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A STUDY OF DIFFERENTIAL IDENTITIES ON σ-PRIME RINGS

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Abstract. Let $\mathcal{R}$ be a σ-prime ring with involution σ. The main objective of this paper is to describe the structure of the σ-prime ring $\mathcal{R}$ with involution σ satisfying certain differential identities involving three derivations ψ_1, ψ_2 and ψ_3 such that $\psi_1[t_1, \sigma(t_1)] + [\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$. Further, some other related results have also been discussed.

1. Introduction

Throughout this paper, $\mathcal{R}$ will be used to describe an associative ring, and $\mathcal{J}_Z$ is centre of ring $\mathcal{R}$. For any $t_1, t_2 \in \mathcal{R}$, the notation $[t_1, t_2]$ illustrates the commutator $t_1t_2 - t_2t_1$, and $t_1 \circ t_2$ illustrates the anti-commutator $t_1t_2 + t_2t_1$. $\mathcal{R}$ is called 2-torsion free if $2t_1 = 0$ implies $t_1 = 0$. We use this basic identities $[t_1t_2, t_3] = t_1[t_2, t_3]+[t_1, t_3]t_2$ and $[t_1, t_2t_3] = [t_1, t_2]t_3+t_2[t_1, t_3]$ for all $t_1, t_2, t_3 \in \mathcal{R}$ as and when required. Recall that an involution is an anti-automorphism of order 2. Throughout, $(\mathcal{R}, \sigma)$ means a ring with involution σ. A ring $(\mathcal{R}, \sigma)$ is called σ-prime if $a\mathcal{R}b = a\mathcal{R}\sigma(b) = (0)$ or $\sigma(a)\mathcal{R}b = a\mathcal{R}b = (0)$ implies $a = 0$ or $b = 0$. Every $(\mathcal{R}, \sigma)$ prime ring is a σ-prime ring but converse is not true in general. Let $S = \mathcal{R} \times \mathcal{R}^0$, where $\mathcal{R}^0$ is the opposite ring of $\mathcal{R}$. The mapping σ on S as $\sigma(t_1, t_2) = (t_2, t_1)$. Thus S is a σ-prime ring but S is not a prime ring. "An element t_1 in $(\mathcal{R}, \sigma)$ is said to be hermitian if $\sigma(t_1) = t_1$ and skew-hermitian if $\sigma(t_1) = -(t_1)$". Let $\mathcal{J}_H$ be the set of hermitian elements and $\mathcal{J}_S$ be the set of skew-hermitian elements of $\mathcal{R}$. If $char(\mathcal{R}) \neq 2$, then every $t_1 \in \mathcal{R}$ can be uniquely expressed as $2t_1 = h + k$, where $h \in \mathcal{J}_H$ and $k \in \mathcal{J}_S$. If $\mathcal{J}_Z \subseteq \mathcal{J}_H$, then σ is said to be first kind and it is called second kind if $\mathcal{J}_S \cap \mathcal{J}_Z \neq \{0\}$. Any $t_1 \in \mathcal{R}$ is called normal if it commutes with its image under involution σ, and if every elements of $\mathcal{R}$ are normal, then $\mathcal{R}$ is called a normal ring (see [6]).

A mapping ψ on $\mathcal{R}$ is termed as derivation on $\mathcal{R}$ if $\psi(t_1+t_2) = \psi(t_1)+\psi(t_2)$ and $\psi(t_1t_2) = \psi(t_1)t_2 + t_1\psi(t_2)$ for all $t_1, t_2 \in \mathcal{R}$. An additive map $g : \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation associated with a derivation $d : \mathcal{R} \to \mathcal{R}$

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if $g(t_1t_2) = g(t_1)t_2 + t_1d(t_2)$ holds for all $t_1, t_2 \in \mathcal{R}$, as an special case if $d = 0$, then g becomes left centralizer on $\mathcal{R}$. A map $f : \mathcal{R} \to \mathcal{R}$ is called centralizing on $\mathcal{R}$ if $[f(t_1), t_1] \in \mathcal{J}_Z$ holds for all $t_1 \in \mathcal{R}$. In particular, if $[f(t_1), t_1] = 0$ holds for all $t_1 \in \mathcal{R}$, then f is said to be commuting. The narrative of centralising and commuting maps dates back to 1955, when Divinsky proved that if a simple artinian ring has commuting non-trivial automorphisms, then it is commutative. After few years, Posner [14] established that the presence of a nonzero centralizing derivation on a prime ring implies commutativity of rings. The study of centralizing (resp. commuting) derivations and various generalizations of concept of centralizing (resp. commuting) maps are the main concepts emerging directly from Posner's result, with many applications in various areas. Recently, a number of algebraists demonstrated the commutativity theorem for prime and semi-prime rings with or without involution, accepting identities on automorphism, derivations, left centralizers and generalized derivations (see [1, 2, 4, 5, 7–9, 13]).

Very recently, Ali and Dar [2] start the study of σ-centralizing derivation in prime rings with involution and proved σ-version of classical results of Posner [14], and they proved that "Let $\mathcal{R}$ be a prime ring with involution σ such that $char(\mathcal{R}) \neq 2$. Let Ψ be a nonzero derivation of $\mathcal{R}$ such that $[\psi(x), \sigma(x)] \in \mathcal{J}_Z$ for all $x \in \mathcal{R}$ and $\psi(\mathcal{J}_S \cap \mathcal{J}_Z) \neq \{0\}$. Then $\mathcal{R}$ is commutative". Further, this result was extended by Najjar et al. [13] for the second kind involution instead of condition $\psi(\mathcal{J}_S \cap \mathcal{J}_Z) \neq \{0\}$. Recently Alahmadi et al. [1] generalized above result for generalized derivation and they proved that "Let $\mathcal{R}$ be a prime ring with involution σ of the second kind such that $char(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a nonzero generalized derivation F associated with a derivation d such that $[F(t), \sigma(t)] \in \mathcal{J}_Z$ for all $t \in \mathcal{R}$, then $\mathcal{R}$ is commutative". In this direction a lot of work have been done in the recent years (see for reference [10–12] where further references can be found). In [3], Ali et al. proved that "Let $\mathcal{R}$ be a prime ring with involution σ of the second kind such that char$(\mathcal{R}) \neq 2$. Let δ_1 and δ_2 be derivations of $\mathcal{R}$ such that at least one of them is nonzero and satisfying the identity $[\delta_1(t), \delta_1(\sigma(t))] + \delta_2(t \circ \sigma(t)) = 0$ for all $t \in \mathcal{R}$. Then $\mathcal{R}$ is a commutative integral domain". Our motivation for this manuscript comes from the types of identities studied by Ali et al. in [3] and motivated by these types of identities, here we study commutativity of σ-prime rings with the help of identities involving three different derivation taken together in our results and some other results are discussed under the same σ-prime ring. To prove our main results, we need some lemmas as well as some facts, so we start with the proof of these lemmas and facts.

2. Main results

Lemma 2.1. *Let $\mathcal{R}$ be a σ-prime ring and ψ be a derivation on $\mathcal{R}$. For some $a \in \mathcal{R}$, $a\psi(t_1) = 0$ for all $t_1 \in \mathcal{R}$ and σ, ψ commute then either $a = 0$ or $\psi = 0$.*

Proof. Let $a\psi(t_1) = 0$ for all $t_1 \in \mathcal{R}$. Replacing t_1 by t_1t_2, then $0 = a\psi(t_1t_2) = a\psi(t_1)t_2 + at_1\psi(t_2) = at_1\psi(t_2)$ for all $t_1, t_2 \in \mathcal{R}$. Now replacing t_2 by $\sigma(t_2)$ we have $at_1\psi(t_2) = 0 = at_1\psi(\sigma(t_2))$ for all $t_1, t_2 \in \mathcal{R}$ since σ and ψ commute with each other. So we have $a\mathcal{R}\psi(t_2) = a\mathcal{R}\sigma(\psi(t_2)) = (0)$. By the definition of σ-prime rings we have either $a = 0$ or $\psi(t_2) = 0$ for all $t_2 \in \mathcal{R}$, which implies $\psi = 0$. □

Lemma 2.2. *Let b and ab be in centre of σ-prime ring $\mathcal{R}$, and σ, ψ commute. If $b \neq 0$, then $a \in \mathcal{J}_Z$.*

Proof. Since $b, ab \in \mathcal{J}_Z$, then $0 = [ab, r] = a[b, r] + [a, r]b = [a, r]b$ for all $a \in \mathcal{R}$, then $I_a(r)b = 0$ so by Lemma 2.1, we have either $b = 0$ or $I_a = 0$, since $b = 0$ is not possible by given condition. Later case implies that $a \in \mathcal{J}_Z$. □

Lemma 2.3. *Let $\mathcal{R}$ be a σ-prime ring of $char(\mathcal{R}) \neq 2$. Then $\mathcal{R}$ is 2-torsion free.*

Proof. Let, $u \in \mathcal{R}$ and $2u = 0$ suggest, $2u(vw) = 0$ for all $v, w \in \mathcal{R}$ and $u\mathcal{R}(2w) = 0$ for all $w \in \mathcal{R}$. Since $char(\mathcal{R}) \neq 2$ and $\mathcal{R} \neq (0)$ there exists $0 \neq p \in \mathcal{R}$ such that $2p \neq 0$, forces $u\mathcal{R}(2p) = (0) = u\mathcal{R}\sigma(2p)$, by the definition of σ-prime rings we have, either $u = 0$ or $2p = 0$ second case is not possible by the assumption and first case implies $\mathcal{R}$ is 2-torsion free. □

Lemma 2.4. *In σ-prime ring, $\mathcal{J}_Z \cap \mathcal{J}_H$ and $\mathcal{J}_Z \cap \mathcal{J}_S$ are free from zero-divisor.*

Proof. Let $a, b \in \mathcal{J}_Z \cap \mathcal{J}_H$ such that $ab = 0$, which implies $abu = 0$ for all $u \in \mathcal{R}$. This provides us $a\mathcal{R}b = (0) = a\mathcal{R}\sigma(b)$. So by definition of σ-prime ring, we have either $a = 0$ or $b = 0$. Similarly we can show that for $\mathcal{J}_Z \cap \mathcal{J}_S$. □

Lemma 2.5. *Let $\mathcal{R}$ be a σ-prime ring with involution σ, which is of second kind, with $char(\mathcal{R}) \neq 2$. If $t_1^2 \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof. $t_1^2 \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$, after linearizing we get, $t_1t_2 + t_2t_1 \in \mathcal{J}_Z$ for all $t_1, t_2 \in \mathcal{R}$. Since σ is of second kind, there exists $0 \neq c \in \mathcal{J}_Z \cap \mathcal{J}_S$. Replacing t_2 by c and using $\text{char}(\mathcal{R}) \neq 2$, we have $t_1c \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$ $[t_1c, r] = 0$ for all $r \in \mathcal{R}$, which implies $[t_1, r]c = 0$. Now by using Lemma 2.2, we get $[t_1, r] = 0$ for all $t_1, r \in \mathcal{R}$, implies $\mathcal{R}$ is commutative. □

Fact 2.6. Let $\mathcal{R}$ be a 2-torsion free σ-prime ring with involution σ which is of second kind. Then $\mathcal{R}$ is commutative if $\mathcal{R}$ is normal.

Proof. Since $\mathcal{R}$ is normal, i.e., $hk = kh$ where $h \in \mathcal{J}_H$ and $k \in \mathcal{J}_S$, respectively. Taking any $t_1 \in \mathcal{R}$ then $t_1 - \sigma(t_1) \in \mathcal{J}_S$.

$$h(t_1 - \sigma(t_1)) = (t_1 - \sigma(t_1))h \quad \text{for all } t_1 \in \mathcal{R} \text{ and } h \in \mathcal{J}_H. \tag{2.1}$$

Taking $s \in \mathcal{J}_S \cap \mathcal{J}_Z$, then $s(t_1 + \sigma(t_1)) \in \mathcal{J}_S$ for all $t_1 \in \mathcal{R}$, so by normality of $\mathcal{R}$ we have $hs(t_1 + \sigma(t_1)) = s(t_1 + \sigma(t_1))h$ for all $t_1 \in \mathcal{R}$ and $h \in \mathcal{J}_H$.

$$s\{h(t_1 + \sigma(t_1)) - (t_1 + \sigma(t_1))h\} = 0 \quad \text{for all } t_1 \in \mathcal{R} \text{ and for all } h \in \mathcal{J}_H. \tag{2.2}$$

So by Lemma 2.4, we have either $s = 0$ or $h(t_1 + \sigma(t_1)) = (t_1 + \sigma(t_1))h$. First case is not possible since σ is of second kind and later case together with (2.1) gives $ht_1 = t_1h$ for all $t_1 \in \mathcal{R}$ and $h \in \mathcal{J}_H$. Substituting t_2 for t_1, we obtain

$$ht_2 = t_2h \text{ for all } t_2 \in \mathcal{R} \text{ and } h \in \mathcal{J}_H. \tag{2.3}$$

Replacing h by $t_1 + \sigma(t_1)$ in (2.3), we get

$$\{t_1 + \sigma(t_1)\}t_2 = t_2\{t_1 + \sigma(t_1)\} \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.4}$$

Now we take $s \in \mathcal{J}_S \cap \mathcal{J}_Z$, then $s(t_1 - \sigma(t_1)) \in \mathcal{J}_H$ and using (2.3) we have $s\{(t_1 - \sigma(t_1))t_2 - t_2(t_1 - \sigma(t_1))\} = 0$ for all $t_1 \in \mathcal{R}$. By Lemma 2.4, we have either $s = 0$ or $(t_1 - \sigma(t_1))t_2 = t_2(t_1 - \sigma(t_1))$. The first case is not possible since σ is of second kind and the later case implies

$$(t_1 - \sigma(t_1))t_2 = t_2(t_1 - \sigma(t_1)) \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.5}$$

Using (2.4) together with (2.5), we get $t_1t_2 = t_2t_1$ for all $t_1, t_2 \in \mathcal{R}$, which implies the commutativity of $\mathcal{R}$. □

Fact 2.7. Let $\mathcal{R}$ be a σ-prime ring with involution σ which is of second kind. Then σ is centralizing if and only if $\mathcal{R}$ is commutative.

Proof. Let

$$[t_1, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.6}$$

Replacing t_1 by $t_1 + t_2$ in (2.6), we get

$$[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}, \tag{2.7}$$

$$[[t_1, \sigma(t_2)], t_1] + [[t_2, \sigma(t_1)], t_1] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.8}$$

Displacing t_2 by t_2t_1 in (2.8), we get

$$\begin{aligned}&[[t_1, t_2], t_1]t_1 + \sigma(t_1)[[\sigma(t_2), \sigma(t_1)], t_1]\\ &\quad + [\sigma(t_1), t_1][\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}.\end{aligned} \tag{2.9}$$

Using (2.8) in (2.9), we get

$$\begin{aligned}&[[t_1, t_2], t_1]t_1 - \sigma(t_1)[[t_2, t_1], t_1]\\ &\quad + [\sigma(t_1), t_1][\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}.\end{aligned} \tag{2.10}$$

Taking t_2t_1 for t_2 in above equation, we attain

$$\begin{aligned}&[[t_1, t_2], t_1]t_1^2 - \sigma(t_1)[[t_2, t_1], t_1]t_1\\ &\quad + [\sigma(t_1), t_1]\sigma(t_1)[\sigma(t_2), \sigma(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}.\end{aligned} \tag{2.11}$$

Using (2.10) in (2.11) and replacing t_1 by $\sigma(t_1)$ and t_2 by $\sigma(t_2)$, we have

$$[t_1, \sigma(t_1)]\{t_1[t_2, t_1] - [t_2, t_1]\sigma(t_1)\} = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.12}$$

Exchanging t_2 by t_2t_1 in (2.12), we capture

$$[t_1, \sigma(t_1)]\{t_1[t_2, t_1]t_1 - [t_2, t_1]t_1\sigma(t_1)\} = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.13}$$

Invoking (2.12) in (2.13), we obtain

$$[t_1, \sigma(t_1)][t_2, t_1]\{-t_1\sigma(t_1) + \sigma(t_1)t_1\} = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.14}$$

Applying the hypothesis σ is centralizing

$$[t_1, \sigma(t_1)]^2\mathcal{R}[t_2, t_1] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.15}$$

Replacing t_1 by $\sigma(t_1)$ and t_2 by $\sigma(t_2)$ in (2.15), we get

$$[t_1, \sigma(t_1)]^2\mathcal{R}[t_2, t_1] = 0 = [t_1, \sigma(t_1)]^2\mathcal{R}\ \sigma\{[t_2, t_1]\} \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.16}$$

So by definition of σ-prime ring, we get

$$[t_1, \sigma(t_1)]^2 = 0 \text{ or } [t_1, t_2] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.17}$$

Later case suggests that $\mathcal{R}$ is commutative, by first case we have

$$[t_1, \sigma(t_1)]^2 = 0 \text{ for all } t_1 \in \mathcal{R}. \tag{2.18}$$

Since $[t_1, \sigma(t_1)] \in \mathcal{J}_Z \cap \mathcal{J}_H$ and $\mathcal{J}_Z \cap \mathcal{J}_S$ is free from zero-divisor in σ-prime ring, we get

$$[t_1, \sigma(t_1)] = 0 \text{ for all } t_1 \in \mathcal{R}. \tag{2.19}$$

Because $\mathcal{R}$ is normal and by Fact 2.6, $\mathcal{R}$ is commutative. Converse part can be directly hold. □

Fact 2.8. Let $\mathcal{R}$ be a σ-prime ring with involution σ of second kind with $char(\mathcal{R}) \neq 2$. Then $t_1 \circ \sigma(t_1) \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$ if and only if $\mathcal{R}$ is commutative.

Proof. By the given condition,

$$t_1 \circ \sigma(t_1) \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.20}$$

Replacing t_1 by $t_1 + t_2$ in the last relation

$$t_1 \circ \sigma(t_2) + t_2 \circ \sigma(t_1) \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.21}$$

Last relation further implies

$$[t_1 \circ \sigma(t_2), r] + [t_2 \circ \sigma(t_1), r] = 0 \text{ for all } t_1, t_2, r \in \mathcal{R}. \tag{2.22}$$

Replacing t_2 by $\sigma(t_2)$ in (2.22), we found

$$[t_1 \circ t_2, r] + [\sigma(t_2) \circ \sigma(t_1), r] = 0 \text{ for all } t_1, t_2, r \in \mathcal{R}. \tag{2.23}$$

Substituting t_2 in place of t_1 in (2.23), we get

$$[t_1^2, r] + [\sigma(t_1)^2, r] = 0 \text{ for all } t_1, r \in \mathcal{R}. \tag{2.24}$$

Consider $t_2 \in \mathcal{J}_Z \setminus \{0\}$ and taking $t_1{}^2$ for t_1 in (2.23), we have

$$[t_1^2, r]t_2 + [\sigma(t_1)^2, r]\sigma(t_2) = 0 \text{ for all } t_1, r \in \mathcal{R}. \tag{2.25}$$

Making use of (2.24) in (2.25), we obtain

$$[t_1^2, r]\{t_2 - \sigma(t_2)\} = 0 \text{ for all } t_1, t_2, r \in \mathcal{R}, \tag{2.26}$$

$\{t_2 - \sigma(t_2)\} \in \mathcal{J}_S \cap \mathcal{J}_Z$. By using Lemma 2.4, we have either $[t_1^2, r] = 0$ or $\{t_2 - \sigma(t_2)\} = 0$. The later case is not possible since σ is of second kind. The first case implies

$$[t_1^2, r] = 0 \text{ for all } t_1, r \in \mathcal{R}. \tag{2.27}$$

So, $t_1^2 \in \mathcal{J}(\mathcal{R})$ for all $t_1 \in \mathcal{R}$. Using Lemma 2.5, $\mathcal{R}$ is commutative. □

Fact 2.9. Let $\mathcal{R}$ be a σ-prime ring with $char(\mathcal{R}) \neq 2$, and $\psi \neq 0$ is centralizing derivation on $\mathcal{R}$, and σ and ψ commute with each other. Then $\mathcal{R}$ is commutative.

Proof. By the given condition,

$$[t_1, \psi(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.28}$$

Taking $t_1 + t_2$ in place of t_1, using (2.28), we obtain

$$[t_1, \psi(t_2)] + [t_2, \psi(t_1)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.29}$$

Replacing t_1^2 in place of t_2 in above equation, we have

$$[t_1, \psi(t_1^2)] + [t_1^2, \psi(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.30}$$

By the definition of derivation, we have

$$[t_1, \psi(t_1^2)] = [t_1^2, \psi(t_1)] \text{ for all } t_1 \in \mathcal{R}. \tag{2.31}$$

Invoking (2.30) in (2.31) and applying $char(\mathcal{R}) \neq 2$, we have

$$[t_1^2, \psi(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}, \tag{2.32}$$

$$[[t_1^2, \psi(t_1)], r] = 0 \text{ for all } t_1, r \in \mathcal{R}. \tag{2.33}$$

Using (2.28) and $char(\mathcal{R}) \neq 2$, we obtain

$$[t_1, \psi(t_1)][t_1, r] = 0 \text{ for all } t_1, r \in \mathcal{R}. \tag{2.34}$$

Exchanging r by ru where $u \in \mathcal{R}$ and using (2.34), we get

$$[t_1, \psi(t_1)] \ \mathcal{R} \ [t_1, u] = 0 \text{ for all } t_1, u \in \mathcal{R}. \tag{2.35}$$

Since u is an arbitrary elements of $\mathcal{R}$, we take $\psi(t_1)$ in place of u

$$[t_1, \psi(t_1)] \ \mathcal{R} \ [t_1, \psi(t_1)] = 0 \text{ for all } t_1 \in \mathcal{R}. \tag{2.36}$$

Every σ-prime ring is a semiprime ring. Now by semiprimeness of $\mathcal{R}$, we have $[t_1, \psi(t_1)] = 0$ for all $t_1 \in \mathcal{R}$. On linearization we obtain

$$[t_1, \psi(t_2)] + [t_2, \psi(t_1)] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.37}$$

Further implies

$$[t_1, \psi(t_2)] = [\psi(t_1), t_2] \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.38}$$

Now define $\psi_c(t_1) = [t_1, c]$ for all $t_1 \in \mathcal{R}$ is called an inner derivation, then (2.38) implies

$$\psi_{\psi(t_2)}(t_1) = \psi_{t_2} \circ \psi(t_1), \tag{2.39}$$

where ψ is a derivation and ψ_{t_2} is an inner derivation.

Posner's first theorem for σ-prime rings states that iterate of derivation is a derivation if at least one of them is 0 (see [4, Theorem 3.1]), so we have either $\psi_{t_2} = 0$ or $\psi = 0$. The later case is not possible by our assumption. The first case implies $t_2 \in \mathcal{J}_Z$ for all $t_2 \in \mathcal{R}$. Hence $\mathcal{R}$ is commutative. □

Theorem 2.10. *Let $\mathcal{R}$ be a σ-prime ring with involution σ such that $char(\mathcal{R}) \neq 2$, let ψ_1, ψ_2 and ψ_3 be derivations on $\mathcal{R}$ such that at least one of them is nonzero satisfying the identity $\psi_1[t_1, \sigma(t_1)] + [\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$. If ψ_1, ψ_2 and ψ_3 commute with σ, then $\mathcal{R}$ is commutative.*

Proof. By given hypothesis, ψ_1, ψ_2 and ψ_3 are derivations on $\mathcal{R}$ such that

$$\psi_1[t_1, \sigma(t_1)] + [\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.40}$$

We examine and split up the proof in the following cases.

Case (i) : If $\psi_1 = 0$, then we consider that

$$[\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.41}$$

Substituting $\sigma(t_1)$ for t_1 in (2.41), we obtain

$$-[\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(\sigma(t_1)), t_1] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.42}$$

By using (2.41) and (2.42), we achieve

$$[\psi_3(\sigma(t_1)), t_1] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.43}$$

Linearizing (2.43), we obtain

$$\begin{aligned} &[\psi_3(t_1), \sigma(t_2)] + [\psi_3(t_2), \sigma(t_1)] \\ &+ [\psi(\sigma(t_1)), t_2] + [\psi(\sigma(t_2)), t_1] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.44}$$

Replacing t_2 by $t_2 h$ where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$ gives

$$\begin{aligned} &h\{[\psi_3(t_1), \sigma(t_2)] + [\psi_3(t_2), \sigma(t_1)] + [\psi(\sigma(t_1)), t_2] + [\psi(\sigma(t_2)), t_1]\} \\ &+ \psi_3(h)\{[t_2, \sigma(t_1)] + [\sigma(t_2), t_1]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.45}$$

Using (2.44) in (2.45), we acquire

$$\psi_3(h)\{[t_2, \sigma(t_1)] + [\sigma(t_2), t_1]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.46}$$

Above relation further implies that

$$\psi_3(h)[[t_2, \sigma(t_1)] + [\sigma(t_2), t_1], r] = 0 \text{ for all } t_1, t_2, r \in \mathcal{R}. \tag{2.47}$$

Since ψ_3 and σ commute with each other, $\psi_3(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. Then by Lemma 2.4, we have either $[[t_2, \sigma(t_1)] + [\sigma(t_2), t_1], r] = 0$ or $\psi_3(h) = 0$. Later case is not possible since σ is of second kind. The first case implies

$$[t_2, \sigma(t_1)] + [\sigma(t_2), t_1]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.48}$$

Changing t_2 by $t_2 s$ where $0 \neq s \in \mathcal{J}_Z \cap \mathcal{J}_S$, we receive

$$s\{[t_2, \sigma(t_1)] - [\sigma(t_2), t_1]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.49}$$

By Lemma 2.2, we gain

$$[t_2, \sigma(t_1)] - [\sigma(t_2), t_1] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.50}$$

By using (2.48) and (2.50), we achieve

$$[t_2, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.51}$$

Taking t_2 for t_1 in (2.51), we have

$$[t_1, \sigma(t_2)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.52}$$

By Fact 2.7, $\mathcal{R}$ is commutative.
Case (ii) : If $\psi_2 = 0$, then we have

$$\psi_1[t_1, \sigma(t_1)] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.53}$$

Changing t_1 by $\sigma(t_1)$ in the last equation, we obtain

$$-\psi_1[t_1, \sigma(t_1)] + [\psi_3(\sigma(t_1)), t_1] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.54}$$

Invoking (2.53) in (2.54), we find

$$[\psi_3(t_1), \sigma(t_1)] + [\psi_3(\sigma(t_1)), t_1] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.55}$$

Equation (2.55) is same as (2.43). So by same argument $\mathcal{R}$ is commutative.
Case (iii) : If $\psi_3 = 0$, then we have

$$\psi_1[t_1, \sigma(t_1)] + [\psi_2(t_1), \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.56}$$

Linearizing above,

$$\begin{aligned} &\psi_1[t_1, \sigma(t_2)] + \psi_1[t_2, \sigma(t_1)] \\ &+ [\psi_2(t_1), \psi_2(\sigma(t_2))] + [\psi_2(t_2), \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.57}$$

Taking t_2h in place of t_2 where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$ and using (2.57), we get

$$\begin{aligned} &\psi_1(h)\{[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)]\} \\ &+ \psi_2(h)\{[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.58}$$

Substitute t_2s by t_2 where $0 \neq s \in \mathcal{J}_Z \cap \mathcal{J}_S$, by using Lemma 2.2, we obtain

$$\begin{aligned} &\psi_1(h)\{-[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)]\} \\ &+ \psi_2(h)\{-[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.59}$$

Equation (2.58) together with (2.59) gives us

$$\psi_1(h)\{[t_2, \sigma(t_1)]\} + \psi_2(h)\{[t_2, \psi_2(\sigma(t_1))]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.60}$$

Displacing t_1 by $\sigma(t_1)$ in above relation, we find

$$\psi_1(h)\{[t_2, t_1]\} + \psi_2(h)\{[t_2, \psi_2(t_1)]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.61}$$

In particular taking $t_2 = t_1$ we gain

$$\psi_2(h)[t_1, \psi_2(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.62}$$

Last relation further implies

$$\psi_2(h)\ [[t_1, \psi_2(t_1)], r] = 0 \text{ for all } t_1, r \in \mathcal{R}. \tag{2.63}$$

Since σ commutes with ψ_2, $\psi_2(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. So by Lemma 2.4, we have either $\psi_2(h) = 0$ or $[[t_1, \psi_2(t_1)], r] = 0$. First case is not possible because σ is of second kind. The later case implies

$$[t_1, \psi_2(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.64}$$

By Fact 2.9, $\mathcal{R}$ is commutative.

Case (iv) : If $\psi_1 = \psi_2 = 0$ and $\psi_3 \neq 0$, we have $[\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$. Change t_1 by $t_1 + t_2$ in the last equation

$$[\psi_3(t_1), \sigma(t_2)] + [\psi_3(t_2), \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.65}$$

Changing t_2 by $t_2 h$ where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$, we get

$$\begin{aligned} &h\{[\psi_3(t_1), \sigma(t_2)] + [\psi_3(t_2), \sigma(t_1)]\} \\ &+ \psi_3(h)\{[t_2, \sigma(t_1)]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.66}$$

Using (2.65) in (2.66) and Lemma 2.2, we obtain

$$\psi_3(h)\{[t_2, \sigma(t_1)]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}, \tag{2.67}$$

$$\psi_3(h)\ [[t_2, \sigma(t_1)], r] = 0 \text{ for all } t_1, t_2, r \in \mathcal{R}. \tag{2.68}$$

Since σ commutes with ψ_3, $\psi_3(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. So by Lemma 2.4, we have either $\psi_3(h) = 0$ or $[[t_2, \sigma(t_1)], r] = 0$. First case is not possible because σ is of second kind. The later case implies

$$[t_2, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.69}$$

In particular taking $t_2 = t_1$ in above equation, we obtain

$$[t_1, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.70}$$

By Fact 2.7, $\mathcal{R}$ is commutative.

Case (v) : Suppose $\psi_2 = \psi_3 = 0$ and $\psi_1 \neq 0$, we have $\psi_1[t_1, \sigma(t_1)] \in \mathcal{J}_Z$ for all $t_1 \in \mathcal{R}$. Changing t_1 by $t_1 + t_2$ in the last relation, we achieve

$$\psi_1[t_1, \sigma(t_2)] + \psi_1[t_2, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.71}$$

Replacing t_2 by $t_2 h$ where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$, we get

$$\begin{aligned} &h\{\psi_1[t_1, \sigma(t_2)] + \psi_1[t_2, \sigma(t_1)]\} \\ &+ \psi_1(h)\{[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.72}$$

Using (2.71) in (2.72) and Lemma 2.2, we obtain

$$\psi_1(h)\{[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}, \tag{2.73}$$

$$\psi_1(h)[[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)], r] = 0 \text{ for all } t_1, t_2, r \in \mathcal{R}. \tag{2.74}$$

Since σ commutes with ψ_1, $\psi_1(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. So by Lemma 2.4, we have either $\psi_1(h) = 0$ or $[[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)], r] = 0$. First case is not possible because σ is of second kind. The later case implies

$$[t_1, \sigma(t_2)] + [t_2, \sigma(t_1)]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.75}$$

In particular taking $t_2 = t_1$ and using $char(\mathcal{R}) \neq 2$, we gain

$$[t_1, \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.76}$$

By Fact 2.7, $\mathcal{R}$ is commutative.

Case (vi) : Let $\psi_1 = \psi_3 = 0$ and $\psi_2 \neq 0$. Then we find

$$[\psi_2(t_1), \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.77}$$

Replacing t_1 by $t_1 + t_2$ in last relation, we achieve

$$[\psi_2(t_1), \psi_2(\sigma(t_2))] + [\psi_2(t_2), \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.78}$$

Displacing t_2 by $t_2 h$, where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$, we gain

$$\begin{aligned} &h\{[\psi_2(t_1), \psi_2(\sigma(t_2))] + [\psi_2(t_2), \psi_2(\sigma(t_1))]\} \\ &+ \psi_2(h)\{[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.79}$$

Using equation (2.78) in (2.79) and by Lemma 2.2, we obtain

$$\psi_2(h)\{[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))]\} \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}, \tag{2.80}$$

$$\psi_2(h)[[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))], r] = 0 \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.81}$$

Since σ commutes with ψ_2, $\psi_2(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. So by Lemma 2.4, we have either $\psi_2(h) = 0$ or $[[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))], r] = 0$. First case is not possible because σ is of second kind. The later case implies

$$[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.82}$$

Replacing t_2 by $t_2 s$, where $0 \neq s \in \mathcal{J}_Z \cap \mathcal{J}_S$, and by Lemma 2.2, we find

$$-[\psi_2(t_1), \sigma(t_2)] + [t_2, \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.83}$$

By (2.82) and (2.83) and $char(\mathcal{R}) \neq 2$, we have

$$[t_2, \psi_2(\sigma(t_1))] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.84}$$

Changing t_1 by $\sigma(t_1)$ in the last equation, we get

$$[t_2, \psi_2(t_1))] \in \mathcal{J}_Z \text{ for all } t_1, t_2 \in \mathcal{R}. \tag{2.85}$$

In particular taking $t_2 = t_1$ in above, we achieve

$$[t_1, \psi_2(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.86}$$

By Fact 2.9, $\mathcal{R}$ is commutative.

Case (vii) : If $\psi_1 \neq 0$, $\psi_2 \neq 0$ and $\psi_3 \neq 0$, then we have

$$\psi_1[t_1, \sigma(t_1)] + [\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(t_1), \sigma(t_1)] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.87}$$

Replacing t_1 by $\sigma(t_1)$ in above, we obtain

$$-\psi_1[t_1, \sigma(t_1)] - [\psi_2(t_1), \psi_2(\sigma(t_1))] + [\psi_3(\sigma(t_1)), t_1] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.88}$$

Using equation (2.87) and (2.87), we find

$$[\psi_3(t_1), \sigma(t_1)] + [\psi_3(\sigma(t_1)), t_1] \in \mathcal{J}_Z \text{ for all } t_1 \in \mathcal{R}. \tag{2.89}$$

Above equation is same as (2.43). So by same argument $\mathcal{R}$ is commutative. □

Theorem 2.11. *Let $\mathcal{R}$ be a σ-prime ring with involution σ of the second kind such that char$(\mathcal{R}) \neq 2$. Let ψ be derivation on $\mathcal{R}$ which commutes with σ. Then the following are equivalent*:

(1) $\psi(t_1) \circ \psi(\sigma(t_1)) - t_1 \circ \sigma(t_1) \in \mathcal{J}_Z$ *for all* $t_1 \in \mathcal{R}$;
(2) $\psi(t_1) \circ \psi(\sigma(t_1)) + t_1 \circ \sigma(t_1) \in \mathcal{J}_Z$ *for all* $t_1 \in \mathcal{R}$;
(3) $\mathcal{R}$ *is commutative.*

Moreover, if $\psi \neq 0$ *and* $\psi(t_1) \circ \psi(\sigma(t_1)) \in \mathcal{J}_Z$ *for all* $t_1 \in \mathcal{R}$, $\mathcal{R}$ *is commutative.*

Proof. If $\psi = 0$, then by Fact 2.8, $\mathcal{R}$ is commutative. Assume that $\psi \neq 0$.

(1) $\Rightarrow$ (3) By the given condition

$$\psi(t_1) \circ \psi(\sigma(t_1)) - t_1 \circ \sigma(t_1) \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.90}$$

Replacing t_1 by $t_1 + t_2$ in above

$$\begin{aligned} &\psi(t_1) \circ \psi(\sigma(t_2)) + \psi(t_2) \circ \psi(\sigma(t_1)) \\ &- t_1 \circ \sigma(t_2) - t_2 \circ \sigma(t_1) \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.91}$$

Taking $t_2 h$ in place of t_2 where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$. By using (2.91), we get

$$[\psi(t_1) \circ \sigma(t_2) + t_2 \circ \psi(\sigma(t_1)), r] \ \psi(h) = 0 \quad \text{for all} \quad t_1, t_2, r \in \mathcal{R}. \tag{2.92}$$

Since σ commutes with ψ, $\psi(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. So by Lemma 2.4, we have either $\psi(h) = 0$ or $[\psi(t_1) \circ \sigma(t_2) + t_2 \circ \psi(\sigma(t_1)), r] = 0$. First case is not possible because σ is of second kind. The later case implies

$$\psi(t_1) \circ \sigma(t_2) + t_2 \circ \psi(\sigma(t_1)) \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.93}$$

Changing t_2 by $\sigma(t_2)$, we gain

$$\psi(t_1) \circ t_2 + \sigma(t_2) \circ \psi(\sigma(t_1)) \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.94}$$

Changing t_2 by $t_2 s$, where $0 \neq s \in \mathcal{J}_Z \cap \mathcal{J}_S$, by using Lemma 2.2, we have

$$\psi(t_1) \circ t_2 - \sigma(t_2) \circ \psi(\sigma(t_1)) \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.95}$$

By using equation (2.94) in (2.95), we obtain

$$\psi(t_1) \circ t_2 \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.96}$$

Last relation further implies

$$[\psi(t_1) \circ t_2, r] = 0 \quad \text{for all} \quad t_1, t_2, r \in \mathcal{R}. \tag{2.97}$$

Interchanging t_2 by $t_2 r$ and using (2.97), we obtain

$$[t_2[\psi(t_1), r], r] = 0 \quad \text{for all} \quad t_1, t_2, r \in \mathcal{R}. \tag{2.98}$$

Exchanging t_2 by $t_2 u$ and using above relation, we have

$$[u, r] t_2 [\psi(t_1), r] = 0 \quad \text{for all} \quad t_1, t_2, r, u \in \mathcal{R}, \tag{2.99}$$

$$[u, r] \ \mathcal{R} \ [\psi(t_1), r] = \sigma([u, r]) \ \mathcal{R} \ [\psi(t_1), r] = 0 \quad \text{for all} \quad t_1, r, u \in \mathcal{R}. \tag{2.100}$$

By definition of σ-prime rings we have either $[u,r]=0$ or $[\psi(t_1),r]=0$. The first case implies commutativity of $\mathcal{R}$. Later case implies

$$[\psi(t_1),r]=0 \text{ for all } t_1,r\in\mathcal{R}. \tag{2.101}$$

In particular taking $r=t_1$

$$[\psi(t_1),t_1]=0 \text{ for all } t_1\in\mathcal{R}. \tag{2.102}$$

By Fact 2.9, $\mathcal{R}$ is commutative.

(2) $\Rightarrow$ (3) We have by hypothesis.

$$\psi(t_1)\circ\psi(\sigma(t_1))+t_1\circ\sigma(t_1)\in\mathcal{J}_Z \text{ for all } t_1\in\mathcal{R}. \tag{2.103}$$

Linearizing above

$$\begin{aligned}&\psi(t_1)\circ\psi(\sigma(t_2))+\psi(t_2)\circ\psi(\sigma(t_1))\\&+t_1\circ\sigma(t_2)+t_2\circ\sigma(t_1)\in\mathcal{J}_Z \text{ for all } t_1,t_2\in\mathcal{R}.\end{aligned} \tag{2.104}$$

Taking t_2h in place of t_2, where $0\neq h\in\mathcal{J}_Z\cap\mathcal{J}_H$, by using (2.104), we get

$$[\psi(t_1)\circ\sigma(t_2)+t_2\circ\psi(\sigma(t_1)),r]\ \psi(h)=0 \text{ for all } t_1,t_2,r\in\mathcal{R}. \tag{2.105}$$

Last relation is same as (2.92). So by same argument $\mathcal{R}$ is commutative.

If $\psi\neq 0$ and $\psi(t_1)\circ\psi(\sigma(t_1))\in\mathcal{J}_Z$ for all $t_1\in\mathcal{R}$. Changing t_1 by t_1+t_2, we have

$$\psi(t_1)\circ\psi(\sigma(t_2))+\psi(t_2)\circ\psi(\sigma(t_1))\in\mathcal{J}_Z \text{ for all } t_1,t_2\in\mathcal{R}. \tag{2.106}$$

Changing t_2 by t_2h, where $0\neq h\in\mathcal{J}_Z\cap\mathcal{J}_H$, in the last relation, we have

$$[\psi(t_1)\circ\sigma(t_2)+t_2\circ\psi(\sigma(t_1)),r]\ \psi(h)=0 \text{ for all } t_1,t_2,r\in\mathcal{R}. \tag{2.107}$$

Above equation is same as (2.92). So by same argument $\mathcal{R}$ is commutative.

(2) $\Rightarrow$ (3) can be done easily by using the same steps of proof as we did in (1) $\Rightarrow$ (3) case. □

Theorem 2.12. *Let $\mathcal{R}$ be a σ-prime ring with involution σ of the second kind such that char($\mathcal{R}$) $\neq 2$. Let ψ be a derivation on $\mathcal{R}$ which commutes with σ. Then the following are equivalent*:

(1) $[\psi(t_1),\psi(\sigma(t_1))]-t_1\circ\sigma(t_1)\in\mathcal{J}_Z$ *for all* $t_1\in\mathcal{R}$;
(2) $[\psi(t_1),\psi(\sigma(t_1))]+t_1\circ\sigma(t_1)\in\mathcal{J}_Z$ *for all* $t_1\in\mathcal{R}$;
(3) $\psi(t_1)\circ\psi(\sigma(t_1))-[t_1,\sigma(t_1)]\in\mathcal{J}_Z$ *for all* $t_1\in\mathcal{R}$;
(4) $\psi(t_1)\circ\psi(\sigma(t_1))+[t_1,\sigma(t_1)]\in\mathcal{J}_Z$ *for all* $t_1\in\mathcal{R}$;
(5) $\mathcal{R}$ *is commutative.*

Proof. (1) $\Rightarrow$ (5) By the given hypothesis

$$[\psi(t_1),\psi(\sigma(t_1))]-t_1\circ\sigma(t_1)\in\mathcal{J}_Z \text{ for all } t_1\in\mathcal{R}. \tag{2.108}$$

Changing t_1 by t_1+t_2 in above equation

$$\begin{aligned}&[\psi(t_1),\psi(\sigma(t_2))]+[\psi(t_2),\psi(\sigma(t_1))]\\&-t_1\circ\sigma(t_2)-t_2\circ\sigma(t_1)-\in\mathcal{J}_Z \text{ for all } t_1,t_2\in\mathcal{R}.\end{aligned} \tag{2.109}$$

Changing t_2 by t_2h where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$ and using (2.109), we achieve

$$\{[\psi(t_1), \sigma(t_2)] + [t_2, \psi(\sigma(t_1))]\} \; \psi(h) \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}, \tag{2.110}$$

$$[[\psi(t_1), \sigma(t_2)] + [t_2, \psi(\sigma(t_1))], r] \; \psi(h) = 0 \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.111}$$

Since σ commutes with ψ, $\psi(h) \in \mathcal{J}_Z \cap \mathcal{J}_H$. So by Lemma 2.4, we have either $\psi(h) = 0$ or $[[\psi(t_1), \sigma(t_2)] + [t_2, \psi(\sigma(t_1))], r] = 0$. First case is not possible because σ is of second kind. The later case implies

$$[\psi(t_1), \sigma(t_2)] + [t_2, \psi(\sigma(t_1))] \; \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.112}$$

Exchanging t_2 by $\sigma(t_2)$ in (2.112), we get

$$[\psi(t_1), t_2] + [\sigma(t_2), \psi(\sigma(t_1))] \; \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.113}$$

Changing t_2 by t_2s, where $0 \neq s \in \mathcal{J}_Z \cap \mathcal{J}_S$, by using Lemma 2.2, we

$$[\psi(t_1), t_2] - [\sigma(t_2), \psi(\sigma(t_1))] \; \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.114}$$

By using (2.113) and (2.114), we achieve

$$[\psi(t_1), t_2] \; \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.115}$$

In particular taking $t_2 = t_1$, we have

$$[\psi(t_1), t_1] \; \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.116}$$

By Fact 2.9, $\mathcal{R}$ is commutative.

(2) $\Rightarrow$ (5) By the given hypothesis

$$[\psi(t_1), \psi(\sigma(t_1))] + t_1 \circ \sigma(t_1) \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.117}$$

Linearizing above

$$\begin{aligned} &[\psi(t_1), \psi(\sigma(t_2))] + [\psi(t_2), \psi(\sigma(t_1))] \\ &+ t_1 \circ \sigma(t_2) + t_2 \circ \sigma(t_1) + \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \end{aligned} \tag{2.118}$$

Changing t_2 by t_2h, where $0 \neq h \in \mathcal{J}_Z \cap \mathcal{J}_H$, by using (2.118), we achieve

$$\{[\psi(t_1), \sigma(t_2)] + [t_2, \psi(\sigma(t_1))]\} \; \psi(h) \in \mathcal{J}_Z \quad \text{for all} \quad t_1, t_2 \in \mathcal{R}. \tag{2.119}$$

Above equation is similar to (2.110), so by same argument $\mathcal{R}$ is commutative.

(3) $\Rightarrow$ (5) By the given hypothesis

$$\psi(t_1) \circ \psi(\sigma(t_1)) - [t_1, \sigma(t_1)] \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.120}$$

Changing t_1 by $\sigma(t_1)$ in the last equation, we have

$$\psi(t_1) \circ \psi(\sigma(t_1)) + [t_1, \sigma(t_1)] \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.121}$$

Using (2.120) in (2.121), we gain

$$[t_1, \sigma(t_1)] \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.122}$$

By Fact 2.7, $\mathcal{R}$ is commutative.

(4) $\Rightarrow$ (5) By the given hypothesis

$$\psi(t_1) \circ \psi(\sigma(t_1)) + [t_1, \sigma(t_1)] \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.123}$$

Changing $\sigma(t_1)$ in the last equation, we have

$$\psi(t_1) \circ \psi(\sigma(t_1)) - [t_1, \sigma(t_1)] \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.124}$$

Using (2.123) in (2.124), we gain

$$[t_1, \sigma(t_1)] \in \mathcal{J}_Z \quad \text{for all} \quad t_1 \in \mathcal{R}. \tag{2.125}$$

By Fact 2.7, $\mathcal{R}$ is commutative. □

Although it is commonly known that the centre of a prime ring is free of zero divisor, but in σ-prime rings centre is not free from zero divisor. The following example explains that the centre of a σ-prime ring is not free zero-divisor.

Example 2.13. Let us consider $\mathcal{R} = \left\{ \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] \middle| a, b, \in \mathbb{Z} \right\}$, and define σ on $\mathcal{R}$ as $\sigma\left(\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]\right) = \left[\begin{smallmatrix} b & 0 \\ 0 & a \end{smallmatrix}\right]$. It is easy to verify that $\mathcal{R}$ is a σ-prime ring with involution σ. For any non zero a, $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathcal{J}_Z(\mathcal{R})$, and for any non zero b, $\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right] \in \mathcal{R}$, $\left[\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. This shows that the centre of σ-prime ring is not free from zero-divisor.

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