KYUNGPOOK Math. J. 63(2023), 313-324
https://doi.org/10.5666/KMJ.2023.63.2.313
pISSN 1225-6951 eISSN 0454-8124

Generalized Ricci Solitons on $N(\kappa)$-contact Metric Manifolds

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Abstract. In the present paper, we study generalized Ricci solitons on $N(\kappa)$-contact metric manifolds, in particular, we consider when the potential vector field is the concircular vector field. We also consider generalized gradient Ricci solitons, and verify our results with an example.

1. Introduction

The notion of a κ-nullity distribution (in brief, KND) on a Riemannian manifold (RM, in short) was coined by Tanno [12]. A KND in a RM $\mathbf{M}$ is described by

$$N(\kappa) : q \longrightarrow N_q(\kappa) = \{V_3 \in T_q\mathbf{M} : R(V_1, V_2)V_3 = \kappa[g(V_2, V_3)V_1 - g(V_1, V_3)V_2]\},$$

for vector fields $V_1, V_2 \in T_q\mathbf{M}$, κ being a real number, and $T_q\mathbf{M}$ being the tangent space of $\mathbf{M}$ at q. A (2m+1)-dimensional contact metric manifold (CMM, in short) is called $N(\kappa)$-contact metric manifold (NCMM, in short) if the Reeb vector field θ satisfies KND. So, for a NCMM, we have

$$R(V_1, V_2)\theta = \kappa\{\tau(V_2)V_1 - \tau(V_1)V_2\}. \tag{1.1}$$

For $\kappa = 1$, the manifold is a Sasakian manifold and when $\kappa = 0$ it is locally isometric to the product of an m-dimensional manifold of scalar curvature 4 with a flat $(m+1)$-dimensional manifold, provided $m>1$. If $m = 1$ and $\kappa = 0$, the manifold is flat [1]. NCMMs have been studied by many authors [1, 2, 3, 5].

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Received November 25, 2022; revised May 31, 2023; accepted June 1, 2023.
2020 Mathematics Subject Classification: 53C15, 53D25.
Key words and phrases:Nullity distribution, contact manifolds, $N(\kappa)$-contact metric manifolds, Ricci solitons, generalized Ricci solitons.
Urmila Biswas is financially supported by UGC, NTA Ref. no-201610057626.

Hamilton [8] introduced the famous geometric flow, named as Ricci flow, which is a kind of pseudo parabolic heat equation defined on a RM as

$$\frac{\partial g(t)}{\partial t} = -2Ric(t), \tag{1.2}$$

where g and Ric indicate, respectively, the Riemannian metric and the $(0, 2)$ Ricci tensor.

A Ricci soliton (RS, in short) is a fixed point of Ricci flow (RF, in short) equation (1.2). At the same time it is also a generalization of the Einstein metric. A RS on a $(2m + 1)$-dimensional RM is given by

$$(\mathcal{L}_E g)(V_1, V_2) + 2Ric(V_1, V_2) = 2\lambda g(V_1, V_2),$$

$\mathcal{L}$ being the Lie-derivative operator and λ is a constant. The nature of a RS is described by the value of λ, that means, a RS is shrinking if $\lambda{>}0$, it is steady if $\lambda = 0$, for $\lambda{<}0$ it is expanding. For more about RSs, one can see the papers [11, 13].

In [10], the authors extended the idea of an RS to a generalized Ricci soliton (GRS, in short). On a RM it is given by

$$(\mathcal{L}_E g)(V_1, V_2) + 2aRic(V_1, V_2) + 2bE^{\sharp}(V_1)E^{\sharp}(V_2) = 2\lambda g(V_1, V_2), \tag{1.3}$$

where a, b, $\lambda \in \mathbb{R}$ and $E^{\sharp}$ is the canonical 1-form related with E i.e., $E^{\sharp}(V_1) = g(V_1, E)$. Similar to a RS, a GRS is shrinking or steady or expanding according as λ takes positive, zero or negative value. Here the potential vector field (PVF, in short) is termed as

- Homothetic vector field if $a = b = 0$
- Killing vector field if $a = b = \lambda = 0$

and the equation (1.3) is called

- Ricci soliton when $a = 1$ and $b = 0$
- Einstein-Weyl equation when $a = \frac{1}{n-1}$ and $b = 1$.

GRS on different types of RS have been discussed by many authors like Ghosh and De [6, 7], Kumara, Naik and Venkatesha [9].

If the PVF be taken as the gradient of a smooth function, the GRS reduces into generalized gradient Ricci soliton (GGRS, in short). Thus the GGRS on a RM $\mathbf{M}$ is given by

$$\nabla^2\psi(V_1, V_2) + aRic(V_1, V_2) + b(V_1\psi)(V_2\psi) = \lambda g(V_1, V_2), \tag{1.4}$$

where ∇^2 being the Hessian operator and ψ is a smooth function on $\mathbf{M}$.

In a RM a vector field E is called concircular vector field [4] if

$$\nabla_{V_1} E = fV_1, \tag{1.5}$$

for any vector field V_1 on the manifold, where ∇ is the Levi-Civita connection and f is a smooth function.

The paper is embodied as follows: after a brief review of literature, we give some basic definition and curvature properties of NCMMs in the Section 2. In Section 3, we deduce certain characterizations of GRSs on NCMMs. The next section deals with generalized gradient Ricci solitons. In the last section, we give an example to support our results.

2. Preliminaries

A $(2m+1)$-dimensional differentiable manifold $\mathbb{N}$ endowed with a tensor field ϕ of type $(1,1)$, a vector field θ, a 1-form τ satisfying [5]

$$\phi^2(V_1) = -V_1 + \tau(V_1)\theta, \quad \tau(\theta) = 1, \tag{2.1}$$

for any vector field $V_1 \in \chi(\mathbf{N})$, the set of all vector fields on $\mathbf{N}$, is known as an almost contact manifold. An almost contact manifold is called almost contact metric manifold if it admits a Riemannian metric g such that

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \tau(V_1)\tau(V_2). \tag{2.2}$$

As a consequence of (2.2) and (2.3), we get the following:

$$\phi\theta = 0, \quad g(V_1, \theta) = \tau(V_1), \quad \tau(\phi V_1) = 0,$$

$$g(\phi V_1, V_2) = -g(V_1, \phi V_2),$$

$$(\nabla_{V_1}\tau)(V_2) = g(\nabla_{V_1}\theta, V_2),$$

for any vector fields V_1, $V_2 \in \chi(\mathbf{N})$.

An almost contact metric manifold is called CMM whenever the almost contact metric structure (ϕ, θ, τ, g) satisfies the following condition [5]

$$g(V_1, \phi V_2) = d\tau(V_1, V_2),$$

for every vector fields V_1, $V_2 \in \chi(\mathbf{N})$. For a CMM $\mathbf{N}$, we determine a symmetric $(1,1)$-tensor field h by $h = \frac{1}{2}\mathcal{L}_\theta\phi$, $\mathcal{L}_\theta\phi$ indicates the Lie differentiation of ϕ in the direction θ and satisfying the following conditions

$$h\theta = 0, \quad h\phi + \phi h = 0, \quad tr(h) = tr(h\phi) = 0,$$

$$\nabla_{V_1}\theta = -\phi V_1 - \phi h V_1. \tag{2.3}$$

For a NCMM $\mathbf{N}$ of dimension $2m+1, m \geq 1$, we have [5]

$$h^2 = (\kappa - 1)\phi^2,$$

$$(\nabla_{V_1}\phi)(V_2) = g(V_1 + hV_1, V_2)\theta - \tau(V_2)(V_1 + hV_1), \tag{2.4}$$

$$R(V_1,V_2)\theta=\kappa\{\tau(V_2)V_1-\tau(V_1)V_2\}, \tag{2.5}$$

$$R(\theta,V_1)V_2=\kappa\{g(V_1,V_2)\theta-\tau(V_2)V_1\},$$

$$\begin{aligned}Ric(V_1,V_2)=&2(m-1)\{g(V_1,V_2)+g(hV_1,V_2)\}\\&+\{2m\kappa-2(m-1)\}\tau(V_1)\tau(V_2),\end{aligned} \tag{2.6}$$

$$Ric(V_1,\theta)=2m\kappa\tau(V_1), \tag{2.7}$$

$$(\nabla_{V_1}\tau)(V_2)=g(V_1+hV_1,\phi V_2), \tag{2.8}$$

$$\begin{aligned}(\nabla_{V_1}h)(V_2)=&\{(1-\kappa)g(V_1,\phi V_2)+g(V_1,h\phi V_2)\}\theta\\&+\tau(V_2)\{h(\phi V_1+\phi hV_1)\},\end{aligned} \tag{2.9}$$

$$r=2m(2m-2+\kappa), \tag{2.10}$$

for every vector fields V_1, V_2, $\in\chi(\mathbf{N})$; R, Ric and r are the Riemannian curvature, Ricci tensor and scalar curvature, respectively.
Lemma 2.1. *In a (2m+1)-dimensional NCMM* $\boldsymbol{N}$*, the following holds*

$$\begin{aligned}(\nabla_{V_1}h\phi)V_2=&(\kappa-1)(g(V_1,V_2)\theta-2\tau(V_1)\tau(V_2)\theta\\&+\tau(V_2)V_1)-g(V_1,hV_2)\theta-\tau(V_2)hV_1,\end{aligned} \tag{2.11}$$

for any vector fields V_1*,* V_2 *on the manifold.*

Proof. By a straightforward calculation, we obtain

$$(\nabla_{V_1}h\phi)V_2=(\nabla_{V_1}h)\phi V_2+h(\nabla_{V_1}\phi)V_2. \tag{2.12}$$

Using (2.4) and (2.9) in the previous relation, the desired result is obtained. □

3. Generalized Ricci Solitons on $N(\kappa)$-contact Metric Manifolds

Let $\mathbf{N}$ be a NCMM of dimension $(2m+1)$ admitting generalized Ricci soliton. Applying covariant derivative on (1.3) in the direction V_3, we obtain

$$\begin{aligned}(\nabla_{V_3}\mathcal{L}_Eg)(V_1,V_2)=&-2a(\nabla_{V_3}Ric)(V_1,V_2)-2b(g(\nabla_{V_3}E,V_1)E^\sharp(V_2)\\&+E^\sharp(V_1)g(\nabla_{V_3}E,V_2)).\end{aligned} \tag{3.1}$$

According to Yano [14], we infer

$$\begin{aligned}(\mathcal{L}_E\nabla_{V_3}g-\nabla_{V_3}\mathcal{L}_Eg-\nabla_{[E,V_3]}g)(V_1,V_2)=&-g((\mathcal{L}_E\nabla)(V_3,V_1),V_2)\\&-g((\mathcal{L}_E\nabla)(V_3,V_2),V_1).\end{aligned}$$

As $\nabla g = 0$ and from the above equation, we have

$$(\nabla_{V_3}\mathcal{L}_E g)(V_1, V_2) = g((\mathcal{L}_E\nabla)(V_3, V_1), V_2) + g((\mathcal{L}_E\nabla)(V_3, V_2), V_1).$$

Due to symmetry property of $\mathcal{L}_E\nabla$, the above equation reduces to

$$\begin{aligned} 2g((\mathcal{L}_E\nabla)(V_3, V_1), V_2) =&(\nabla_{V_3}\mathcal{L}_E g)(V_1, V_2) + (\nabla_{V_1}\mathcal{L}_E g)(V_3, V_2) \\ &-(\nabla_{V_2}\mathcal{L}_E g)(V_3, V_1). \end{aligned} \tag{3.2}$$

Using (3.1) in (3.2), we get

$$\begin{aligned} 2g((\mathcal{L}_E\nabla)(V_3, V_1), V_2) =& -2a[(\nabla_{V_3}Ric)(V_1, V_2) + (\nabla_{V_1}Ric)(V_3, V_2) \\ &-(\nabla_{V_2}Ric)(V_3, V_1)] - 2b[g(\nabla_{V_3}E, V_1)E^\sharp(V_2) \\ &+E^\sharp(V_1)g(\nabla_{V_3}E, V_2) + g(\nabla_{V_1}E, V_3)E^\sharp(V_2) \\ &+E^\sharp(V_3)g(\nabla_{V_1}E, V_2) - g(\nabla_{V_2}E, V_3)E^\sharp(V_1) \\ &-E^\sharp(V_3)g(\nabla_{V_2}E, V_1)]. \end{aligned} \tag{3.3}$$

Assuming that the PVF E as concircular vector field. Then using (1.5) in (3.3), we get

$$\begin{aligned} 2g((\mathcal{L}_E\nabla)(V_3, V_1), V_2) =& -2a[(\nabla_{V_3}Ric)(V_1, V_2) + (\nabla_{V_1}Ric)(V_3, V_2) \\ &-(\nabla_{V_2}Ric)(V_3, V_1)] - 4bfg(V_3, V_1)E^\sharp(V_2). \end{aligned} \tag{3.4}$$

Differentiating (2.6) covariantly with respect to V_3 and utilizing (2.3), one obtains

$$\begin{aligned} (\nabla_{V_3}Ric)(V_1, V_2) =&2\kappa(g(V_3, \phi V_1)\tau(V_2) - g(V_3, \phi V_2)\tau(V_1)) \\ &+2m\kappa(g(V_3, h\phi V_1)\tau(V_2) + g(V_3, h\phi V_2)\tau(V_1)). \end{aligned} \tag{3.5}$$

Applying (3.5) in (3.4), we obtain

$$g((\mathcal{L}_E\nabla)(V_3, V_1), V_2) = -4ma\kappa g(V_3, h\phi V_1)\tau(V_2) - 2bfg(V_3, V_1)E^\sharp(V_2), \tag{3.6}$$

which implies

$$(\mathcal{L}_E\nabla)(V_3, V_1) = -4ma\kappa g(V_3, h\phi V_1)\theta - 2bfg(V_3, V_1)E. \tag{3.7}$$

Differentiating above equation covariantly with respect to V_2 and applying (1.5), we infer

$$\begin{aligned} (\nabla_{V_2}\mathcal{L}_E\nabla)(V_3, V_1) =& -4ma\kappa(g(V_3, (\nabla_{V_2}h\phi)V_1)\theta \\ &-g(V_3, h\phi V_1)(\phi V_2 + \phi h V_2)) \\ &-2b(V_2 f)g(V_3, V_1)E - 2bf^2 g(V_3, V_1)V_2. \end{aligned} \tag{3.8}$$

Due to Yano ([14], p-23), we have

$$(\mathcal{L}_E R)(V_1, V_2)V_3 = (\nabla_{V_1}\mathcal{L}_E\nabla)(V_2, V_3) - (\nabla_{V_2}\mathcal{L}_E\nabla)(V_1, V_3). \tag{3.9}$$

Applying (3.8) in the above equation, we have

$$
\begin{aligned}
(\mathcal{L}_E R)(V_1,V_2)V_3 =& -4ma\kappa(g((\nabla_{V_1}h\phi)V_2-(\nabla_{V_2}h\phi)V_1,V_3)\theta\\
&-g(h\phi V_2,V_3)(\phi V_1+\phi hV_1)\\
&+g(h\phi V_1,V_3)(\phi V_2+\phi hV_2))\\
&-2b(V_1f)g(V_3,V_2)E+2b(V_2f)g(V_3,V_1)E\\
&-2bf^2(g(V_2,V_3)V_1-g(V_1,V_3)V_2).
\end{aligned}
\tag{3.10}
$$

Using (2.11) in (3.10), we get

$$
\begin{aligned}
(\mathcal{L}_E R)(V_1,V_2)V_3 =& -4ma\kappa((\kappa-1)(g(V_1,V_3)\tau(V_2)\theta\\
&-g(V_2,V_3)\tau(V_1)\theta)-g(hV_1,V_3)\tau(V_2)\theta\\
&+g(hV_2,V_3)\tau(V_1)\theta-g(h\phi V_2,V_3)(\phi V_1+\phi hV_1)\\
&\quad+g(h\phi V_1,V_3)(\phi V_2+\phi hV_2))\\
&-2b(V_1f)g(V_3,V_2)E+2b(V_2f)g(V_3,V_1)E\\
&-2bf^2(g(V_2,V_3)V_1-g(V_1,V_3)V_2).
\end{aligned}
\tag{3.11}
$$

Setting $V_2=V_3=\theta$ in the above equation, we obtain

$$
(\mathcal{L}_E R)(V_1,\theta)\theta=-2b(V_1f)E+2b(\theta f)\tau(V_1)E-2bf^2(V_1-\tau(V_1)\theta). \tag{3.12}
$$

Taking Lie derivative of $R(V_1,\theta)\theta=\kappa(V_1-\tau(V_1)\theta)$ along the vector field E, we get

$$
\begin{aligned}
(\mathcal{L}_E R)(V_1,\theta)\theta =& -\kappa((\mathcal{L}_E\tau)(V_1)\theta-\tau(V_1)\mathcal{L}_E\theta)\\
&-R(V_1,\mathcal{L}_E\theta)\theta-R(V_1,\theta)\mathcal{L}_E\theta.
\end{aligned}
\tag{3.13}
$$

Putting $V_2=\theta$ in (1.3) and using (2.7), we infer

$$
(\mathcal{L}_E\tau)(V_1)=g(V_1,\mathcal{L}_E\theta)-2(2ma\kappa-\lambda)\tau(V_1)\theta-2bE^\sharp(V_1)E^\sharp(\theta). \tag{3.14}
$$

Applying (3.14) in (3.13), we have

$$
\begin{aligned}
(\mathcal{L}_E R)(V_1,\theta)\theta =& -\kappa(g(V_1,\mathcal{L}_E\theta)-2(2ma\kappa-\lambda)\tau(V_1)\theta\\
&-2bE^\sharp(V_1)E^\sharp(\theta)-\tau(V_1)\mathcal{L}_E\theta)\\
&-R(V_1,\mathcal{L}_E\theta)\theta-R(V_1,\theta)\mathcal{L}_E\theta.
\end{aligned}
\tag{3.15}
$$

Comparing (3.12) and (3.15), we obtain

$$
\begin{aligned}
&R(V_1,\mathcal{L}_E\theta)\theta+R(V_1,\theta)\mathcal{L}_E\theta\\
=&2b(V_1f)E-2b(\theta f)\tau(V_1)E\\
&+2bf^2(V_1-\tau(V_1)\theta)-\kappa(g(V_1,\mathcal{L}_E\theta)\\
&-2(2ma\kappa-\lambda)\tau(V_1)\theta-2bE^\sharp(V_1)E^\sharp(\theta)\\
&-\tau(V_1)\mathcal{L}_E\theta).
\end{aligned}
\tag{3.16}
$$

Contracting the above equation, we get

$$\begin{aligned} Ric(\mathcal{L}_E\theta,\theta) =& b(Ef)-b(\theta f)\tau(E)+2mbf^2 \\ &+\kappa((2ma\kappa-\lambda)+b(\theta(E))^2). \end{aligned} \tag{3.17}$$

Using (2.7) in the above equation, we get

$$\begin{aligned} g(\mathcal{L}_E\theta,\theta) =& \frac{1}{2m\kappa}[b(Ef)-b(\theta f)\tau(E)+2mbf^2 \\ &+\kappa\{(2ma\kappa-\lambda)+b(\tau(E))^2\}]. \end{aligned} \tag{3.18}$$

Again, putting $V_1=V_2=\theta$ in (1.3), we obtain

$$g(\mathcal{L}_E\theta,\theta) = -\lambda+2ma\kappa+b(\tau(E))^2. \tag{3.19}$$

Comparing (3.18) and (3.19) and putting $E=\theta$, we infer

$$f^2 = \frac{(2m-1)\kappa}{2mb}(2ma\kappa-\lambda+b). \tag{3.20}$$

Thus we may assert the following theorem.

Theorem 3.1. *If a (2m+1)-dimensional NCMM admits GRS where the PVF being the concircular vector field, then* $f^2=\frac{(2m-1)\kappa}{2mb}(2ma\kappa-\lambda+b)$.

Let us consider the PVF be pointwise collinear with the Reeb vector field θ, i.e., $E=\psi\theta$, ψ being a smooth function on the manifold. Then, from (1.2), we have

$$\begin{aligned} &\psi g(\nabla_{V_1}\theta,V_2)+(V_1\psi)\tau(V_2)+\psi g(V_1,\nabla_{V_2}\theta) \\ &+(V_2\psi)\tau(V_2)+2aRic(V_1,V_2)+2b\psi^2\tau(V_1)\tau(V_2)=2\lambda g(V_1,V_2). \end{aligned} \tag{3.21}$$

Using (2.3) in (3.21), we infer

$$\begin{aligned} &-2\psi g(\phi hV_1,V_2)+(V_1\psi)\tau(V_2)+(V_2\psi)\tau(V_1)+2aRic(V_1,V_2) \\ &+2b\psi^2\tau(V_1)\tau(V_2)=2\lambda g(V_1,V_2). \end{aligned} \tag{3.22}$$

Setting $V_2=\theta$ in the foregoing equation, we obtain

$$(V_1\psi)+(\theta\psi)\tau(V_1)=2(\lambda-2ma\kappa-b\psi^2)\tau(V_1). \tag{3.23}$$

Again, putting $V_1=\theta$ in the above equation, we have

$$(\theta\psi)=\lambda-2ma\kappa-b\psi^2. \tag{3.24}$$

Applying (3.24) in (3.23), we get

$$(V_1\psi)=(\lambda-2ma\kappa-b\psi^2)\tau(V_1), \tag{3.25}$$

which implies

$$d\psi=(\lambda-2ma\kappa-b\psi^2)\tau. \tag{3.26}$$

Taking exterior derivative if (3.26) and then taking wedge product with τ, we get

$$(\lambda - 2ma\kappa - b\psi^2)\tau \wedge d\tau = 0. \tag{3.27}$$

Since $\tau \wedge d\tau$ is the volume element, $\tau \wedge d\tau \neq 0$. Thus, from (3.27), we infer

$$\psi^2 = \frac{\lambda - 2ma\kappa}{b}, \tag{3.28}$$

which indicates that ψ is a constant. Thus we assert the following

Theorem 3.2. *If a (2m+1)-dimensional NCMM admits GRS and the PVF is pointwise collinear with the Reeb vector field, then the potential vector field is a constant multiple of the Reeb vector field.*

Let us suppose that the PVF be the Reeb vector field θ, then from (1.2), we infer

$$-2g(\phi h V_1, V_2) + 2aRic(V_1, V_2) + 2b\tau(V_1)\tau(V_2) = 2\lambda g(V_1, V_2). \tag{3.29}$$

Assume an orthonormal frame field $\{e_i\}$, $i = 1, 2, \ldots, (2m+1)$ at any point on the manifold and contracting V_1 and V_2, yields

$$-tr(\phi h) + ar + b = (2m+1)\lambda. \tag{3.30}$$

Since $tr(\phi h) = 0$, we obtain from above equation

$$2ma(2m - 2 + \kappa) + b = (2m+1)\lambda, \tag{3.31}$$

where we used (2.10). Again putting $V_1 = V_2 = \theta$ in (3.29) and using (2.7), we have

$$2ma\kappa + b = \lambda. \tag{3.32}$$

Comparing (3.31) and (3.32), we get

$$\kappa = -\frac{b}{2ma} - \frac{m-1}{m}. \tag{3.33}$$

Thus we may assert the following

Theorem 3.3. *If a (2m+1)-dimensional $N(\kappa)$-contact metric manifold admits generalized Ricci soliton and the potential vector is the Reeb vector field θ, then $\kappa = -\frac{b}{2ma} - \frac{m-1}{m}$.*

4. Generalized Gradient Ricci Solitons on $N(\kappa)$-contact Metric Manifolds

In the current section we investigate the behaviour of generalized gradient Ricci solitons on $N(\kappa)$-contact metric manifolds.

Let us consider that a $(2m+1)$-dimensional NCMM admitting generalized gradient Ricci solitons. Then equation (1.4) can be written as

$$\nabla_{V_1} D\psi = \lambda V_1 - aQV_1 - b(V_1\psi)D\psi, \tag{4.1}$$

where D indicates the gradient operator. With the help of (2.6), the above equation reduces to

$$\begin{aligned}\nabla_{V_1} D\psi =&\lambda V_1 - 2a(m-1)(V_1 + hV_1)\\ &-a(2m\kappa - 2(m-1))\tau(V_1)\theta - b(V_1\psi)D\psi.\end{aligned} \tag{4.2}$$

Differentiating covariantly of the equation (4.2), we infer

$$\begin{aligned}\nabla_{V_2}\nabla_{V_1} D\psi =&(\lambda - 2a(m-1))\nabla_{V_2} V_1 - 2a(m-1)\nabla_{V_2} hV_1\\ &-a(2m\kappa - 2(m-1))(\nabla_{V_2}\tau(V_1)\theta + \tau(V_1)\nabla_{V_2}\theta)\\ &-b(V_2(V_1\psi))D\psi - b(V_1\psi)\nabla_{V_2} D\psi.\end{aligned} \tag{4.3}$$

Altering V_1 and V_2 in the foregoing equation, we have

$$\begin{aligned}\nabla_{V_1}\nabla_{V_2} D\psi =&(\lambda - 2a(m-1))\nabla_{V_1} V_2 - 2a(m-1)\nabla_{V_1} hV_2\\ &-a(2m\kappa - 2(m-1))(\nabla_{V_1}\tau(V_2)\theta + \tau(V_2)\nabla_{V_1}\theta)\\ &-b(V_1(V_2\psi))D\psi - b(V_2\psi)\nabla_{V_1} D\psi.\end{aligned} \tag{4.4}$$

Also, from (4.2), we get

$$\begin{aligned}\nabla_{[V_1,V_2]} D\psi =&(\lambda - 2a(m-1))[V_1, V_2] - 2a(m-1)h[V_1, V_2]\\ &-a(2m\kappa - 2(m-1))\tau([V_1, V_2])\theta - b([V_1, V_2]\psi)D\psi,\end{aligned} \tag{4.5}$$

Using equations (4.3)-(4.5), we have

$$\begin{aligned}R(V_1, V_2)D\psi =& -2a(m-1)\{(\nabla_{V_1} h)V_2 - (\nabla_{V_2} h)V_1\}\\ &-a(2m\kappa - 2(m-1))\{(\nabla_{V_1}\tau)(V_2)\theta - (\nabla_{V_2}\tau)(V_1)\theta\\ &+\tau(V_2)\nabla_{V_1}\theta - \tau(V_1)\nabla_{V_2}\theta\}\\ &-b\{(V_2\psi)\nabla_{V_1} D\psi - (V_1\psi)\nabla_{V_2} D\psi\}.\end{aligned} \tag{4.6}$$

Remembering (2.3), (2.8), (2.9) and (4.2), the above equation reduces to

$$\begin{aligned}R(V_1, V_2)D\psi =& -2a(m-1)\{2(1-\kappa)g(V_1, \phi V_2)\theta\\ &+\tau(V_2)(h(\phi V_1 + \phi hV_1)) - \tau(V_1)(h(\phi V_2 + \phi hV_2))\}\\ &-a(2m\kappa - 2(m-1))\{2g(V_1, \phi V_2 + h\phi V_2)\theta\\ &+\tau(V_1)(\phi V_2 + \phi hV_2) - \tau(V_2)(\phi V_1 + \phi hV_1)\}\\ &-b\{-2a(m-1)((V_2\psi)(V_1 + hV_1) - (V_1\psi)(V_2 + hV_2))\\ &-a(2m\kappa - 2(m-1))((V_2\psi)\tau(V_1)\theta - (V_1\psi)\tau(V_2)\theta)\\ &+\lambda((V_2\psi)V_1 - (V_1\psi)V_2)\}.\end{aligned} \tag{4.7}$$

Taking inner product with θ, we have

$$\begin{aligned}g(R(V_1, V_2)D\psi, \theta) =& -4a(m-1)g(V_1, \phi V_2)\\ &-2a(2m\kappa - 2(m-1))g(V_1, \phi V_2 + h\phi V_2)\\ &-b(\lambda - 2am\kappa)((V_2\psi)\tau(V_1)(V_1\psi)\tau(V_2)).\end{aligned} \tag{4.8}$$

Again, taking inner product of (2.5) with $D\psi$, we get

$$g(R(V_1,V_2)\theta, D\psi) = \kappa((V_1\psi)\tau(V_2) - (V_2\psi)\tau(V_1)). \tag{4.9}$$

Equations (4.8) and (4.9) together give

$$\begin{aligned}&(\kappa + b(\lambda - 2am\kappa))((V_2\psi)\tau(V_1) - (V_1\psi)\tau(V_2))\\ =& -4a(m-1)(1-\kappa)g(V_1,\phi V_2) - 2a(2m\kappa - 2(m-1))g(V_1,\phi V_2 + h\phi V_2).\end{aligned} \tag{4.10}$$

Setting $V_2 = \theta$ in (4.10), we infer

$$(\kappa + b(\lambda - 2am\kappa))((\theta\psi)\tau(V_1) - (V_1\psi)) = 0, \tag{4.11}$$

which indicates that either $\lambda = 2am\kappa - \frac{\kappa}{b}$ or $D\psi = (\theta\psi)\theta$. Thus we conclude that

Theorem 4.1. *If a $(2m+1)$-dimensional NCMM admits GGRS, then either $\lambda = 2am\kappa - \frac{\kappa}{b}$ or the potential vector field is pointwise collinear with the characteristic vector field θ.*

5. Example

In [5], De at el. initiated an example of a $N(\kappa)$-contact metric manifold. Following that example we construct the following.

Let us consider the manifold $M = \{x_1, x_2, x_3 \in \mathbb{R}^3 : x_3 \neq 0\}$ of dimension 3, where (x_1, x_2, x_3) are standard co-ordinates in $\mathbb{R}^3$. We choose the vector fields W_1, W_2 and W_3 which satisfy

$$[W_1, W_2] = 3W_3, \quad [W_1, W_3] = W_2, \quad [W_2, W_3] = 2W_1.$$

The Riemannian metric tensor g is considered as

$$g(W_i, W_i) = 1, \qquad i = 1, 2, 3$$

and

$$g(W_i, W_j) = 0, \qquad i \neq j.$$

The 1-form τ is defined by

$$\tau(V) = g(V, W_1),$$

for every vector field V on M. The $(1,1)$ type tensor field ϕ is given by

$$\phi(W_1) = 0, \quad \phi(W_2) = W_3, \quad \phi(W_3) = -W_2.$$

Then we find that

$$\tau(W_1) = 1, \quad \phi^2 V_1 = -V_1 + \tau(V_1)W_1,$$

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) - \tau(V_1)\tau(V_2), \quad d\tau(V_1, V_2) = g(V_1, \phi V_2),$$

for every vector fields V_1, V_2 on M. Thus (ϕ, W_1, τ, g) defines a contact structure.

Due to Koszul's famous formula, we obtain the following

$$\nabla_{W_1}W_1 = 0, \quad \nabla_{W_1}W_2 = 0, \quad \nabla_{W_1}W_3 = 0,$$

$$\nabla_{W_2}W_2 = 0, \quad \nabla_{W_2}W_1 = -3W_3, \quad \nabla_{W_2}W_3 = 3W_1,$$

$$\nabla_{W_3}W_3 = 0, \quad \nabla_{W_3}W_1 = -W_2, \quad \nabla_{W_3}W_2 = W_1.$$

From the above expressions of ∇, we obtain

$$hW_1 = 0, \quad hW_2 = 2W_2, \quad hW_3 = -2W_3.$$

We also have

$$R(W_1, W_2)W_2 = -3W_1, \quad R(W_2, W_1)W_1 = -3W_2, \quad R(W_2, W_3)W_3 = 3W_2,$$

$$R(W_3, W_2)W_2 = 3W_3, \quad R(W_1, W_3)W_3 = -3W_1, \quad R(W_3, W_1)W_1 = -3W_3,$$

$$R(W_1, W_2)W_3 = 0, \quad R(W_2, W_3)W_1 = 0, \quad R(W_1, W_3)W_2 = 0.$$

Thus the manifold is an $N(\kappa)$-contact metric manifold with $\kappa = -3$.

On contraction of curvature tensor, we infer

$$Ric(W_1, W_1) = -6, \quad Ric(W_2, W_2) = 0, \quad Ric(W_3, W_3) = 0.$$

The scalar curvature r of the manifold is given by

$$r = Ric(W_1, W_1) + Ric(W_2, W_2) + Ric(W_3, W_3) = -6.$$

Let the potential vector field $E = W_1$, then

$$(\mathcal{L}_{W_1}g)(W_1, W_1) = 0, \quad (\mathcal{L}_{W_1}g)(W_2, W_2) = 0, \quad (\mathcal{L}_{W_1}g)(W_3, W_3) = 0.$$

The above data indicates that the given manifold admits generalized Ricci soliton with $\lambda = 0$ and hence the soliton is steady type. Also, from (1.3), we see that $6a = b$ and these data satisfy equation (3.33). Hence **Theorem 3.3** is verified.

Acknowledgements. The authors are thankful to the referee for his/her valuable suggestions towards the improvement of the paper.

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