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RIEMANN SOLITONS ON (κ, μ)-ALMOST COSYMPLECTIC MANIFOLDS

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Abstract. In this paper, we study almost cosymplectic manifolds with nullity distributions admitting Riemann solitons and gradient almost Riemann solitons. First, we consider Riemann soliton on (κ, μ)-almost cosymplectic manifold M with $\kappa < 0$ and we show that the soliton is expanding with $\lambda = \frac{\kappa}{2n-1}(4n-1)$ and M is locally isometric to the Lie group G_ρ. Finally, we prove the non-existence of gradient almost Riemann soliton on a (κ, μ)-almost cosymplectic manifold of dimension greater than 3 with $\kappa < 0$.

1. Introduction

In 2016, Hirică and Udrişte [10] introduced and studied the notion of Riemann soliton as an analog of Ricci soliton. Since then, it attracted many attentions in differential geometry of almost contact Riemannian geometry. A Riemann soliton is regarded as the generalization of the space of constant curvature, and also is a special solution to Riemann flow (see [15, 16]). A solution $g(t)$ of the non-linear evolution PDE:

$$\frac{\partial}{\partial t} G(t) = -2R(g(t)), \quad t \in \mathbb{R}$$

is called the *Riemann flow*, where $G = \frac{1}{2} g \odot g$, R is the Riemann curvature tensor associated to the metric g and $\odot$ is Kulkarni-Nomizu product. Some results in the Riemann flow resembles the case of Hamilton's Ricci flow [9] (for details, see [16]). A Riemannian manifold (M, g) is called a *Riemann soliton* if there are a smooth vector field V and a scalar $\lambda \in \mathbb{R}$ such that

$$2R + \lambda g \odot g + g \odot \pounds_V g = 0 \tag{1}$$

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on M, where $\pounds_V g$ is the Lie derivative of the metric g and $\odot$ is Kulkarni-Nomizu product defined by (see Besse [1])

$$(p \odot q)(X, Y, U, W) = p(X, W)q(Y, U) + p(Y, U)q(X, W) - p(X, U)q(Y, W) - p(Y, W)q(X, U). \tag{2}$$

The Riemann soliton also corresponds as a fixed point of the Riemann flow, and they can be viewed as a dynamical system, on the space of Riemannian metric modulo diffeomorphism. A Riemann soliton is said to be shrinking, steady, and expanding accordingly as λ is negative, zero, and positive, respectively. If the potential vector field V is the gradient of some function u on M, then the Riemann soliton equation becomes

$$R + \frac{1}{2}\lambda(g \odot g) + (g \odot Hess\ u) = 0, \tag{3}$$

where, $Hess\ u$ denotes the Hessian of the smooth function u and characterizes what is called a *gradient Riemann soliton.* A Riemann soliton on any compact Riemannian manifold is always a gradient Riemann soliton. If the potential vector field V vanishes identically, then the Riemann soliton becomes trivial, and in this case manifold is of constant sectional curvature. If λ in equations (1) and (3) is a smooth function, then g is called almost Riemann soliton and almost gradient Riemann soliton, respectively. On a Sasakian manifold, the concept of Riemann soliton becomes the Sasaki-Riemann soliton. In [10], it is proved that if Sasakian manifold admits a Riemann soliton whose soliton vector field V is pointwise collinear with ξ (or a gradient Riemann soliton and potential function is harmonic), then it is Sasakian-space-form. Later, the characterization of Riemann soliton in terms of infinitesimal harmonic transformation was carried out by Stepanov and Tsyganok [14]. The problem of studying Riemann solitons in the context of contact geometry was initiated by Devaraja et al. [6]. In particular, they studied Riemann soliton (g, V) with V as contact vector field on a Sasakian manifold (M, g) and proved that in this case the manifold M is either of constant curvature $+1$ (and V is Killing) or D-homothetically fixed η-Einstein manifold (and V leaves the structure tensor ϕ invariant). Further, they also shown that if a compact K-contact manifold whose metric g is a gradient almost Riemann soliton, then it is Sasakian and isometric to a unit sphere S^{2n+1}. More recently, Venkatesha et al. [17] classified certain class of almost Kenmotsu manifold which admits a Riemann soliton and almost gradient Riemann soliton.

On the other hand, it is remark that one of the another important class of research in almost contact manifolds is almost cosymplectic manifolds. Nowadays, many attention have been paid towards the study of geometry of almost cosymplectic manifolds. By an *almost cosymplectic manifold*, we mean a smooth manifold of $(2n + 1)$-dimension equipped with a closed 1-form η and a closed 2-form ω such that $\eta \wedge \omega^n$ is a volume form. The concept was first introduced by Goldberg and Yano [8] in 1969. The products of almost Kaehler manifolds and the real line $\mathbb{R}$ or the circle S^1 are the simplest examples of almost cosymplectic manifolds. At this point, we refer the papers [5, 11, 12] and the references therein to reader for a wide and detailed overview of the results on almost cosymplectic manifolds.

The present paper is organized as follows: Section 2 is concerned with the basic formulas and properties of almost cosymplectic manifolds. In Section 3, the Riemann soliton on a (κ,μ)-almost cosymplectic manifold is being considered and obtained some interesting results. In the last section, we prove the non-existence of gradient almost Riemann soliton on a (κ,μ)-almost cosymplectic manifold of dimension greater than 3 with $\kappa < 0$.

2. Almost cosymplectic manifolds

Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold. An *almost contact structure* [2] on M^{2n+1} is a triple (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is a characteristic or Reeb vector field and η is a 1-form satisfying

$$\phi^2 = -I + \eta\otimes\xi, \qquad \eta(\xi) = 1, \qquad \phi\xi = 0, \qquad \eta\cdot\phi = 0. \tag{4}$$

The first and one of the remaining three relations in (4) imply the other two relations in (4). In general, a smooth manifold M^{2n+1} together with the almost contact structure (ϕ,ξ,η) is said to be an almost contact manifold. An almost contact structure (ϕ,ξ,η) is said to be normal if the corresponding complex structure J on $M^{2n+1}\times\mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$, and f is a smooth function on $M^{2n+1}\times\mathbb{R}$.

If an almost contact manifold (M^{2n+1},ϕ,ξ,η) admits a Riemannian metric g satisfying

$$g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{5}$$

for any vector fields X and Y, then the manifold is called an *almost contact metric manifold* and is denoted by $(M^{2n+1},\phi,\xi,\eta,g)$ or simply M^{2n+1}. Then from (4) and (5) it can be easily deduced that

$$g(\phi X, Y) = -g(X,\phi Y), \qquad g(X,\xi) = \eta(X).$$

The fundamental 2-form ω associate with the almost contact metric structure M^{2n+1} is defined by $\omega(X,Y) = g(X,\phi Y)$ for any vector fields X and Y.

An almost contact metric manifold M^{2n+1} is said to be an *almost cosymplectic manifold* [3, 8] if both η and ω are closed, that is,

$$d\eta = 0, \;\; d\omega = 0.$$

A normal almost cosymplectic manifold is a *cosymplectic manifold* and characterized, through Levi-Civita connection by $(\nabla_X\phi)Y = 0$, or equivalently, $\nabla\omega = 0$.

Let M^{2n+1} be an almost cosymplectic manifold. We define the operators h and h', $h := \frac{1}{2}\mathcal{L}_\xi \phi$ and $h' := h \cdot \phi$ where $\mathcal{L}_\xi$ is Lie-derivative along ξ. The $(1,1)$-type tensor fields h and h' are symmetric and satisfy:

$$h\xi = h'\xi = 0, \quad tr(h) = tr(h') = 0, \quad h \cdot \phi = -\phi \cdot h, \tag{6}$$

$$\nabla_\xi \phi = 0, \quad \nabla \xi = h', \quad div\xi = 0, \tag{7}$$

$$S(\xi, \xi) + ||h||^2 = 0. \tag{8}$$

Here tr and div denote the trace and divergence operators with respect to the metric g, respectively. The Ricci tensor S is defined by $S(X,Y) = tr\{Z \to R(Z,X)Y\}$, and the Ricci operator Q is defined by $g(QX,Y) = S(X,Y)$. If, in addition, we put $l = R(\cdot, \xi)\xi$, then we also have

$$\phi l \phi - l = 2h^2, \tag{9}$$

where the Riemannian curvature tensor R is defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}$.

On an almost cosymplectic manifold M^{2n+1}, using the condition $h \cdot \phi = -\phi \cdot h$, we obtain $(\pounds_\xi g)(X,Y) = 2g(h'X,Y)$. This means that the Reeb vector field is Killing if and only if the (1, 1)-type tensor field h vanishes.

In addition, an almost cosymplectic manifold M^{2n+1} is said to be a (κ, μ)-*almost cosymplectic manifold* [7] if the characteristic vector field ξ belongs to (κ, μ)-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa\left(\eta(Y)X - \eta(X)Y\right) + \mu\left(\eta(Y)hX - \eta(X)hY\right) \tag{10}$$

for any vector fields X and Y, where κ, μ are smooth functions on M^{2n+1} and h is the $(1,1)$-tensor field defined by $2h := \pounds_\xi \phi$. Using (10) we have $l = -\kappa\phi^2 + \mu h$, and taking this into (9) gives that

$$h^2 = \kappa \phi^2. \tag{11}$$

By (11), we find easily that $\kappa \leq 0$ and $\kappa = 0$ if and only if M^{2n+1} is a cosymplectic manifold, thus in the following we always suppose $\kappa < 0$. Moreover, if $\mu = 0$, then M is called as an $N(\kappa)$-*almost cosymplectic manifold* (see Dacko [4]) and in such a case we have the following:

Theorem 2.1 ([4, Theorem 4]). *An $N(\kappa)$-almost cosymplectic manifold for some $\kappa < 0$ is locally isomorphic to a solvable non-nilpotent Lie group G_ρ endowed with the almost cosymplectic structure as follows*:

$$\phi E_0 = 0, \qquad \phi E_i = E_{n+i}, \qquad \phi E_{n+i} = -E_i,$$
$$\xi = E_0, \qquad g(E_i, E_j) = \delta_{ij}, \qquad \eta(\cdot) = g(\cdot, \xi),$$

where $\{E_i\}_{i=1}^{2n+1}$ is the basis of Lie algebra of G_ρ and $\rho = \sqrt{-\kappa}$.

Further, on a (κ, μ)-almost cosymplectic manifold M^{2n+1} of dimension greater than 3 with $\kappa < 0$ the Ricci operator is given by: [18]

$$Q = \mu h + 2n\kappa \eta \otimes \xi, \tag{12}$$

where κ is a non-zero constant and μ is a smooth function satisfying $d\mu\wedge\eta=0$. It follows from (12) that

$$S(\xi,\xi)=2n\kappa. \tag{13}$$

If κ and μ are constants, then the relations (12) and (13) are still valid on M^{2n+1} of greater than or equal to 3 (see Endo [7]).

Now, we state the relations held on every $(2n+1)$-dimensional (κ,μ)-almost cosymplectic manifold M^{2n+1} (see Proposition 9 of [13] for the case $\alpha=0$ and $\nu=0$) which will be used in next sections:

$$(\nabla_\xi h)X=\mu h'XY, \tag{14}$$

$$(\nabla_X\phi)Y=g(hX,Y)\xi-\eta(Y)hX, \tag{15}$$

$$\begin{aligned}(\nabla_X h)Y-(\nabla_Y h)X&=\kappa\{2g(\phi X,Y)\xi-\eta(X)\phi Y+\eta(Y)\phi X\}\\&\quad+\mu\{\eta(X)h'Y-\eta(Y)h'X\},\end{aligned} \tag{16}$$

$$(\nabla_X h')Y-(\nabla_Y h')X=\kappa\{\eta(Y)X-\eta(X)Y\}+\mu\{\eta(Y)hX-\eta(X)hY\} \tag{17}$$

for any vector fields X and Y.

3. Riemann solitons on (κ,μ)-almost cosymplectic manifolds

In this section, we consider a Riemann soliton (g,V) on (κ,μ)-almost cosymplectic manifold M^{2n+1} with $\kappa<0$ and first we prove the following:

Lemma 3.1. *Let M^{2n+1} be a (κ,μ)-almost cosymplectic manifold with κ (<0) and μ as constants. If (g,V) is a Riemann soliton, then*

$$\mu^2=\lambda(2n-1)+\kappa(4n-1). \tag{18}$$

Proof. By using the definition of Kulkarni-Nomizu product in the Riemann soliton equation (1), it follows that

$$\begin{aligned}&2R(X,Y,U,W)+2\lambda\{g(X,W)g(Y,U)-g(X,U)g(Y,W)\}\\&+\{g(X,W)(\pounds_V g)(Y,U)+g(Y,U)(\pounds_V g)(X,W)\\&-g(X,W)(\pounds_V g)(Y,W)-g(Y,W)(\pounds_V g)(X,U)\}=0.\end{aligned} \tag{19}$$

Contracting the preceding equation over X and W yields

$$(\pounds_V g)(Y,U)=-\frac{2}{2n-1}\{(2n\lambda+divV)g(Y,U)+S(Y,U)\}. \tag{20}$$

Differentiating (20) along X gives

$$(\nabla_X\pounds_V g)(Y,Z)=-\frac{2}{2n-1}\{X(divV)g(Y,Z)+(\nabla_X S)(Y,Z)\}. \tag{21}$$

Contracting (20) and making use of (12) yields

$$divV=-\frac{1}{2}\{\lambda(2n+1)+\kappa\}. \tag{22}$$

Fetching (22) in (21) we get

$$(\nabla_X \pounds_V g)(Y, Z) = -\frac{2}{2n-1}(\nabla_X S)(Y, Z). \tag{23}$$

We can easily deduce that

$$(\nabla_X \pounds_V g)(Y, Z) = g((\pounds_V \nabla)(X, Y), Z) + g((\pounds_V \nabla)(X, Z), Y), \tag{24}$$

with the help of the following identity [19]:

$$\begin{aligned} &(\pounds_V \nabla_X g - \nabla_X \pounds_V g - \nabla_{[V,X]} g)(Y, Z) \\ &= -g((\pounds_V \nabla)(X, Y), Z) - g((\pounds_V \nabla)(X, Z), Y). \end{aligned}$$

Moreover, by virtue of (24) we derive

$$\begin{aligned} &g((\pounds_V \nabla)(X, Y), Z) \\ &= \frac{1}{2}(\nabla_X \pounds_V g)(Y, Z) + \frac{1}{2}(\nabla_Y \pounds_V g)(Z, X) - \frac{1}{2}(\nabla_Z \pounds_V g)(X, Y). \end{aligned} \tag{25}$$

Using (23) in (25) we get

$$\begin{aligned} &g((\pounds_V \nabla)(X, Y), Z) \\ &= -\frac{1}{2n-1}\{(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) - (\nabla_Z S)(X, Y)\}. \end{aligned} \tag{26}$$

In terms of Ricci tensor, (12) can be equivalently written as

$$S(X, Y) = \mu g(hX, Y) + 2n\kappa\eta(X)\eta(Y). \tag{27}$$

Taking covariant derivative to (27) and recalling (7) leads to

$$(\nabla_Z S)(X, Y) = \mu g((\nabla_Z h)X, Y) + 2n\kappa\{g(X, h'Z)\eta(Y) + g(Y, h'Z)\eta(X)\}. \tag{28}$$

Plugging (28) in (26) yields

$$\begin{aligned} &g((\pounds_V \nabla)(X, Y), Z) \\ &= \frac{\mu}{2n-1}\{g((\nabla_Z h)X - (\nabla_X h)Z, Y) - g((\nabla_Y h)Z, X)\} \\ &\quad - \frac{4n\kappa}{2n-1}\{g(h'X, Y)\eta(Z)\}. \end{aligned} \tag{29}$$

Now, by recalling (16) we find that

$$\begin{aligned} &(X, Y) \\ &= \frac{\mu}{2n-1}\Big[-\kappa\{2\eta(Y)\phi X + \eta(X)\phi Y + g(\phi X, Y)\xi\} \\ &\quad - \mu\{\eta(X)h'Y - g(h'X, Y)\xi\} - (\nabla_Y h)X\Big] - \frac{4n\kappa}{2n-1} g(h'X, Y)\xi. \end{aligned} \tag{30}$$

Setting $Y = \xi$ in (30) deduces to

$$(\pounds_V \nabla)(X, \xi) = \frac{\mu}{2n-1}\{-2\kappa\phi X - \mu h'X\}. \tag{31}$$

Now, differentiating (31) along Y and using (17) and (15) we obtain

$$\begin{aligned}&(\nabla_Y \pounds_V \nabla)(X,\xi) + (\pounds_V \nabla)(X, h'Y)\\ =& -\frac{2\mu\kappa}{2n-1}\{g(hY,X)\xi - \eta(X)hY\} - \frac{\mu^2}{2n-1}(\nabla_Y h')X.\end{aligned}$$

Substituting this in the well known identity

$$(\pounds_V R)(X,Y)Z = (\nabla_X \pounds_V \nabla)(Y,Z) - (\nabla_Y \pounds_V \nabla)(X,Z), \tag{32}$$

we have

$$\begin{aligned}(\pounds_V R)(X,Y)\xi =& \frac{2\mu\kappa}{2n-1}\{\eta(Y)hX - \eta(X)hY\}\\ &+ \frac{\mu^2}{2n-1}\{(\nabla_Y h')X - (\nabla_X h')Y\}\\ &+ (\pounds_V \nabla)(X,h'Y) - (\pounds_V \nabla)(Y,h'X).\end{aligned} \tag{33}$$

Now, plugging $Y=\xi$ in (33) and making use of the last relation, (6), (17) and (31) we find

$$(\pounds_V R)(X,\xi)\xi = \frac{1}{2n-1}\{4\kappa\mu hX + 2\kappa\mu^2\phi^2 X - \mu^3 hX\}. \tag{34}$$

Now contracting (34) with respect to X and using the second relation of (6) with the fact that $tr\phi^2 = -2n$ we have

$$\begin{aligned}(\pounds_V S)(\xi,\xi) &= \frac{1}{2n-1}\{2\kappa\mu^2(-2n)\}\\ &= -\frac{4n\kappa\mu^2}{2n-1}.\end{aligned} \tag{35}$$

From (13), we find $(\pounds_V S)(\xi,\xi) = -4n\kappa\eta(\pounds_V\xi)$. Taking $Y=U=\xi$ in (20) gives $\eta(\pounds_V\xi) = \frac{1}{2n-1}\{(2n\lambda + divV) + 2n\kappa\}$. Thus, we have

$$(\pounds_V S)(\xi,\xi) = -\frac{4n\kappa}{2n-1}\{(2n\lambda + divV) + 2n\kappa\}. \tag{36}$$

By virtue of (35), (36) and (22) we get (18). This completes the proof. □

Theorem 3.2. *Let (M^{2n+1},g) be a (κ,μ)-almost cosymplectic manifold with κ (<0) and μ as constants. If g is a Riemann soliton, then the soliton is expanding with $\lambda = \frac{\kappa}{2n-1}(4n-1)$ and M^{2n+1} is locally isometric to the above Lie group G_ρ.*

Proof. Taking $Y=\xi$ in (10) gives

$$R(X,\xi)\xi = \kappa\{X - \eta(X)\xi\} + \mu hX.$$

Now taking Lie-derivative to the above equation along V gives

$$\begin{aligned}&(\pounds_V R)(X,\xi)\xi + R(X,\pounds_V\xi)\xi + R(X,\xi)\pounds_V\xi\\ =& -\kappa\{(\pounds_V\eta)(X)\xi + \eta(X)\pounds_V\xi\} + \mu(\pounds_V h)X.\end{aligned} \tag{37}$$

Since $\eta(X) = g(X, \xi)$, we have

$$(\pounds_V \eta)(X) = g(\pounds_V \xi, X) - \frac{2}{2n-1}\{(2n\lambda + divV) + 2n\kappa\}\eta(X).$$

Substituting this in (37) and using (10) we get

$$\begin{aligned}&(\pounds_V R)(X, \xi)\xi \\ =\ & -2\kappa\eta(\pounds_V \xi)X - 2\mu\eta(\pounds_V \xi)hX + \mu\eta(X)h\pounds_V \xi + \mu g(hX, \pounds_V \xi)\xi \\ &+ \frac{2\kappa}{2n-1}\{(2n\lambda + divV) + 2n\kappa\}\eta(X)\xi + \mu(\pounds_V h)X.\end{aligned}$$

Since $\eta(\pounds_V \xi) = \frac{1}{2n-1}\{(2n\lambda + divV) + 2n\kappa\}$ we have

$$\begin{aligned}(\pounds_V R)(X, \xi)\xi = &\frac{2\kappa}{2n-1}\{2n\lambda + divV + 2n\kappa\}(-X + \eta(X)\xi) \\ &- \frac{2\mu}{2n-1}(2n\lambda + divV + 2n\kappa)hX \\ &+ \mu\{\eta(X)h\pounds_V \xi + g(hX, \pounds_V \xi)\xi + (\pounds_V h)X\}.\end{aligned} \tag{38}$$

Comparing (38) with (34) and using (18) we have

$$\frac{4\kappa\mu hX}{2n-1} = \mu\{\eta(X)h\pounds_V \xi + g(hX, \pounds_V \xi)\xi + (\pounds_V h)X\}. \tag{39}$$

Now replacing X by hX in (39) gives one equation and operating (39) by h gives another equation. Adding the obtained two equations gives

$$\begin{aligned}&\frac{8\kappa\mu h^2 X}{2n-1} \\ &= \kappa\mu\{-\eta(X)\pounds_V \xi - g(\pounds_V \xi, X)\xi + \frac{2}{2n-1}(2n\lambda + divV + 2n\kappa)\eta(X)\xi\} \\ &\quad + \mu\{(\pounds_V h)hX + h(\pounds_V h)X\}.\end{aligned} \tag{40}$$

From (11), we have

$$h^2 X = \kappa\{-X + \eta(X)\xi\}.$$

Taking Lie-derivative to this gives

$$\begin{aligned}&(\pounds_V h)hX + h(\pounds_V h)X \\ &= \kappa\{g(\pounds_V \xi, X)\xi - \frac{2}{2n-1}(2n\lambda + divV + 2n\kappa)\eta(X)\xi + \eta(X)\pounds_V \xi\}.\end{aligned}$$

Substituting the above equation in (40) gives

$$8\mu\kappa h^2 X = 0.$$

Tracing it gives $8\mu\kappa||h^2|| = 0$. But, from (8) and (13), we have

$$8\mu\kappa(-2n\kappa) = 0.$$

Since $\kappa < 0$, we have $\mu = 0$. Thus from (18) we have

$$\lambda = -\frac{\kappa}{2n-1}(4n-1).$$

Thus $\lambda > 0$ and so expanding and the proof finishes by employing Theorem 2.1. □

4. Gradient almost Riemann solitons on (κ,μ)-almost cosymplectic manifolds

Suppose that in a (κ,μ)-almost cosymplectic manifold M, the metric g admits a gradient almost Riemann soliton. Then the contraction of soliton equation defined by (3) with the potential function u can be exhibited as

$$\nabla_Y Du = -\frac{1}{2n-1}QY - \frac{1}{2n-1}(2n\lambda + \Delta u)Y, \tag{41}$$

where D is the gradient operator of g on M, and $\Delta u = divDu$, and Δ is the Laplacian operator. By straightforward computations, using the well known expression of the curvature tensor:

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[x,Y]}Z$$

and the repeated use of equation (41) gives

$$\begin{aligned} R(X,Y)Du = {} & \frac{1}{2n-1}\{(\nabla_Y Q)X - (\nabla_X Q)Y\} \\ & + \frac{1}{2n-1}\{Y(2n\lambda + \Delta u)X - X(2n\lambda + \Delta u)Y\}. \end{aligned} \tag{42}$$

Applying equation (12) in (42) gives that

$$\begin{aligned} & R(X,Y)Du \\ = {} & \frac{1}{2n-1}\{\mu((\nabla_Y h)X - (\nabla_X h)Y) + 2n\kappa(\eta(X)h'Y - \eta(Y)h'X) \\ & + Y(\mu)hX - X(\mu)hY\} \\ & + \frac{1}{2n-1}\{Y(2n\lambda + \Delta u)X - X(2n\lambda + \Delta u)Y\}. \end{aligned}$$

Using (16) in the above equation, we obtain

$$\begin{aligned} & (2n-1)R(X,Y)Du \\ = {} & \kappa\mu(\eta(X)\phi Y - \eta(Y)\phi X + 2g(X,\phi Y)\xi) \\ & + (2n\kappa - \mu^2)(\eta(X)h'Y - \eta(Y)h'X) \\ & + \{Y(\mu)hX - X(\mu)hY\} + \{Y(2n\lambda + \Delta u)X - X(2n\lambda + \Delta u)Y\}. \end{aligned} \tag{43}$$

Taking inner product of the previous equation with ξ we obtain

$$\begin{aligned} & (2n-1)g(R(X,Y)Du,\xi) \\ = {} & 2\kappa\mu g(X,\phi Y) + \{Y(2n\lambda + \Delta u)\eta(X) - X(2n\lambda + \Delta u)\eta(Y)\}. \end{aligned} \tag{44}$$

From (10), we have

$$\begin{aligned} & g(R(X,Y)Du,\xi) \\ = {} & \kappa\{(Yu)\eta(X)) - (Xu)\eta(Y)\} + \mu\{g(hY,Du)\eta(X) - g(hX,Du)\eta(Y)\}. \end{aligned} \tag{45}$$

Replacing X with ξ in (44) and (45) and then comparing with other yields that

$$\frac{1}{2n-1}D(2n\lambda + \Delta u) = \kappa\{Du - (\xi u)\xi\} + \mu hDu + \frac{1}{2n-1}\xi(2n\lambda + \Delta u)\xi. \tag{46}$$

Contracting (43) over X, we get

$$S(Y, Du) = \frac{2n}{2n-1}Y(2n\lambda + \Delta u),$$

where we have used $d\mu \wedge \eta = 0$. Making use of (12) in the foregoing equation gives that

$$\frac{2n}{2n-1}Y(2n\lambda + \Delta u) = \mu hDu + 2n\kappa\xi(u)\xi. \tag{47}$$

In the contrast of above equation with (46) we obtain

$$2n\kappa(Du - 2\xi(u)\xi) + (2n-1)\mu hDu + \frac{2n}{2n-1}\xi(2n\lambda + \Delta u)\xi = 0. \tag{48}$$

Taking inner product of (47) with ξ we get

$$\kappa\xi(u) = \frac{1}{2n-1}\xi(2n\lambda + \Delta u). \tag{49}$$

Using (49) in (48) we have

$$2n\kappa\{-Du + \xi(u)\xi\} = (2n-1)\mu hDu. \tag{50}$$

Moreover, setting $X = \phi X$ and $Y = \phi Y$ in the equation (44), and noting that $g(R(\phi X, \phi Y)Du, \xi) = 0$ (it follows from (10)) and $h\phi = \phi h$ we have $2\kappa\mu g(\phi X, Y) = 0$. Replacing X by ϕX gives

$$2\kappa\mu g(\phi^2 X, Y) = 0.$$

Taking into the account of $tr(\phi^2) = -2n$, the contraction of foregoing equation with respect to X and Y gives that $2\kappa\mu = 0$. Since $\kappa < 0$, we have $\mu = 0$. So the equation (50) gives

$$Du - \xi(u)\xi = 0. \tag{51}$$

Taking covariant differentiation of (51) along an arbitrary vector field X on M^{2n+1} together with (4), (7) entails that

$$\nabla_X Du = X(\xi(u))\xi + \xi(u)h'X.$$

Comparing the value of QX from relation (12) and the last equation, we compute

$$(2n\lambda + \Delta u)X + (2n-1)X(\xi(u))\xi + (2n-1)\xi(u)h'X + 2n\kappa\eta(X)\xi = 0.$$

Tracing this over X we have

$$(2n+1)(2n\lambda + \Delta u)X + (2n-1)\xi(\xi(u)) + 2n\kappa = 0. \tag{52}$$

Substituting Y by ξ in (41) and then taking scalar product with ξ yields $(2n-1)\xi(\xi(u)) = -2n\kappa - (2n\lambda + \Delta u)$, which together with (52) gives

$$2n\lambda + \Delta u = 0. \tag{53}$$

Making use of (53) in (44) and taking into fact that $\mu = 0$ we obtain

$$g(R(X,Y)Du, \xi) = 0. \tag{54}$$

On the other hand, it follows from (10) that

$$g(R(X,Y)Du, \xi) = \kappa(\eta(Y)X(u) - \eta(X)Y(u)). \tag{55}$$

In view of $\kappa < 0$, from (54) and (55) we have

$$\eta(Y)X(u) - \eta(X)Y(u) = 0. \tag{56}$$

In view of $\mu = 0$, we obtain from (50) that $\xi(u) = 0$. Putting $X = \xi$ in the above relation and using $\xi(u) = 0$ we obtain that u is constant. From relation (41) and (53) we obtain $Q = 0$. It gives $\kappa = 0$. But, this contradicts the hypothesis that $\kappa < 0$. Thus, we arrive at the following:

Theorem 4.1. *There exist no gradient almost Riemann soliton on a (κ, μ)-almost cosymplectic manifold of dimension greater than 3 with $\kappa < 0$.*

As we pointed out earlier, if κ and μ are constants then (12) is still valid on M^{2n+1} of dimension ≥ 3. Thus, Theorem 4.1 implies the following:

Corollary 4.2. *There exist no gradient almost Riemann soliton on a (κ, μ)-almost cosymplectic manifold with κ (< 0) and μ as constants.*

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