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AREAS OF POLYGONS WITH VERTICES FROM LUCAS SEQUENCES ON A PLANE

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Abstract. Area problems for triangles and polygons whose vertices have Fibonacci numbers on a plane were presented by A. Shriki, O. Liba, and S. Edwards et al. In 2017, V. P. Johnson and C. K. Cook addressed problems of the areas of triangles and polygons whose vertices have various sequences. This paper examines the conditions of triangles and polygons whose vertices have Lucas sequences and presents a formula for their areas.

1. Introduction

Let F_n and L_n be the Fibonacci numbers and Lucas numbers defined as

$$\begin{aligned} F_0 &= 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n,\ n \geq 2, \\ L_0 &= 2,\ L_1 = 1,\ L_{n+2} = L_{n+1} + L_n,\ n \geq 2. \end{aligned}$$

The study of triangles for Fibonacci numbers and Lucas numbers is an interesting topic. A triangle whose individual sides have Fibonacci numbers as their lengths and whose area has a natural number is called a *Fibonacci triangle*. A conjecture is that a Fibonacci triangle whose three sides are $F_5 = 5, F_5 = 5$ and $F_6 = 8$ is the only Fibonacci triangle discovered thus far, and that there are no other Fibonacci triangles (see [2], [3]).

Meanwhile, studies on the area of a triangle whose vertices have Fibonacci numbers on a $\mathbb{R}^2$ plane were conducted. In 2015, Edwards presented the following problem in *The Fibonacci Quarterly* (see [1, B–1172]).

Show that the area of the triangle whose vertices have coordinates

$$(F_n, F_{n+k}),\ (F_{n+2k}, F_{n+3k}),\ (F_{n+4k}, F_{n+5k})$$

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is

$$\frac{5F_k^4L_k}{2} \text{ if } k \text{ is even and } \frac{F_k^2L_k^3}{2} \text{ if } k \text{ is odd.}$$

Also, find the area of the triangle whose vertices have coordinates

$$(L_n, L_{n+k}),\ (L_{n+2k}, L_{n+3k}),\ (L_{n+4k}, L_{n+5k}).$$

This problem can be said to be a triangle version of the problem for polygons $(F_1, F_2), (F_3, F_4), \ldots, (F_{2n-1}, F_{2n})$ whose vertices have Fibonacci numbers published in *The Fibonacci Quarterly* in 2015 (see [9, B–1167]). The result of B–1172 was presented by Johnson et al. (see [4]) and the result of B–1167 was obtained by Kwong [6]. In addition, Johnson and Cook [5] found the areas of triangles and polygons whose vertices have Pell numbers or Pell-Lucas numbers with the same indices and showed that Jacobsthal numbers and Jacobsthal-Lucas numbers with the same indices lie on a straight line.

In this paper, the areas of triangles and polygons such as (F_n, F_{m+k}), (F_{n+l}, F_{m+l}), (F_{n+2k}, F_{m+2l}) and (F_n, L_{m+k}), (F_{n+l}, L_{m+l}), (F_{n+2k}, L_{m+2l}), whose vertices are generalized, are calculated.

2. The Binet formula for Lucas sequences

Given two integers p and q, a function $f_{p,q}(n)$ is defined by the recurrence relations:

$$f_{p,q}(n+2) = p \cdot f_{p,q}(n+1) - q \cdot f_{p,q}(n)$$

for $n \geq 0$. Let $D = p^2 - 4q \neq 0$. Then $f_{p,q}(n)$ can be expressed as the follows with the Binet formula (see [7, p. 114]).

$$f_{p,q}(n) = A\alpha^n + B\beta^n,$$

where $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$, $\beta = \frac{p-\sqrt{p^2-4q}}{2}$ and

$$\begin{bmatrix} A \\ B \end{bmatrix} = \frac{1}{\beta - \alpha} \begin{bmatrix} \beta & -1 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} f_{p,q}(0) \\ f_{p,q}(1) \end{bmatrix}.$$

When $f_{p,q}(0) = 0$ and $f_{p,q}(1) = 1$, let $f_{p,q}(n)$ be $U_n(p,q)$, when $f_{p,q}(0) = 2$ and $f_{p,q}(1) = p$, let $f_{p,q}(n)$ be $V_n(p,q)$. In this case, $U_n(p,q)$ and $V_n(p,q)$ are called Lucas sequences. Then the general terms of $U_n(p,q)$ and $V_n(p,q)$ are as follows:

$$U_n(p,q) = \frac{\alpha^n - \beta^n}{\sqrt{D}},\ V_n(p,q) = \alpha^n + \beta^n.$$

As is well known, $U_n(1,-1)$ is a Fibonacci number F_n and $V_n(1,-1)$ is a Lucas number L_n. Table 1 shows the Lucas sequences for (p,q).

In this paper, when p and q are given, the following is defined.

$$\begin{aligned} &p \neq 0,\ q \neq 0, \\ &D := p^2 - 4q > 0, \\ &f(n) := f_{p,q}(n), \end{aligned}$$

Table 1. Table for specific names of Lucas sequences

(p,q)	U_n	V_n
$(1,-1)$	Fibonacci number: F_n	Lucas number: L_n
$(2,-1)$	Pell number: P_n	Pell-Lucas number: Q_n
$(1,-2)$	Jacobsthal number: J_n	Jacobsthal-Lucas number: j_n

$$U_n := U_n(p,q),$$
$$V_n := V_n(p,q).$$

In addition, let the two functions $f(n)$ and $g(n)$ have the same recurrence formula and be increasing functions. The general terms of these functions are expressed as follows:

$$f(n) = A_1\alpha^n + B_1\beta^n,\ g(n) = A_2\alpha^n + B_2\beta^n$$

for some real numbers A_1, B_1, A_2, B_2.

3. Area of a triangle whose vertices have Lucas sequences

Let there be three points, $P_1 = (f(n), g(m)), P_2 = (f(n+l), g(m+l))$ and $P_3 = (f(n+2l), g(m+2l))$. Calculate the cross product $\overrightarrow{T}$ of the two vertors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_2P_3}$. If $\overrightarrow{T} = \overrightarrow{0}$, then three points P_1, P_2 and P_3 are on a straight line, and if $\overrightarrow{T} \neq \overrightarrow{0}$, then a triangle will be formed when three points P_1, P_2 and P_3 have been connected and the area will be $\frac{\|\overrightarrow{T}\|}{2}$. Let $\overrightarrow{k} = (0,0,1)$.

$$\begin{aligned}
\overrightarrow{T} &= \begin{vmatrix} f(n+l)-f(n) & g(m+l)-g(m) \\ f(n+2l)-f(n+l) & g(m+2l)-g(m+l) \end{vmatrix} \overrightarrow{k} \\
&= \begin{vmatrix} A_1\alpha^n(\alpha^l-1)+B_1\beta^n(\beta^l-1) & A_2\alpha^m(\alpha^l-1)+B_2\beta^m(\beta^l-1) \\ A_1\alpha^{n+l}(\alpha^l-1)+B_1\beta^{n+l}(\beta^l-1) & A_2\alpha^{m+l}(\alpha^l-1)+B_2\beta^{m+l}(\beta^l-1) \end{vmatrix} \overrightarrow{k} \\
&= \begin{vmatrix} \alpha^l-1 & \beta^l-1 \\ \alpha^l(\alpha^l-1) & \beta^l(\beta^l-1) \end{vmatrix} \begin{vmatrix} A_1\alpha^n & A_2\alpha^m \\ B_1\beta^n & B_2\beta^m \end{vmatrix} \overrightarrow{k} \\
&= (\alpha\beta)^n(\alpha^l-1)(\beta^l-1) \begin{vmatrix} 1 & 1 \\ \alpha^l & \beta^l \end{vmatrix} \begin{vmatrix} A_1 & A_2\alpha^{m-n} \\ B_1 & B_2\beta^{m-n} \end{vmatrix} \overrightarrow{k}.
\end{aligned}$$

Hence,

$$\overrightarrow{T} = \sqrt{D}q^n(q^l - V_l + 1)U_l(A_2B_1\alpha^{m-n} - A_1B_2\beta^{m-n})\overrightarrow{k}. \tag{3.1}$$

From the above formula, the following theorem can be obtained.

Theorem 3.1. *The three points $P = (f(n), g(m)), Q = (f(n+l), g(m+l))$ and $R = (f(n+2l), g(m+2l))$ lie on a straight line if and only if $q^l - V_l + 1 = 0$ or $A_2B_1\alpha^{m-n} - A_1B_2\beta^{m-n} = 0$.*

Corollary 3.2. *The following three points lie on a straight line.*

(i) $P(J_n, J_m),\ Q(J_{n+2l}, J_{m+2l}),\ R(J_{n+4l}, J_{m+4l}),$

(ii) $P(j_n, j_m)$, $Q(j_{n+2l}, j_{m+2l})$, $R(j_{n+4l}, j_{m+4l})$,
(iii) $P(J_n, j_m)$, $Q(J_{n+2l}, j_{m+2l})$, $R(J_{n+4l}, j_{m+4l})$.

Proof. Since $q = 2$ and $V_l = 2^l + (-1)^l$ in Jocobsthal numbers J_n and Jacobsthal-Lucas numbers j_n, we obtain

$$q^{2l} - V_{2l} + 1 = 2^{2l} - (2^{2l} + 1) + 1 = 0.$$

Hence, three points P, Q and R lie on a straight line. □

Now we show the areas of the triangles associated with the Lucas sequences U_n and V_n.

Theorem 3.3. *Let $q^l - V_l + 1 \neq 0$ and $m > n$. Then the following are satisfied.*
(i) *The area S_1 of a triangle having three vertices (U_n, U_m), (U_{n+l}, U_{m+l}) and (U_{n+2l}, U_{m+2l}) is*

$$S_1 = \frac{1}{2}|q^n(q^l - V_l + 1)U_l U_{m-n}|.$$

(ii) *The area S_2 of a triangle having three vertices (V_n, V_m), (V_{n+l}, V_{m+l}) and (V_{n+2l}, V_{m+2l}) is*

$$S_2 = \frac{1}{2}|Dq^n(q^l - V_l + 1)U_l U_{m-n}| = D \cdot S_1.$$

(iii) *The area S_3 of a triangle having three vertices (U_n, V_m), (U_{n+l}, V_{m+l}) and (U_{n+2l}, V_{m+2l}) is*

$$S_3 = \frac{1}{2}|\sqrt{D}q^n(q^l - V_l + 1)U_l U_{m-n}| = \sqrt{D} \cdot S_1.$$

Proof. (i) First, it is shown that three points (U_n, U_m), (U_{n+l}, U_{m+l}) and (U_{n+2l}, U_{m+2l}) do not lie on a straight line. Since $m > n$ in the assumption, it follows that

$$A_2 B_1 \alpha^{m-n} - A_1 B_2 \beta^{m-n} = \frac{1}{D}(\alpha^{m-n} - \beta^{m-n}) = \frac{1}{\sqrt{D}} U_{m-n} \neq 0.$$

Hence (U_n, U_m), (U_{n+l}, U_{m+l}) and (U_{n+2l}, U_{m+2l}) are the vertices of the triangle. Now let us find the area of this triangle. Using (3.1), we get

$$\overrightarrow{T} = \frac{1}{\sqrt{D}} q^n(q^l - V_l + 1)U_l(\alpha^{m-n} - \beta^{m-n})\overrightarrow{k}.$$

Thus the area of this triangle is

$$\frac{1}{2}|q^n(q^l - V_l + 1)U_l U_{m-n}|.$$

The results of (ii) and (iii) can be also obtained in a similar way. □

Using the above theorem, the areas of triangles whose vertices have Fibonacci numbers or Lucas numbers can be obtained as follows.

Corollary 3.4. *Let n and m be nonnegative integers with $m > n$ and let l be a positive integer. Then the following hold.*

(i) *The area S_1 of a triangle having three vertices (F_n, F_m), (F_{n+l}, F_{m+l}) and (F_{n+2l}, F_{m+2l}) is*

$$S_1 = \begin{cases} \frac{1}{2}F_{2l}F_{m-n}, & \text{if } l \text{ is odd}, \\ \frac{1}{2}L_{l/2}^2 F_l F_{m-n}, & \text{if } l \equiv 2 \pmod 4, \\ \frac{5}{2}F_{l/2}^2 F_l F_{m-n}, & \text{if } l \equiv 0 \pmod 4. \end{cases}$$

(ii) *The area S_2 of a triangle having three vertices (L_n, L_m), (L_{n+l}, L_{m+l}) and (L_{n+2l}, L_{m+2l}) is*

$$S_2 = \begin{cases} \frac{5}{2}F_{2l}F_{m-n}, & \text{if } l \text{ is odd}, \\ \frac{5}{2}L_{l/2}^2 F_l F_{m-n}, & \text{if } l \equiv 2 \pmod 4, \\ \frac{25}{2}F_{l/2}^2 F_l F_{m-n}, & \text{if } l \equiv 0 \pmod 4. \end{cases}$$

(iii) *The area S_3 of a triangle having three vertices (F_n, L_m), (F_{n+l}, L_{m+l}) and (F_{n+2l}, L_{m+2l}) is*

$$S_3 = \begin{cases} \frac{\sqrt{5}}{2}F_{2l}F_{m-n}, & \text{if } l \text{ is odd}, \\ \frac{\sqrt{5}}{2}L_{l/2}^2 F_l F_{m-n}, & \text{if } l \equiv 2 \pmod 4, \\ \frac{5\sqrt{5}}{2}F_{l/2}^2 F_l F_{m-n}, & \text{if } l \equiv 0 \pmod 4. \end{cases}$$

Proof. (i) Taking $q = -1$, $U_n = F_n$ and $V_n = L_n$ in (i) of Theorem 3.3, the area S_1 of the triangle having three vertices (F_n, F_m), (F_{n+l}, F_{m+l}) and (F_{n+2l}, F_{m+2l}) can be obtained as follows:

$$S_1 = \frac{1}{2}|((-1)^l - L_l + 1)F_l F_{m-n}|.$$

Suppose that l is odd. Then

$$\begin{aligned} S_1 &= \frac{1}{2}L_l F_l F_{m-n} \\ &= \frac{1}{2}F_{2l}F_{m-n}. \end{aligned}$$

Suppose that l is even. Then

$$S_1 = \frac{1}{2}(L_l - 2)F_l F_{m-n}.$$

If $l \equiv 2 \pmod 4$, then

$$L_l - 2 = \alpha^l + 2(\alpha\beta)^{l/2} + \beta^l = L_{l/2}^2$$

and

$$S_1 = \frac{1}{2}L_{l/2}^2 F_l F_{m-n}.$$

If $l \equiv 0 \pmod 4$, then

$$L_l - 2 = \alpha^l - 2(\alpha\beta)^{l/2} + \beta^l = 5F_{l/2}^2$$

TABLE 2. The areas of triangle associated with Lucas sequences

Vertices	odd l	$l \equiv 2 \pmod 4$	$l \equiv 0 \pmod 4$
$(F_n, F_m), (F_{n+l}, F_{m+l}), (F_{n+2l}, F_{m+2l})$	$\frac{1}{2}F_{2l}F_{m-n}$	$\frac{1}{2}L_{l/2}^2F_lF_{m-n}$	$\frac{1}{2}F_{l/2}^2F_lF_{m-n}$
$(L_n, L_m), (L_{n+l}, L_{m+l}), (L_{n+2l}, L_{m+2l})$	$\frac{5}{2}F_{2l}F_{m-n}$	$\frac{5}{2}L_{l/2}^2F_lF_{m-n}$	$\frac{25}{2}F_{l/2}^2F_lF_{m-n}$
$(F_n, L_m), (F_{n+l}, L_{m+l}), (F_{n+2l}, L_{m+2l})$	$\frac{\sqrt{5}}{2}F_{2l}F_{m-n}$	$\frac{5\sqrt{5}}{2}L_{l/2}^2F_lF_{m-n}$	$\frac{\sqrt{5}}{2}F_{l/2}^2F_lF_{m-n}$
$(P_n, P_m), (P_{n+l}, P_{m+l}), (P_{n+2l}, P_{m+2l})$	$\frac{1}{2}P_{2l}P_{m-n}$	$\frac{1}{2}Q_{l/2}^2P_lP_{m-n}$	$4P_{l/2}^2P_lP_{m-n}$
$(Q_n, Q_m), (Q_{n+l}, Q_{m+l}), (Q_{n+2l}, Q_{m+2l})$	$4P_{2l}P_{m-n}$	$4Q_{l/2}^2P_lP_{m-n}$	$32P_{l/2}^2P_lP_{m-n}$
$(P_n, Q_m), (P_{n+l}, Q_{m+l}), (P_{n+2l}, Q_{m+2l})$	$\sqrt{2}P_{2l}P_{m-n}$	$\sqrt{2}Q_{l/2}^2P_lP_{m-n}$	$8\sqrt{2}P_{l/2}^2P_lP_{m-n}$
$(J_n, J_m), (J_{n+l}, J_{m+l}), (J_{n+2l}, J_{m+2l})$	$2^nJ_{2l}J_{m-n}$	0	0
$(j_n, j_m), (j_{n+l}, j_{m+l}), (j_{n+2l}, j_{m+2l})$	$9 \cdot 2^nJ_{2l}J_{m-n}$	0	0
$(J_n, j_m), (J_{n+l}, j_{m+l}), (J_{n+2l}, j_{m+2l})$	$3 \cdot 2^nJ_{2l}J_{m-n}$	0	0

and

$$S_1 = \frac{5}{2}F_{l/2}^2F_lF_{m-n}.$$

From Theorem 3.3, this can be seen that $S_2 = 5S_1$ and $S_3 = \sqrt{5}S_1$. Therefore, this theorem is true. □

Remark 3.5. In Corollary 3.4, let $m = n + k$ and $l = 2k$. Then the area of a triangle having three vertices (F_n, F_{n+k}), (F_{n+2k}, F_{n+3k}) and (F_{n+4k}, F_{n+5k}) is

$$\begin{cases} \frac{1}{2}L_k^3F_k, & \text{if } k \text{ is odd,} \\ \frac{5}{2}L_kF_k^4, & \text{if } k \text{ is even.} \end{cases}$$

This is the result of Johnson and Cook (see [5]).

Table 2 shows the area of a triangle having Fibonacci numbers, Lucas numbers, Pell numbers, Pell-Lucas numbers, Jacobsthal numbers, or Jacobsthal-Lucas numbers as vertices.

4. Area of polygons whose vertices have Lucas sequences

Let the vertices of the t-polygon $P_i = (x_i, y_i)$ $(i = 0, \ldots, t-1)$ be connected in order and P_{t-1} is connected to P_0. In [8], the area of this polygon is

$$S = \frac{1}{2}\left|\sum_{i=0}^{t-2} x_iy_{i+1} + x_{t-1}y_0 - \sum_{i=0}^{t-2} x_{i+1}y_i - x_0y_{t-1}\right|. \tag{4.1}$$

Theorem 4.1. *Let m and n be distinct integers, and l and t are positive numbers with $t \geq 3$. Assume that the points $P_i = (f(n+il), g(m+il))$ connected sequentially to $i = 0, \ldots, t-1$ form a t-polygon. Then the area S of this t-polygon is*

$$S = \begin{cases} \frac{\sqrt{D}}{2}|q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - \frac{q^{(t-1)l}-1}{q^l-1}U_l)|, & \text{if } q^l \neq 1, \\ \frac{\sqrt{D}}{2}|q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - (t-1)U_l)|, & \text{if } q^l = 1. \end{cases}$$

Proof. In (4.1), we calculate the following part.

$$\begin{aligned}
T = &\sum_{i=0}^{t-2}\Big(f(n+il)g(m+(i+1)l) - f(n+(i+1)l)g(m+il)\Big) \\
&\quad + f(n+(t-1)l)g(m) - f(n)g(m+(t-1)l) \\
= &\sum_{i=0}^{t-2}\Big((A_1\alpha^{n+il} + B_1\beta^{n+il})(A_2\alpha^{m+(i+1)l} + B_2\beta^{m+(i+1)l}) \\
&\qquad - (A_1\alpha^{n+(i+1)l} + B_1\beta^{n+(i+1)l})(A_2\alpha^{m+il} + B_2\beta^{m+il})\Big) \\
&+ (A_1\alpha^{n+(t-1)l} + B_1\beta^{n+(t-1)l})(A_2\alpha^{m} + B_2\beta^{m}) \\
&\quad - (A_1\alpha^{n} + B_1\beta^{n})(A_2\alpha^{m+(t-1)l} + B_2\beta^{m+(t-1)l}) \\
= &\sum_{i=0}^{t-2} q^{m+il}(\alpha^l - \beta^l)(-A_1B_2\alpha^{n-m} + A_2B_1\beta^{n-m}) \\
&\quad + q^m(\alpha^{(t-1)l} - \beta^{(t-1)l})(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m}) \\
= &\sqrt{D}q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - U_l\sum_{i=0}^{t-2}q^{il}).
\end{aligned}$$

If $q^l = 1$, then

$$T = \sqrt{D}q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - (t-1)U_l)$$

and if $q^l \neq 1$, then

$$T = \sqrt{D}q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - \frac{q^{(t-1)l}-1}{q^l-1}U_l).$$

Therefore, the area of the t-polygon is

$$\begin{aligned}
S &= \frac{|T|}{2} \\
&= \begin{cases} \frac{\sqrt{D}}{2}|q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - \frac{q^{(t-1)l}-1}{q^l-1}U_l)|, & \text{if } q^l \neq 1, \\ \frac{\sqrt{D}}{2}|q^m(A_1B_2\alpha^{n-m} - A_2B_1\beta^{n-m})(U_{(t-1)l} - (t-1)U_l)|, & \text{if } q^l = 1. \end{cases}
\end{aligned}$$

□

But suppose that there are distinct t points $P_i = (x_i, y_i)$ $(i = 0, \ldots, t-1)$ on the plane. This cannot be guaranteed that the shape formed when these points are connected in sequence and P_{t-1} is connected to P_0 is a t-polygon. Clearly, $(J_n, J_m), (J_{n+2l}, J_{m+2l}), \ldots, (J_{n+2(t-1)l}, J_{m+2(t-2)l})$ cannot form any polygon because they lie on a straight line. To solve this, we use the geometric property of cross products.

For all $i = 1, \ldots, t-2$, we have

(i) If a component of $\overrightarrow{k}$ of $\overrightarrow{P_0P_{t-1}} \times \overrightarrow{P_0P_i}$ is greater than 0, then point P_i will be located above the straight line $\overline{P_0P_{t-1}}$,

(ii) If a component of $\overrightarrow{k}$ of $\overrightarrow{P_0P_{t-1}} \times \overrightarrow{P_0P_i}$ is less than 0, then point P_i will be located below the straight line $\overline{P_0P_{t-1}}$,

(iii) If a component of $\overrightarrow{k}$ of $\overrightarrow{P_0P_{t-1}} \times \overrightarrow{P_0P_i}$ is equal to 0, then point P_i will be located on the straight line $\overline{P_0P_{t-1}}$.

From the above, we can prove the following.

Lemma 4.2. *Let p be a positive integer and $q = -1$ and let $A_1B_2\alpha^n\beta^m - A_2B_1\alpha^m\beta^n \neq 0$. If t points $P_i = (f(n+il), g(m+il))$ are sequentially connected for $i = 0, \ldots, t-1$, then a t-polygon will be formed, except for $p = 1$, $t = 4$ and $l = 1$.*

Proof. Suppose that p is a positive integer and $q = -1$. Then

$$\begin{aligned}
&\overrightarrow{P_0P_{t-1}} \times \overrightarrow{P_0P_i}\\
&= \begin{vmatrix} f(n+(t-1)l) - f(n) & g(m+(t-1)l) - g(m) \\ f(n+il) - f(n) & g(m+il) - g(m) \end{vmatrix} \overrightarrow{k}\\
&= \begin{vmatrix} A_1\alpha^n(\alpha^{(t-1)l}-1) + B_1\beta^n(\beta^{(t-1)l}-1) & A_2\alpha^m(\alpha^{(t-1)l}-1) + B_2\beta^m(\beta^{(t-1)l}-1) \\ A_1\alpha^n(\alpha^{il}-1) + B_1\beta^n(\beta^{il}-1) & A_2\alpha^m(\alpha^{il}-1) + B_2\beta^m(\beta^{il}-1) \end{vmatrix} \overrightarrow{k}\\
&= \begin{vmatrix} \alpha^{(t-1)l}-1 & \beta^{(t-1)l}-1 \\ \alpha^{il}-1 & \beta^{il}-1 \end{vmatrix} \begin{vmatrix} A_1\alpha^n & A_2\alpha^m \\ B_1\beta^n & B_2\beta^m \end{vmatrix} \overrightarrow{k}.
\end{aligned}$$

By the assumption, since

$$\begin{vmatrix} A_1\alpha^n & A_2\alpha^m \\ B_1\beta^n & B_2\beta^m \end{vmatrix} \neq 0$$

and m and n are fixed, its sign is determined. Thus we just need to calculate the sign of the following.

$$\begin{aligned}
&\begin{vmatrix} \alpha^{(t-1)l}-1 & \beta^{(t-1)l}-1 \\ \alpha^{il}-1 & \beta^{il}-1 \end{vmatrix}\\
&= (\alpha\beta)^{il}(\alpha^{(t-i-1)l} - \beta^{(t-i-1)l}) - (\alpha^{(t-1)l} - \beta^{(t-1)l}) + (\alpha^{il} - \beta^{il})\\
&= D((-1)^{il}U_{(t-i-1)l} - U_{(t-1)l} + U_{il}).
\end{aligned}$$

If il is odd, then $(-1)^{il}U_{(t-i-1)l} - U_{(t-1)l} + U_{il} = -U_{(t-i-1)l} - U_{(t-1)l} + U_{il} < 0$. Suppose that il is even. Then

$$(-1)^{il}U_{(t-i-1)l} - U_{(t-1)l} + U_{il} = U_{(t-i-1)l} - U_{(t-1)l} + U_{il}.$$

By the recurrence formula,

$$\begin{aligned}
U_{(t-1)l} &= pU_{(t-1)l-1} + U_{(t-1)l-2}\\
&\geq U_{(t-1)l-1} + U_{(t-1)l-2}\\
&\geq U_{(t-1-i)l} + U_{il}.
\end{aligned}$$

Hence

$$U_{(t-i-1)l} - U_{(t-1)l} + U_{il} \leq 0.$$

Also, we obtain that $U_{(t-i-1)l} - U_{(t-1)l} + U_{il} = 0$ if and only if $p = 1$ and $(i, t, l) = (2, 4, 1)$.

Hence except for $p = 1$, $t = 4$ and $l = 1$, the sign of the component of $\overrightarrow{k}$ of $\overrightarrow{P_0P_{t-1}} \times \overrightarrow{P_0P_i}$ is constant regardless of i. This means that for $i = 1, \ldots, t-2$, all points P_i are on one side of straight line $\overline{P_0P_{t-1}}$. In addition, since $f(n)$ and $g(n)$ increase with respect to n, if points P_i $(i = 0, \ldots, t-1)$ are connected in sequence, a t-polygon will be formed. □

From Lemma 4.2, this can be seen that if points (F_{n+il}, F_{m+il}) are connected sequentially for $i = 0, \ldots, t-1$, a t-polygon will be formed except for $p = 1$, $t = 4$ and $l = 1$.

For a positive integer p, let $U_n = pU_{n-1} + U_{n-2}$, $V_n = pV_{n-1} + V_{n-2}$. For $i = 0, \ldots, t-1$, the following t points P_i are classified as follows.

Case 1) $P_i = (U_{n+il}, U_{m+il})$,

Case 2) $P_i = (V_{n+il}, V_{m+il})$,

Case 3) $P_i = (U_{n+il}, V_{m+il})$.

Here, we assume $m > n$. If $(p, t, l) \neq (1, 4, 1)$ is excluded, this can be seen from Lemma 4.2 that if points P_i are connected sequentially, a t-polygon will be formed. Let the area of the polygon in Case 1) be S_1, the area of the polygon in Case 2) be S_2, and the area of the polygon in Case 3) be S_3. Then we can get S_1 from Theorem 4.1.

$$S_1 = \begin{cases} \frac{1}{2}U_{m-n}U_{(t-1)l}, & \text{if } l \text{ and } t \text{ are odd,} \\ \frac{1}{2}U_{m-n}(U_{(t-1)l} - U_l), & \text{if } l \text{ is odd and } t \text{ is even,} \\ \frac{1}{2}U_{m-n}|U_{(t-1)l} - (t-1)U_l\,|, & \text{if } l \text{ is even.} \end{cases} \tag{4.2}$$

Calculating in a similar way

$$S_2 = D \cdot S_1 \quad \text{and} \quad S_3 = \sqrt{D} \cdot S_1.$$

From (4.2), the following result is obtained.

Corollary 4.3. *Let m and n be nonnegative integers with $m > n$ and let l and $t(t \geq 3)$ be positive numbers. Then the area of a polygon, whose vertices are $(F_n, F_m), (F_{n+l}, F_{m+l}), \ldots, (F_{n+(t-1)l}, F_{m+(t-1)l})$ is*

$$S = \begin{cases} \frac{1}{2}F_{m-n}F_{(t-1)l}, & \text{if } l \text{ and } t \text{ are odd,} \\ \frac{1}{2}F_{m-n}(F_{(t-1)l} - F_l), & \text{if } l \text{ is odd and } t \text{ is even,} \\ \frac{1}{2}F_{m-n}|F_{(t-1)l} - (t-1)F_l|, & \text{if } l \text{ is even,} \end{cases}$$

except for $t = 4$ and $l = 1$.

Table 3 shows the areas of t-polygons associated with Lucas sequences.

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TABLE 3. The areas of t-polygons associated with Lucas sequences

Vertices	odd l and t	odd l and even t	even l
$P_i = (F_{n+il}, F_{m+il})$	$\frac{1}{2}F_{m-n}F_{(t-1)l}$	$\frac{1}{2}F_{m-n}(F_{(t-1)l} - F_l)$	$\frac{1}{2}F_{m-n}\lvert F_{(t-1)l} - (t-1)F_l\rvert$
$P_i = (L_{n+il}, L_{m+il})$	$\frac{5}{2}F_{m-n}F_{(t-1)l}$	$\frac{5}{2}F_{m-n}(F_{(t-1)l} - F_l)$	$\frac{5}{2}F_{m-n}\lvert F_{(t-1)l} - (t-1)F_l\rvert$
$P_i = (F_{n+il}, L_{m+il})$	$\frac{\sqrt{5}}{2}F_{m-n}F_{(t-1)l}$	$\frac{\sqrt{5}}{2}F_{m-n}(F_{(t-1)l} - F_l)$	$\frac{\sqrt{5}}{2}F_{m-n}\lvert F_{(t-1)l} - (t-1)F_l\rvert$
$P_i = (P_{n+il}, P_{m+il})$	$\frac{1}{2}P_{m-n}P_{(t-1)l}$	$\frac{1}{2}P_{m-n}(P_{(t-1)l} - P_l)$	$\frac{1}{2}P_{m-n}\lvert P_{(t-1)l} - (t-1)P_l\rvert$
$P_i = (Q_{n+il}, Q_{m+il})$	$4P_{m-n}P_{(t-1)l}$	$4P_{m-n}(P_{(t-1)l} - P_l)$	$4P_{m-n}\lvert P_{(t-1)l} - (t-1)P_l\rvert$
$P_i = (P_{n+il}, Q_{m+il})$	$\sqrt{2}P_{m-n}P_{(t-1)l}$	$\sqrt{2}P_{m-n}(P_{(t-1)l} - P_l)$	$\sqrt{2}P_{m-n}\lvert P_{(t-1)l} - (t-1)P_l\rvert$

References

[1] S. Edwards, *Elementary problems and solutions B–1172*, Fibonacci Quart. **53** (2015), 180–181.

[2] H. Harborth and A. Kemnitz, *Fibonacci triangles*, in Applications of Fibonacci numbers, Vol. 3 (Pisa, 1988), 129–132, Kluwer Acad. Publ., Dordrecht, 1990.

[3] H. Harborth, A. Kemnitz, and N. Robbins, *Non-existence of Fibonacci triangles*, Congr. Numer. **114** (1996), 29–31.

[4] P. Johnson, *Elementary problems and solutions B-1172*, Fibonacci Quart. **54** (2016), 273–275.

[5] V. P. Johnson and C. K. Cook, *Areas of triangles and other polygons with vertices from various sequences*, Fibonacci Quart. **55** (2017), no. 5, 86–95.

[6] H. Kwong, *Elementary problems and solutions B–1167*, Fibonacci Quart. **54** (2016), 180–181.

[7] S. Lipschutz and M. Lipson, *Schaum's outline of Discrete Mathematics*, Third Edition, McGRAW–HILL, 2007.

[8] Shoelace formula, *Wikipedia*, https://en.wikipedia.org/wiki/Shoelace_formula, 2022.

[9] A. Shriki and O. Liba, *Elementary problems and solutions B–1167*, Fibonacci Quart. **53** (2015), 179.

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