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CERTAIN STUDY OF GENERALIZED *B* CURVATURE TENSOR WITHIN THE FRAMEWORK OF KENMOTSU MANIFOLD

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Abstract. In the present study, we consider some curvature properties of generalized B-curvature tensor on Kenmotsu manifold. Here first we describe certain vanishing properties of generalized B curvature tensor on Kenmostu manifold. Later we formulate generalized B pseudo-symmetric condition on Kenmotsu manifold. Moreover, we also characterize generalized B ϕ-recurrent Kenmotsu manifold.

1. Introduction

In [7], authors Shaikh and Kundu proved the equivalency of various geometric structures obtained by the same curvature restriction on different curvature tensors. For this purpose they have introduced a special type of (0,4) tensor field, called B curvature tensor and further they studied generalized B curvature tensor on a Riemannian manifold and is given by

$$B(U,V)X = a_0R(U,V)X + a_1[S(V,X)U - S(U,X)V + g(V,X)QU - g(U,X)QV] + 2a_2r[g(V,X)U - g(U,X)V], \tag{1}$$

where a_0, a_1 and a_2 are scalars.

The generalized B curvature tensor includes the structures of quasi-conformal, Weyl conformal, conharmonic and concircular curvature tensors:

(i) The quasi-conformal curvature tensor C^* [10] if $a_0 = a$, $a_1 = b$ and $a_2 = \frac{-1}{n}[\frac{a}{n-1} + 2b]$.

(ii) The weyl-conformal curvature tensor $\tilde{C}$ [9] if $a_0 = 1$, $a_1 = \frac{-1}{n-2}$ and $a_2 = \frac{-1}{2(n-1)(n-2)}$.

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(iii) The concircular curvature tensor C [8] if $a_0 = 1$, $a_1 = 0$ and $a_2 = \frac{-1}{2n(n-1)}$.

(iv) The conharmonic curvature tensor P [3] if $a_0 = 1$, $a_1 = \frac{-1}{n-2}$ and $a_2 = 0$.

On the other hand, the structure of Kenmotsu manifold was first developed and studied by Kenmotsu [5] and he proved that a Kenmotsu manifold admitting $R(X,Y) \cdot R = 0$ is a space of negative curvature -1. In [4], authors studied some symmetric properties of Kenmotsu manifold. Later Amit Prakash [6] studied concircularly ϕ-recurrent conditions of Kenmotsu manifold. Recently Ajit Barman and De [1] have formulated Kenmotsu manifold admitting m-projective curvature tensor with different metric connection.

The present paper is structured in the following way: In Section 2, we formulate the preliminaries and fundamental results of Kenmotsu manifold which is needed throughout this study. In Section 3 we shall describe some vanishing conditions of generalized B curvature tensor on Kenmotsu manifold. In particular we consider generalized B flat, generalized $\xi - B$ flat and generalized $\phi - B$ flat Kenmotsu manifold and we found that the manifold reduces to Einstein, space of constant scalar curvature and η-Einstein, respectively. Next we consider generalized B pseudo-symmetric Kenmotsu manifolds, here we proved that the manifold becomes η-Einstein and generalized B pseudo-symmetric Kenmotsu manifold is never reduces to generalized B semi-symmetric manifold. Finally we have determined that if a scalar curvature of generalized B ϕ-recurrent Kenmotsu manifold is constant then the manifold turns into generalized B ϕ-symmetric.

2. Some preliminaries on Kenmotsu manifold

A differentiable manifold is expressed as an almost contact metric manifold [2] if it carries a $(1,1)$-tensor field ϕ, a vector field ξ, a global 1-form η and a Riemanian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \cdot \phi = 0, \tag{2}$$

$$g(\phi U, V) = -g(U, \phi V), \quad \eta(V) = g(V, \xi), \tag{3}$$

$$g(\phi U, \phi X) = g(U, X) - \eta(U)\eta(X). \tag{4}$$

An almost contact metric manifold $M(\phi, \xi, \eta, g)$ becomes a Kenmotsu manifold if the following relations hold [5]:

$$\nabla_V \xi = V - \eta(V)\xi, \tag{5}$$

$$(\nabla_U \phi)(V) = g(\phi U, V)\xi - \eta(V)\phi U, \tag{6}$$

where ∇ denotes the operator of covariant differentiation with respect to Riemannian metric g.

In a Kenmotsu manifold, the following equations hold [5]:

$$(\nabla_U \eta)(V) = g(U, V) - \eta(U)\eta(V), \tag{7}$$

$$R(U,V)X = g(U,X)V - g(V,X)U, \tag{8}$$

$$R(U,V)\xi = \eta(U)V - \eta(V)U, \tag{9}$$

$$S(U,\xi) = -(n-1)\eta(U), \tag{10}$$

$$S(\xi,\xi) = -(n-1), \quad Q\xi = -(n-1)\xi, \tag{11}$$

$$S(\phi U,\phi V) = S(U,V) + (n-1)\eta(U)\eta(V) \tag{12}$$

for any vector fields U, V, X on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type $(0,2)$ such that $g(QU,V) = S(U,V)$.

3. Kenmotsu manifold admitting some vanishing properties of generalized B curvature tensor

In this section we have studied some vanishing properties of generalized B curvature tensor with in the framework of Kenmotsu manifold.

3.1. Kenmotsu manifold admitting $B(U,V)X = 0$

Here first we are consider a Kenmotsu manifold conceding with $B(U,V)X = 0$ for any vector fields U, V and X.

Now it follows from (1) that

$$\begin{aligned} a_0R(U,V)X &+ a_1[S(V,X)U - S(U,X)V + g(V,X)QU - g(U,X)QV] \\ &+ 2a_2r[g(V,X)U - g(U,X)V] = 0. \end{aligned} \tag{13}$$

Now taking an account of (13) we have

$$\begin{aligned} a_0g(R(U,V)X,W) &+ a_1[S(V,X)g(U,W) - S(U,X)g(V,W) \\ &+ g(V,X)S(U,W) - g(U,X)S(V,W)] \\ &+ 2a_2r[g(V,X)g(U,W) - g(U,X)g(V,W)] = 0. \end{aligned} \tag{14}$$

On contracting above expression over U and W, we have

$$S(V,X) = -\frac{r[a_1 + 2a_2(n-1)]}{a_0 + a_1(n-2)}g(V,X). \tag{15}$$

This leads us to the following conclusion:

Theorem 3.1. *An n-dimensional Kenmotsu manifold conceding with $B(U,V)X = 0$ is a Einstein manifold provided that the scalars a_0 and a_1 are not linearly dependent to each other.*

Again contracting (13), we have

$$a_0 + a_1 2(n-1) + a_2 2n(n-1) = 0.$$

From which it concludes the following corollary:

Corollary 3.2. *In an n-dimensional Kenmotsu manifold with $B(U,V)X = 0$, the generalized B curvature tensor is reduces to concircular curvature tensor $\left[a_0 = 1,\ a_1 = 0 \text{ and } a_2 = \frac{-1}{2n(n-1)}\right]$.*

3.2. Kenmotsu manifold admitting $B(U,V)\xi = 0$

Next we have studied Kenmotsu manifolds endowed with $B(U,V)\xi = 0$. It follows from the expression (1) that

$$a_0 R(U,V)\xi + a_1[S(V,\xi)U - S(U,\xi)V + g(V,\xi)QU - g(U,\xi)QV] + 2a_2 r[g(V,\xi)U - g(U,\xi)V] = 0. \tag{16}$$

By virtue of (9) and (10) in (16), one can obtained that

$$a_0[\eta(U)g(V,W) - \eta(V)g(U,W)] + a_1[-(n-1)\eta(V)g(U,W) + (n-1)\eta(U)g(V,W) + \eta(V)S(U,W) - \eta(U)S(V,W)] + 2a_2 r[\eta(V)g(U,W) - \eta(U)g(V,W)] = 0. \tag{17}$$

On contracting above equation, we have

$$\eta(V)[a_0(1-n) + a_1(r-(n-1)(n-2)) + 2a_2 r(n-1)] = 0. \tag{18}$$

Since $\eta(V) \neq 0$, we have found that

$$r = \frac{(n-1)[a_0 + a_1(n-2)]}{a_1 + 2a_2(n-1)}. \tag{19}$$

Hence we can state the following result:

Theorem 3.3. *If a Kenmotsu manifold admitting $B(U,V)\xi = 0$ is of constant scalar curvature on condition, then the scalars a_1 and a_2 are not linearly dependent to each other.*

3.3. Kenmotsu manifold admitting $g(B(\phi U,\phi V)\phi X,\phi W) = 0$

Finally we characterize a Kenmotsu manifold providing with $g(B(\phi U,\phi V)\phi X, \phi W) = 0$ and by taking an account of (1), we get

$$a_0 g(R(\phi U,\phi V)\phi X,\phi W) + a_1[S(\phi V,\phi X)g(\phi U,\phi W) - S(\phi U,\phi X)g(\phi V,\phi W) + g(\phi V,\phi X)S(\phi U,\phi W) - g(\phi U,\phi X)S(\phi V,\phi W)] + 2a_2 r[g(\phi V,\phi X)g(\phi U,\phi W) - g(\phi U,\phi X)g(\phi V,\phi W)] = 0. \tag{20}$$

On contracting foregoing equation over U and W, we obtain

$$S(\phi V,\phi X) = \frac{a_1(n-1-r) - 2a_2 r(n-2)}{a_0 + a_1(n-3)} g(\phi V,\phi X). \tag{21}$$

By considering (4) and (12) in (21), it gives

$$S(V,X) = Ag(V,X) + B\eta(V)\eta(X), \tag{22}$$

where

$$A = \frac{a_1(n-1-r) - 2a_2 r(n-2)}{a_0 + a_1(n-3)},$$

$$B = \frac{-a_0(n-1) - a_1((n-1)(n-2) - r) + 2a_2 r(n-2)}{a_0 + a_1(n-3)}.$$

Thus we can state the following result:

Theorem 3.4. *An n-dimensional Kenmotsu manifold with $g(B(\phi U, \phi V)\phi X, \phi W) = 0$ is a Einstein manifold provided that the scalars a_0 and a_1 are not linearly dependent to each other.*

Replacing V by ϕV and X by ϕX in (22), we obtain

$$(\nabla_Y S)(\phi V, \phi X) = -\frac{(a_1 + 2a_2(n-2))dr(Y)}{a_0 + a_1(n-3)} g(\phi V, \phi X). \tag{23}$$

If we consider a Kenmotsu manifold with constant scalar curvature r, then above equation becomes

$$(\nabla_Y S)(\phi V, \phi X) = 0. \tag{24}$$

Hence we can able to state the following corollary:

Corollary 3.5. *A Kenmotsu manifold satisfies $g(B(\phi U, \phi V)\phi X, \phi W) = 0$ with constant scalar curvature admits an η-parallel Ricci tensor.*

4. Generalized B pseudo-symmetric Kenmotsu manifold

Let us consider a generalized B pseudo-symmetric Kenmotsu manifold, i.e.,

$$(R(U,V) \cdot B)(X,Y)W = L_B[((U \wedge V) \cdot B)(X,Y)W], \tag{25}$$

holds on $U_B = \{x \in M : B \neq 0\}$ at x, where L_B is some function on U_B.

One can be easily seen that from (25) that

$$\begin{aligned}(R(U,\xi) \cdot B)(X,Y)W = L_B[&((U \wedge \xi)(B(X,Y)W)) \\ &- B((U \wedge \xi)X, Y)W - B(X, (U \wedge \xi))W \\ &- B(X,Y)(U \wedge \xi)W].\end{aligned} \tag{26}$$

We can easily obtained from left hand side of (26) that

$$\begin{aligned}&[\eta(B(X,Y)W)U - g(U, B(X,Y)W)\xi - \eta(X)B(U,Y)W \\ &+ g(U,X)B(\xi,Y)W - \eta(W)B(X,Y)U + g(U,W)B(X,Y)\xi \\ &- \eta(Y)B(X,U)W + g(U,Y)B(X,\xi)W].\end{aligned} \tag{27}$$

Similarly right hand side of (26) gives

$$\begin{aligned}&L_B[\eta(B(X,Y)W)U - g(U, B(X,Y)W)\xi - \eta(X)B(U,Y)W \\ &+ g(U,X)B(\xi,Y)W - \eta(W)B(X,Y)U + g(U,W)B(X,Y)\xi \\ &- \eta(Y)B(X,U)W + g(U,Y)B(X,\xi)W].\end{aligned} \tag{28}$$

Now the foregoing equation can takes the form

$$\begin{aligned}&(L_B - 1)[\eta(B(X,Y)W)U - g(U, B(X,Y)W)\xi - \eta(X)B(U,Y)W \\ &+ g(U,X)B(\xi,Y)W - \eta(W)B(X,Y)U + g(U,W)B(X,Y)\xi \\ &- \eta(Y)B(X,U)W + g(U,Y)B(X,\xi)W] = 0.\end{aligned} \tag{29}$$

Plugging $Y = \xi$ in (29) and then by virtue of (1), we obtain either $L_B = 1$ or

$$B(X,U)W = (a_0 - 2a_2r)[g(X,W)U - g(U,W)X] - a_1[g(U,W)\eta(X)\xi - S(U,W)\eta(X)\xi - S(X,U)\eta(W)\xi - ng(U,W)\eta(X)\xi + 2(n-1)g(U,W)X - g(X,W)U + S(X,W)U]. \tag{30}$$

Contraction of equation (30) with respect to X brings into view

$$S(U,W) = lg(U,W) + \eta(U)\eta(W), \tag{31}$$

where $l = -\frac{[a_0(n-1)+a_1(r+n(2n-3))]}{a_0+a_1(n-4)}$ and $m = -\frac{a_1(n-1)}{a_0+a_1(n-4)}$.

Hence we state the following result:

Theorem 4.1. *In a generalized B pseudo-symmetric Kenmotsu manifold, either $L_B = 1$ or the manifold reduces to η-Einstein provided scalars a_0 and a_1 are not linearly dependent to each other.*

Due to Theorem 4.1, we found $L_B = 1$, then by view of (25), we get

$$(R(U,V) \cdot B)(X,Y)W \neq 0.$$

Thus, we obtain the following conclusion:

Corollary 4.2. *A generalized B pseudo-symmetric Kenmotsu manifold is never reduces to generalized B semi-symmetric manifold.*

5. Generalized B ϕ-recurrent Kenmotsu manifold

Definition. A Kenmotsu manifold is called a generalized B ϕ-recurrent manifold, if for every non-zero one form A satisfies

$$\phi^2((\nabla_W B)(U,V)X) = A(W)B(U,V)X \tag{32}$$

for any vector fields $U, V, X, W \in T_pM$.

Now in view of (2) in (32), becomes

$$-(\nabla_W B)(U,V)X + \eta((\nabla_W B)(U,V)X)\xi = A(W)B(U,V)X, \tag{33}$$

from which it follows that

$$- g((\nabla_W B)(U,V)X,Y) + \eta((\nabla_W B)(U,V)X)\eta(Y) = A(W)g(B(U,V)X,Y). \tag{34}$$

Taking an account of (1) in the above equation and then contracting over U and Y turns into

$$- a_1(n-1)(\nabla_W S)(V,X) - [a_1 + a_2(n-1)]d_rW + a_1[(\nabla_W S)(X,\xi)\eta(V) - (\nabla_W S)(V,\xi)\eta(X)] + 2a_2d_rW[g(V,X) - \eta(V)\eta(X)] = A(W)\{(a_0 + a_1(n-2))S(V,X) + (a_1 + 2a_2(n-1))rg(V,X)\}. \tag{35}$$

On plugging $V = X = \xi$ in (35), gives

$$A(W) = \frac{[a_1 + 2a_2(n-1)]d_rW}{a_0(n-1) + a_1[(n-2)(n-1) - r] - 2a_2r(n-1)}. \tag{36}$$

This leads us to the following:

Theorem 5.1. *In an n-dimensional generalized B ϕ-recurrent Kenmotsu manifold, the non-zero* 1*-form A is given by* (36) *provided that* $a_0(n-1) + a_1[(n-2)(n-1) - r] - 2a_2r(n-1) \neq 0$.

Next, if we assume that the scalar curvature of an n-dimensional generalized B ϕ-recurrent Kenmotsu manifold is constant, then $d_rW = 0$.
Hence the equation (36) turns into

$$A(W) = 0. \tag{37}$$

By using (37) in (32), we get

$$\phi^2((\nabla_W B)(X,Y)Z) = 0. \tag{38}$$

Hence we can formulate the following:

Theorem 5.2. *The generalized B ϕ-recurrent Kenmotsu manifold with constant scalar curvature r is reduces to a generalized B ϕ-symmetric manifold.*

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