate neighborhood u, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^\perp))$ such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0 \tag{21}$$

for $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^\perp))$.

2.2. Semi-Riemannian product manifolds

Let (M_1, g_1) and (M_2, g_2) be two m_1 and m_2-dimensional semi-Riemannian manifolds with constant indices q_1 and q_2, respectively. Let $\pi : M_1 \times M_2 \to M_1$ and $\sigma : M_1 \times M_2 \to M_2$ be the projection maps given by $\pi(x, y) = x$ and $\sigma(x, y) = y$ for any $(x, y) \in M_1 \times M_2$. We denote the product manifold by $(\bar{M}, \bar{g}) = (M_1 \times M_2, \bar{g})$, where

$$\bar{g}(X, Y) = g_1(\pi_* X, \pi_* Y) + g_2(\sigma_* X, \sigma_* Y)$$

for any $X, Y \in \Gamma(T\bar{M})$, where $*$ stands for the differential mapping. Then we have

$$\pi_*^2 = \pi_*, \quad \sigma_*^2 = \sigma_*, \quad \pi_*\sigma_* = \sigma_*\pi_* = 0, \quad \pi_* + \sigma_* = I,$$

where I is the identity map of $T(M_1 \times M_2)$. Thus $(\bar{M}, \bar{g})$ is an $(m_1 + m_2)$-dimensional semi-Riemannian product manifold with constant index $(q_1 + q_2)$. The semi-Riemannian product manifold $\bar{M} = M_1 \times M_2$ is characterized by M_1 and M_2, which are totally geodesic submanifolds of $\bar{M}$. If we put $F = \pi_* - \sigma_*$, then $F^2 = I$ and

$$\bar{g}(FX, Y) = \bar{g}(X, FY) \tag{22}$$

for any $X, Y \in \Gamma(T\bar{M})$, where F is called an almost product structure on $T(M_1 \times M_2)$. If we denote the Levi-Civita connection on $\bar{M}$ by $\bar{\nabla}$, then

$$(\bar{\nabla}_X F)Y = 0 \tag{23}$$

for any $X, Y \in \Gamma(T\bar{M})$.

3. Generic lightlike submanifolds

Definition 2. Let $(M, g, S(TM))$ be an r-lightlike submanifold of a semi-Riemannian product manifold $(\bar{M}, \bar{g})$. Then, the screen distribution $S(TM)$ of M is expressed as

$$\begin{aligned} S(TM) &= F(S(TM)^{\perp}) \oplus_{orth} D_0 \\ &= F(Rad(TM)) \oplus F(ltr(TM)) \oplus_{orth} F(S(TM^{\perp})) \oplus_{orth} D_0, \end{aligned} \tag{24}$$

where D_0 is a non-degenerate distribution on M with respect to F, i.e., $F(D_0) = D_0$ and D' is an r-lightlike distribution on $S(TM)$ such that $F(D') \subset tr(TM)$, where $D' = F(ltr(TM)) \oplus_{orth} F(S(TM^{\perp}))$.

Therefore, using Eq. (24), the general decompositions of Eqs. (1) and (3) become

$$TM = D \oplus D', \quad T\bar{M} = D \oplus D' \oplus tr(TM),$$

where D is a $2r$-lightlike distribution on M such that $D = Rad(TM) \oplus_{orth} F(Rad(TM)) \oplus_{orth} D_0$.

Example 3.1. Let M be a submanifold of $(R^8_2, \bar{g})$ given by the equations $x_3 = x_8$ and $x_5 = \sqrt{1 - x_6^2}$, where g is of signature $(+,+,-,+,+,-,+,+)$ with respect to a basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8)$. Then the tangent bundle of M is spanned by

$$Z_1 = \partial x_1, \quad Z_2 = \partial x_2, \quad Z_3 = \partial x_3 + \partial x_8, \quad Z_4 = \partial x_4,$$
$$Z_5 = -x_6 \partial x_5 + x_5 \partial x_6, \quad Z_6 = \partial x_7.$$

Clearly M is a 1-lightlike submanifold with $Rad(TM) = Span\{Z_3\}$ and $FZ_3 = Z_4 + Z_6 \in \Gamma(S(TM))$. Moreover $FZ_1 = Z_2$ and $FZ_2 = Z_1$ and therefore $D_0 = Span\{Z_1, Z_2\}$. By direct calculations, we get $S(TM^{\perp}) = Span\{W = x_5 \partial x_5 - x_6 \partial x_6\}$. Thus, $FW = Z_5$ and hence $FS(TM^{\perp}) \subset S(TM)$. On the other hand, $ltr(TM)$ is spanned by $N = \frac{1}{2}(-\partial x_3 + \partial x_8)$. Then $FN = \frac{1}{2}(-\partial x_4 + \partial x_7) = \frac{1}{2}(-Z_4 + Z_6)$. Hence $D' = \{FN, FW\}$. Thus M is a proper generic lightlike submanifold of R^8_2.

Consider Q, P_1 and P_2 denote the projections from TM to D, $F(ltr(TM))$ and $F(S(TM^{\perp}))$, respectively. Then for $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X, \tag{25}$$

applying F to Eq. (25), we obtain

$$FX = TX + \omega P_1 X + \omega P_2 X \tag{26}$$

and we can write Eq. (26) as

$$FX = TX + \omega X, \tag{27}$$

where TX and ωX are tangential and transversal components of FX, respectively. Similarly,

$$FV = BV \tag{28}$$

for $V \in \Gamma(tr(TM))$, where BV is the section of TM.

Since F is parallel on M, then for $X, Y \in \Gamma(TM)$, using Eqs. (6), (8), (9) and (28), we derive

$$(\nabla_X T)Y = A_{\omega P_1 Y}X + A_{\omega P_2 Y}X + Bh(X,Y), \tag{29}$$

$$D^s(X, \omega P_1 Y) = -\nabla^s_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^s(X, TY), \tag{30}$$

$$D^l(X, \omega P_2 Y) = -\nabla^l_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^l(X, TY). \tag{31}$$

Theorem 3.2. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the induced connection is a metric connection if and only if the following condition holds*

$$\nabla_X FY \in \Gamma(FRad(TM)) \text{ and } Bh(X, FY) = 0$$

for $X \in \Gamma(TM)$ and $Y \in \Gamma(Rad(TM))$.

Proof. Since F is an almost product structure of a semi-Riemannian product manifold $\bar{M}$ therefore we say that $\bar{\nabla}_X Y = \bar{\nabla}_X F^2 Y$ for any $Y \in \Gamma(Rad(TM))$ and $X \in \Gamma(TM)$. Then from Eq. (6), we get $\bar{\nabla}_X Y = F\bar{\nabla}_X FY$ and using Eqs. (4), (27) and (28), we derive

$$\begin{aligned}\nabla_X Y + h(X,Y) &= F(\nabla_X FY + h(X, FY)) \\ &= T\nabla_X FY + \omega \nabla_X FY + Bh(X, FY).\end{aligned}$$

Further on equating the tangential part, the above equation yields

$$\nabla_X Y = T\nabla_X FY + Bh(X, FY). \tag{32}$$

Hence from Eq. (32), $\nabla_X Y \in \Gamma(Rad(TM))$ if and only if

$$\nabla_X FY \in \Gamma(FRad(TM)) \text{ and } Bh(X, FY) = 0,$$

which gives the result. □

Theorem 3.3. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then*

(i) *the distribution D is integrable if and only if*

$$h(FX, Y) = h(X, FY), \quad \forall\ X, Y \in \Gamma(D).$$

(ii) *the distribution D' is integrable if and only if*

$$A_{FZ}V = A_{FV}Z, \quad \forall\ Z, V \in \Gamma(D').$$

Proof. From Eqs. (30) and (31), we get $\omega\nabla_X Y = h(X, TY)$ for any $X, Y \in \Gamma(D)$ which implies $\omega[X,Y] = \omega\nabla_X Y - \omega\nabla_Y X = h(X, TY) - h(TX, Y)$. The distribution D is integrable if and only if $h(X, FY) - h(FX, Y) = 0$, that is, $h(FX, Y) = h(X, FY)$, which proves the first result.

Next from Eq. (29), we have $T\nabla_Z V = -A_{\omega V}Z - Bh(Z, V)$ for any $Z, V \in \Gamma(D')$. Therefore, $T[Z, V] = A_{\omega Z}V - A_{\omega V}Z$, which completes the proof. □

Theorem 3.4. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D defines a totally geodesic foliation in M if and only if $Bh(X,Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Proof. Using the definition of generic lightlike submanifolds, D defines a totally geodesic foliation in M if and only if $\nabla_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. In other words, D defines a totally geodesic foliation in M if and only if

$$g(\nabla_X Y, F\xi) = g(\nabla_X Y, FW) = 0$$

for any $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. From Eqs. (6), (22) and (23), we derive

$$\begin{aligned} g(\nabla_X Y, F\xi) &= \bar{g}(\bar{\nabla}_X Y, F\xi) = \bar{g}(\bar{\nabla}_X FY, \xi) \\ &= \bar{g}(\nabla_X FY + h^l(X, FY) + h^s(X, FY), \xi) \\ &= \bar{g}(h^l(X, FY), \xi) \end{aligned} \tag{33}$$

for $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$. Similarly using Eqs. (6), (22) and (23), we get

$$\begin{aligned} g(\nabla_X Y, FW) &= \bar{g}(\bar{\nabla}_X Y, FW) = \bar{g}(\bar{\nabla}_X FY, W) \\ &= \bar{g}(\nabla_X FY + h^l(X, FY) + h^s(X, FY), W) \\ &= \bar{g}(h^s(X, FY), W) \end{aligned} \tag{34}$$

for $X, Y \in \Gamma(D)$ and $W \in \Gamma(S(TM^\perp))$. It is clear from Eqs. (33) and (34) that D defines a totally geodesic foliation in M if and only if $h^s(X, FY)$ has no components in $(S(TM^\perp))$ and $h^l(X, FY)$ has no components in $ltr(TM)$ for any $X, Y \in \Gamma(D)$. Thus from Eq. (28), we have $Fh(X,Y) = Bh(X,Y) = 0$ for any $X, Y \in \Gamma(D)$. □

Theorem 3.5. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D' defines a totally geodesic foliation in M if and only if $A_{wY} X \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$.*

Proof. From Eq. (29), we have $T\nabla_X Y = -A_{\omega Y} X - Bh(X,Y)$ for any $X, Y \in \Gamma(D')$. If D' defines a totally geodesic foliation in M, then $-A_{\omega Y} X - Bh(X,Y) = 0$, that is, $-A_{\omega Y} X = Bh(X,Y)$ which implies that $A_{\omega Y} X \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$.

Conversely, let $A_{\omega Y} X \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$. Then from Eq. (29), we obtain $T\nabla_X Y = 0$, which further implies that $\nabla_X Y \in \Gamma(D')$. This completes the proof. □

Definition 3. A generic lightlike submanifold of a semi-Riemannian product manifold is called D-geodesic (respectively, D'-geodesic) generic lightlike submanifold if its second fundamental form h satisfies $h(X,Y) = 0$ for $X, Y \in \Gamma(D)$ (respectively, $X, Y \in \Gamma(D')$).

Theorem 3.6. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D defines a totally geodesic foliation in $\bar{M}$ if and only if M is D-geodesic.*

Proof. Let D defines a totally geodesic foliation in a semi-Riemannian product manifold $\bar{M}$ then $\bar{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Then using Eq. (6), for $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$, we get

$$0 = \bar{g}(\bar{\nabla}_X Y, \xi) = \bar{g}(\nabla_X Y + h^l(X, Y) + h^s(X, Y), \xi) = \bar{g}(h^l(X, Y), \xi)$$

and

$$0 = \bar{g}(\bar{\nabla}_X Y, W) = \bar{g}(\nabla_X Y + h^l(X, Y) + h^s(X, Y), W) = \bar{g}(h^s(X, Y), W).$$

Hence we say that $h^l(X, Y) = h^s(X, Y) = 0$ for any $X, Y \in \Gamma(D)$, which implies that M is D-geodesic.

Conversely, let us assume that M is D-geodesic. Then from Eqs. (6) and (23) for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM)$ and $W \in \Gamma(S(TM^\perp))$, we have

$$\bar{g}(\bar{\nabla}_X Y, F\xi) = \bar{g}(\bar{\nabla}_X FY, \xi) = \bar{g}(h^l(X, FY), \xi) = 0$$

and

$$\bar{g}(\bar{\nabla}_X Y, FW) = \bar{g}(\bar{\nabla}_X FY, W) = \bar{g}(h^s(X, FY), W) = 0.$$

Hence $\bar{\nabla}_X Y \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$, which proves the result. □

Theorem 3.7. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is D-geodesic if and only if*

$$g(A_W X, Y) = \bar{g}(D^l(X, W), Y)$$

and

$$\bar{g}(h^l(X, Y), \xi) = -g(FY, \nabla^*_X F\xi)$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Proof. Using the definition of generic lightlike submanifolds, M is D-geodesic if and only if

$$\bar{g}(h^l(X, Y), \xi) = 0 \tag{35}$$

and

$$\bar{g}(h^s(X, Y), W) = 0 \tag{36}$$

for any $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM)$ and $W \in \Gamma(S(TM^\perp))$. Thus for any $X, Y \in \Gamma(D)$, from Eq. (10), we have

$$\bar{g}(h^s(X, Y), W) + \bar{g}(D^l(X, W), Y) = g(A_W X, Y)$$

and further employing Eq. (36), we obtain

$$\bar{g}(D^l(X, W), Y) = g(A_W X, Y),$$

which proves the first part of assertion.

Now for $X, Y \in \Gamma(D)$ and $\xi \in \Gamma(Rad(TM))$, using Eqs. (6) and (12), we get

$$\bar{g}(h^l(X, Y), \xi) = \bar{g}(\bar{\nabla}_X Y, \xi)$$

$$= -\bar{g}(FY, \bar{\nabla}_X F\xi)$$
$$= -g(FY, \nabla_X F\xi) - \bar{g}(FY, h^l(X, F\xi))$$
$$= -g(FY, \nabla^*_X F\xi) - \bar{g}(FY, h^l(X, F\xi)). \tag{37}$$

Since $Y \in \Gamma(D)$, this implies that

$$\bar{g}(FY, h^l(X, F\xi)) = 0$$

and Eq. (37) becomes

$$\bar{g}(h^l(X, Y), \xi) = -g(FY, \nabla^*_X F\xi),$$

which proves the second part of the theorem. □

Theorem 3.8. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is D'-geodesic if and only if $A_W X$ and $A^*_\xi X$ has no components in $\Gamma(F(Rad(TM)) \perp F(S(TM^\perp)))$ for any $X \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM)$ and $W \in \Gamma(S(TM^\perp))$.*

Proof. For any $X, Y \in \Gamma(D')$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$, using Eq. (10), we get

$$\bar{g}(h^s(X, Y), W) = g(A_W X, Y). \tag{38}$$

Next using Eqs. (11) and (13), we get $\bar{g}(h^l(X, Y), \xi) = g(Y, \nabla_X \xi) = g(Y, A^*_\xi X)$, that is,

$$\bar{g}(h^l(X, Y), \xi) = g(Y, A^*_\xi X). \tag{39}$$

Hence, the result follows from Eqs. (38) and (39). □

Definition 4. A generic lightlike submanifold of a semi-Riemannian product manifold is called mixed geodesic (respectively, totally geodesic) generic lightlike submanifold if its second fundamental form h satisfies $h(X, Y) = 0$ for any $X \in \Gamma(D)$ and $Y \in \Gamma(D')$ (respectively, for any $X, Y \in \Gamma(TM)$).

Theorem 3.9. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is mixed geodesic if and only if*

$$A^*_\xi X \in \Gamma(D_0) \perp \Gamma(F(ltr(TM))) \quad \textit{and} \quad A_W X \in \Gamma(D_0) \perp \Gamma(F(ltr(TM)))$$

for any $X \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Proof. From Eq. (11), for any $X \in \Gamma(D)$, $Y \in \Gamma(D')$ and $\xi \in \Gamma(Rad(TM))$, we have $\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, \nabla_X \xi) = 0$, which on employing Eq. (12) gives $\bar{g}(h^l(X, Y), \xi) - g(A^*_\xi X, Y) = 0$. Therefore, we get

$$\bar{g}(h^l(X, Y), \xi) = g(A^*_\xi X, Y). \tag{40}$$

On the other hand, for any $W \in \Gamma(S(TM^\perp))$, using Eq. (10), we derive

$$\bar{g}(h^s(X, Y), W) = g(A_W X, Y). \tag{41}$$

Hence, the assertion follows from Eqs. (40) and (41). □

Theorem 3.10. *Let M be a mixed geodesic generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then the transversal section $V \in \Gamma(FD')$ is D-parallel if and only if $\nabla_X FV \in \Gamma(D)$ for any $X \in \Gamma(D)$.*

Proof. Let $Y \in \Gamma(D')$ such that $FY = wY = V \in \Gamma(F(tr(TM)))$. Then using the hypothesis in Eq. (29), we get $T\nabla_X Y = -A_{\omega Y}X = -A_V X$. Now employing Eq. (7), $\nabla^t_X V = \bar{\nabla}_X V + A_V X = \bar{\nabla}_X FY - T\nabla_X Y$. Since $\bar{\nabla}$ is a metric connection and M is mixed geodesic therefore we get $\nabla^t_X V = \omega\nabla_X Y$, that is, $\nabla^t_X V = \omega\nabla_X FV$, which proves the theorem. □

4. Generic lightlike product manifolds

In this section, we will examine several characterization theorems for a generic lightlike submanifold of a semi-Riemannian product manifold to be a generic lightlike product manifold. To start with, firstly we define a generic lightlike product manifold as follows:

Definition 5. A generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$ is called a generic lightlike product manifold if both the distributions D and D' define totally geodesic foliations in M.

Theorem 4.1. *Let M be a totally geodesic generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Suppose that there exist a transversal vector bundle of M, which is parallel along D' with respect to the Levi-Civita connection on M, that is, $\bar{\nabla}_X V \in \Gamma(tr(TM))$ for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(D')$. Then M is a generic lightlike product manifold.*

Proof. Since M be a totally geodesic generic lightlike submanifold, then $Bh(X,Y) = 0$ for any $X, Y \in \Gamma(D)$. Therefore, the distribution D defines a totally geodesic foliation in M. Now since $\bar{\nabla}_X V \in \Gamma(tr(TM))$ for any $V \in \Gamma(tr(TM))$ and $X \in \Gamma(D')$, therefore Eq. (8) implies that $A_V X = 0$. Then from Eq. (29), we obtain $T\nabla_X Y = 0$ for any $X, Y \in \Gamma(D')$, which further gives $\nabla_X Y \in \Gamma(D')$. Thus, the distribution D' defines a totally geodesic foliation in M. Hence, the proof follows. □

Definition 6. A lightlike submanifold of a semi-Riemannian manifold is said to be irrotational if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$. Thus M is an irrotational lightlike submanifold if and only if $h^l(X,\xi) = 0$ and $h^s(X,\xi) = 0$.

Theorem 4.2. *Let M be an irrotational generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is a generic lightlike product manifold if the following conditions are satisfied:*

(i) $\bar{\nabla}_X U \in \Gamma(S(TM^{\perp}))$ *for any* $X \in \Gamma(TM)$ *and* $U \in \Gamma(tr(TM))$.
(ii) $A^*_\xi Y \in \Gamma(F(S(TM^{\perp})))$ *for any* $Y \in \Gamma(D)$.

Proof. From Eq. (8) with condition (i), we obtain $A_W X = 0$, $D^l(X,W) = 0$ and $\nabla^l_X W = 0$ for any $X \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$. On using Eq. (10), for $X, Y \in \Gamma(D)$ and $W \in \Gamma(S(TM^{\perp}))$, we derive $\bar{g}(h^s(X, Y), W) = 0$. Then, the non-degeneracy of $S(TM^{\perp})$ implies that $h^s(X, Y) = 0$. Hence $Bh^s(X, Y) = 0$. Now let $X, Y \in \Gamma(D)$, $\xi \in \Gamma(Rad(TM))$, then using condition (ii), we get $\bar{g}(h^l(X, Y), \xi) = -g(\nabla_X \xi, Y) = g(A^*_\xi X, Y) = 0$. It implies that $h^l(X, Y) = 0$ and $Bh^l(X, Y) = 0$. Thus, the distribution D defines a totally geodesic foliation in M.

Next let $X, Y \in \Gamma(D')$, then $FY = \omega Y \in \Gamma(tr(TM))$. Using Eq. (29), we obtain $T\nabla_X Y = -Bh(X, Y)$, then comparing the components along D, we get $T\nabla_X Y = 0$, which further implies that $\nabla_X Y \in \Gamma(D)$. Thus, the distribution D defines a totally geodesic foliation in M. Hence, M is a generic lightlike product manifold. □

Theorem 4.3. *Let M be a generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is a generic lightlike product manifold if and only if $(\nabla_X T)Y = 0$ for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$.*

Proof. Let $(\nabla_X T)Y = 0$ for any $X, Y \in \Gamma(D)$ or $X, Y \in \Gamma(D')$. Firstly, let $X, Y \in \Gamma(D)$, then $\omega Y = 0$ and from Eq. (29), we obtain $Bh(X, Y) = 0$ and hence using Theorem 3.4, the distribution D defines a totally geodesic foliation in M. Secondly, let $X, Y \in \Gamma(D')$. Since $BV \in \Gamma(D')$ for any $V \in \Gamma(tr(TM))$, then Eq. (29) implies that $A_{\omega Y} X \in \Gamma(D')$. Hence using Theorem 3.5, we obtain the distribution D' defines a totally geodesic foliation in M. Since both the distribution D and D' define totally geodesic foliation in M, hence M is a generic lightlike product manifold.

Conversely, assume that M is a generic lightlike product manifold, therefore the distribution D and D' define totally geodesic foliation in M. From Eq. (23), for any $X, Y \in \Gamma(D)$, we have $\bar{\nabla}_X FY = F\bar{\nabla}_X Y$, that is, $\nabla_X FY + h(X, FY) = F(\nabla_X Y + h(X, Y))$. Further on comparing the transversal components on both sides, we obtain $h(X, FY) = Fh(X, Y)$. Then $(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y = \bar{\nabla}_X FY - h(X, FY) - F\bar{\nabla}_X Y + Fh(X, Y) = \bar{\nabla}_X FY - F\bar{\nabla}_X Y = 0$ for any $X, Y \in \Gamma(D)$. Since D' defines a totally geodesic foliation in M and using Eq. (23), we get $\bar{\nabla}_X FY = F\bar{\nabla}_X Y$, then comparing the tangential component on both sides, we obtain $-A_{\omega Y} X = Bh(X, Y)$. Further from Eq. (29), we derive $(\nabla_X T)Y = -A_{\omega Y} X - Bh(X, Y) = Bh(X, Y) - Bh(X, Y) = 0$, which implies that $(\nabla_X T)Y = 0$. Hence, the proof is complete. □

Lemma 4.4. *Let M be a totally umbilical generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then D' defines a totally geodesic foliation in M.*

Proof. Let $X, Y \in \Gamma(D')$. Then Eq. (29) implies that $T\nabla_X Y = -A_{\omega Y} X - Bh(X, Y)$, then for any $Z \in \Gamma(D_0)$, we have

$$\begin{aligned} g(T\nabla_X Y, Z) &= -g(A_{\omega Y} X + Bh(X, Y), Z) \\ &= \bar{g}(\bar{\nabla}_X \omega Y, Z) = \bar{g}(\bar{\nabla}_X FY, Z) = \bar{g}(\bar{\nabla}_X Y, FZ) = \bar{g}(\bar{\nabla}_X Y, Z') \\ &= -g(Y, \nabla_X Z'), \end{aligned} \tag{42}$$

where $Z' = FZ \in \Gamma(D_0)$. Since $X \in \Gamma(D'))$ and $Z \in \Gamma(D_0)$, then from Eqs. (30) and (31), we get $\omega P \nabla_X Z = h(X, TZ) = Hg(X, TZ) = 0$. Therefore $\omega P \nabla_X Z = 0$, which implies that $\nabla_X Z \in \Gamma(D)$. Thus Eq. (42) implies that $g(T\nabla_X Y, Z) = 0$, then the non degeneracy of D_0 implies that $T\nabla_X Y = 0$. Hence for $\nabla_X Y \in \Gamma(D')$ for any $X, Y \in \Gamma(D')$. Thus, the result follows. □

Theorem 4.5. *Let M be a totally umbilical generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is a generic lightlike product manifold if and only if $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.*

Proof. Let M be a generic lightlike product manifold therefore the distribution D and D' define totally geodesic foliation in M. Therefore using Theorem 3.4, we have $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Now using the hypothesis for $X \in \Gamma(D')$ and $Y \in \Gamma(D)$, we have $Bh(X, Y) = Bg(X, Y)H = 0$ thus $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$.

Conversely, let $Bh(X, Y) = 0$ for any $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Then for any $X, Y \in \Gamma(D)$, we have $Bh(X, Y) = 0$, which implies that D defines a totally geodesic foliation in M. Now let $X, Y \in \Gamma(D')$. Then from Eq. (29), we have $A_{\omega Y} X = -T\nabla_X Y - Bh(X, Y)$ and using Lemma 4.4, we obtain $TA_{\omega Y} X + \omega A_{\omega Y} X = -Bh(X, Y)$. Thus on comparing the tangential component on both sides, we get $A_{wY} X = 0$, which implies that $A_{wY} X \in \Gamma(D')$, hence by using Theorem 3.5, the distribution D' defines a totally geodesic foliation in M. This completes the proof. □

5. Minimal generic lightlike submanifolds

In [4], Duggal and Bejancu defined a minimal lightlike submanifold M by considering M to be a hypersurface of a 4-dimensional Minkowski space. Later, the general definition of a minimal lightlike submanifold of a semi-Riemannian manifold was given by Bejan and Duggal [1] as follows:

Definition 7. A lightlike submanifold $(M, g, S(TM))$ isometrically immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be minimal if $h^s = 0$ on $Rad(TM)$ and trace $h = 0$, where trace is written with respect to g restricted to $S(TM)$.

Remark 5.1. One may note that Definition 7 is independent of the choice of $S(TM)$ and $S(TM^{\perp})$ but it depends on $tr(TM)$. Further, minimal lightlike submanifolds have been dealt in detail by Duggal and Jin in [12] and Kumar in [18].

Example 5.2. Let $(\bar{M}, \bar{g}) = (R^{10}_2, \bar{g})$ be a semi-Riemannian product manifold with signature $(-, -, +, +, +, +, +, +, +, +)$ with respect to the canonical basis $(\partial x_1, \partial x_2, \partial x_3, \partial x_4, \partial x_5, \partial x_6, \partial x_7, \partial x_8, \partial x_9, \partial x_{10})$.

Let M be a submanifold of R^{10}_2 given by

$$x_1 = u_1, \quad x_2 = u_2, \quad x_3 = u_1, \quad x_4 = u_3, \quad x_5 = u_4 + u_5,$$

$$x_6 = u_4 + u_5, \quad x_7 = \cos u_6 \cosh u_7, \quad x_8 = \cos u_6 \sinh u_7, \quad x_9 = \sin u_6 \cosh u_7,$$
$$x_{10} = \sin u_6 \sinh u_7, \text{ where } u_4, u_6 \in R - \{\tfrac{n\pi}{2}, \ n \in Z\}.$$

Then TM is spanned by

$$\begin{aligned} U_1 &= \partial x_1 + \partial x_3, \ U_2 = \partial x_2, \ U_3 = \partial x_4, \ U_4 = \partial x_5 + \partial x_6, \ U_5 = \partial x_5 + \partial x_6, \\ U_6 &= -\sin u_6 \cosh u_7 \partial x_7 - \sin u_6 \sinh u_7 \partial x_8 + \cos u_6 \cosh u_7 \partial x_9 \\ &\quad + \cos u_6 \sinh u_7 \partial x_{10}, \\ U_7 &= \cos u_6 \sinh u_7 \partial x_7 + \cos u_6 \cosh u_7 \partial x_8 + \sin u_6 \sinh u_7 \partial x_9 \\ &\quad + \sin u_6 \cosh u_7 \partial x_{10}. \end{aligned}$$

Clearly M is a 1-lightlike submanifold with $Rad(TM) = Span\{U_1\}$ and $FU_1 = U_2 + U_3 \in \Gamma(S(TM))$. Moreover $FU_4 = U_5$ therefore $D_0 = \{U_4, U_5\}$. Next we see that FU_6 and FU_7 are orthogonal to TM and therefore we have $S(TM^\perp) = \{FU_6, FU_7\}$. Thus we conclude that M is a proper generic lightlike submanifold of R_2^{10}. The lightlike transversal bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{2}\{-\partial x_1 + \partial x_3\}.$$

Since $FN_1 = -\frac{1}{2}Z_2 - \frac{1}{2}Z_3$, then $ltr(TM) = \{N_1\}$. Now by direct calculations, using the Gauss and Weingartan formulae, we obtain

$$\begin{gathered} h^s(Y, U_1) = h^s(Y, U_2), \quad h^s(Y, U_3) = 0, \\ h^s(Y, U_4) = 0, \quad h^s(Y, U_5) = 0, \quad \forall\ Y \in \Gamma(TM), \\ h^s(U_6, U_6) = \left(\frac{1}{1 + 2\sinh^2 u_7}\right) FU_7, \\ h^s(U_7, U_7) = -\left(\frac{1}{1 + 2\sinh^2 u_7}\right) FU_7. \end{gathered}$$

Thus, the induced connection is a metric connection and M is not totally geodesic, but it is a proper minimal generic lightlike submanifold of R_2^{10}.

Theorem 5.3. *Let M be a totally umbilical generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is minimal if and only if M is totally geodesic.*

Proof. Let M be minimal. Then $h^s(X, Y) = 0$ for any $X, Y \in \Gamma(Rad(TM))$. Since M is totally umbilical therefore $h^l(X, Y) = H^l g(X, Y) = 0$ for any $X, Y \in \Gamma(Rad(TM))$. Now we take an orthonormal basis $\{e_1, e_2, e_3, \ldots, e_{m-r}\}$ of $S(TM)$, then from Eq. (21), we get

$$traceh(e_1, e_2) = \sum_{i=1}^{m-r} \{\varepsilon_i g(e_i, e_i) H^l + \varepsilon_i g(e_i, e_i) H^s\} = (m - r)H^l + (m - r)H^s.$$

Since M is minimal and $ltr(TM) \cap S(TM^\perp) = 0$, we get $H^l = 0$ and $H^s = 0$. Hence M is totally geodesic. The converse part follows directly. □

Theorem 5.4. *Let M be a totally umbilical generic lightlike submanifold of a semi-Riemannian product manifold $\bar{M}$. Then M is minimal if and only if for $V_p \in \Gamma(S(TM^{\perp}))$, one has*

$$trace A_{V_p} = 0 \quad and \quad trace A^*_{\xi_k} = 0 \quad on \quad D_0 \perp FS(TM^{\perp}),$$

where $k \in \{1, 2, \ldots, r\}$ and $p \in \{1, 2, \ldots, m-r\}$.

Proof. Since M is totally umbilical, therefore using Eq. (21), it is clear that $h^s(X, Y) = 0$ for $X, Y \in \Gamma(Rad(TM))$. Now using the definition of generic lightlike submanifolds, we have

$$trace\ h|_{S(TM)} = \sum_{i=1}^{2p} h(Y_i, Y_i) + \sum_{j=1}^{r} h(F\xi_j, F\xi_j) + \sum_{j=1}^{r} h(FN_j, FN_j) + \sum_{l=1}^{m-2(r+p)} h(FV_l, FV_l),$$

where $2p = \dim(D_0)$, $r = \dim(Rad(TM))$ and $m-2(r+p) = \dim(FS(TM^{\perp}))$. Again using Eq. (20), we obtain $h(F\xi_j, F\xi_j) = h(FN_j, FN_j) = 0$. Thus the above equation reduces to

$$\begin{aligned} trace\ h|_{S(TM)} &= \sum_{i=1}^{2p} h(Y_i, Y_i) + \sum_{l=1}^{m-2(r+p)} h(FV_l, FV_l) \\ &= \sum_{i=1}^{2p} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(Y_i, Y_i), \xi_k) N_k \\ &\quad + \sum_{i=1}^{2p} \frac{1}{m-2(r+p)} \sum_{p=1}^{m-2(r+p)} \bar{g}(h^s(Y_i, Y_i), V_p) V_p \\ &\quad + \sum_{l=1}^{m-2(r+p)} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(h^l(FV_l, FV_l), \xi_k) N_k \\ &\quad + \sum_{l=1}^{m-2(r+p)} \frac{1}{m-2(r+p)} \sum_{p=1}^{m-2(r+p)} \bar{g}(h^s(FV_l, FV_l), V_p) V_p, \end{aligned} \tag{43}$$

where $\{V_1, V_2, \ldots, V_{m-2(r+p)}\}$ is an orthonormal basis of $S(TM^{\perp})$. Using Eqs. (10) and (14) in Eq. (43), we obtain

$$\begin{aligned} trace\ h|_{S(TM)} &= \sum_{i=1}^{2p} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} Y_i, Y_i) N_k \\ &\quad + \sum_{i=1}^{2p} \frac{1}{m-2(r+p)} \sum_{p=1}^{m-2(r+p)} \bar{g}(A_{V_p} Y_i, Y_i) V_p \end{aligned}$$

$$+ \sum_{l=1}^{m-2(r+p)} \frac{1}{r} \sum_{k=1}^{r} \bar{g}(A^*_{\xi_k} FV_l, FV_l) N_k$$
$$+ \sum_{l=1}^{m-2(r+p)} \frac{1}{m-2(r+p)} \sum_{p=1}^{m-2(r+p)} \bar{g}(A_{V_p} FV_l, FV_l) V_p.$$

Thus trace $h|_{S(TM)} = 0$ if and only if trace $A_{V_p} = 0$ and trace $A_{\xi_k^*} = 0$ on $D_0 \perp FS(TM^{\perp})$, which proves the theorem. □

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