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Duality of Paranormed Spaces of Matrices Defining Linear Operators from l_p into l_q

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Abstract. Let $1 \leq p, q < \infty$ be fixed, and let $R = [r_{jk}]$ be an infinite scalar matrix such that $1 \leq r_{jk} < \infty$ and $\sup_{j,k} r_{jk} < \infty$. Let $\mathcal{B}(l_p, l_q)$ be the set of all bounded linear operator from l_p into l_q. For a fixed Banach algebra $\mathbf{B}$ with identity, we define a new vector space $S^R_{p,q}(\mathbf{B})$ of infinite matrices over $\mathbf{B}$ and a paranorm G on $S^R_{p,q}(\mathbf{B})$ as follows: let

$$S^R_{p,q}(\mathbf{B}) = \left\{A : A^{[R]} \in \mathcal{B}(l_p, l_q)\right\}$$

and $G(A) = \|A^{[R]}\|^{\frac{1}{M}}_{p,q}$, where $A^{[R]} = [\|a_{jk}\|^{r_{jk}}]$ and $M = \max\{1, \sup_{j,k} r_{jk}\}$. The existance of $S^R_{p,q}(\mathbf{B})$ equipped with the paranorm $G(\cdot)$ including its completeness are studied. We also provide characterizations of β -dual of the paranormed space.

1. Introduction

For any vector space X, we call a function $g : X \mapsto \mathbb{R}^+$ a paranorm on X if g satisfies the following conditions:

1. $g(\theta) = 0$, where θ is the zero element in X,
2. $g(x) = g(-x)$ for all $x \in X$,
3. $g(x + y) \leq g(x) + g(y)$ for all $x, y \in X$,
4. If $\{\alpha_n\}$ is a sequence of scalars with $|\alpha_n - \alpha| \to 0$ and $\{x_n\}$ is a sequence of vectors with $g(x_n - x) \to 0$, then $g(\alpha_n x_n - \alpha x) \to 0$.

A paranormed space is a pair (X, g) of a vector space X and a paranorm g on X. If (X, g) is a paranormed space, then the function $d : X \times X \to \mathbb{R}$ defined by $d(x, y) = g(x-y)$ is a pseudometric on X, and hence it becomes a metric on the set $X/\sim$ of all equivalence classes of elements of X under the equivalence relation $\sim$

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on X defined by $x \sim y \Leftrightarrow d(x,y)=0$. With this consideration, every paranormed space can be regarded as a metric space.

Let $p=\{p_k\}$ be a bounded sequence of real numbers such that $p_k \geq 1$ for all $k \in \mathbb{N}$, and let $M=\max\{1, \sup p_k\}$. It is well-known that the following sequence spaces, defined by Maddox [11] and known as the sequence spaces of Maddox (see further in [20] and [14]):

$$\begin{aligned} c_0(p) &= \{\{x_k\} : |x_k|^{p_k} \to 0 \quad \text{as} \quad k \to \infty\}, \\ c(p) &= \{\{x_k\} : |x_k - l|^{p_k} \to 0 \quad \text{as} \quad k \to \infty \quad \text{for some} \quad l \in \mathbb{R}\}, \\ l_\infty(p) &= \left\{\{x_k\} : \sup_k |x_k|^{p_k} < \infty\right\}, \\ l(p) &= \left\{\{x_k\} : \sum_k |x_k|^{p_k} < \infty\right\}, \end{aligned}$$

are complete paranormed spaces, where the first three spaces are equipped with the paranorm g_1 defined by

$$g_1(\{x_k\}) = \sup_k |x_k|^{\frac{p_k}{M}},$$

and the last one is equipped with the paranorm g_2 defined by

$$g_2(\{x_k\}) = \left(\sum_k |x_k|^{p_k}\right)^{\frac{1}{M}}.$$

We see that when $p_k = p$ for all k, the above sequence spaces of Maddox become the classical Banach sequence spaces

$$\begin{aligned} c_0 &= \{\{x_k\} : |x_k| \to 0 \quad \text{as} \quad k \to \infty\}, \\ c &= \{\{x_k\} : |x_k| \quad \text{is convergent}\}, \\ l_\infty &= \left\{\{x_k\} : \sup_k |x_k| < \infty\right\}, \end{aligned}$$

and

$$l_p = \left\{\{x_k\} : \sum_k |x_k|^p < \infty\right\}.$$

Let E be a subspace of the vector space of all X -valued sequences, called an X -valued sequence space. The α -dual and β -dual of E introduced by Maddox [13] are defined as follows:

$$\begin{aligned} E^\alpha &= \left\{\{A_k\} \subseteq \mathcal{L}(X,Y) : \sum_k A_k x_k \quad \text{converges for all} \quad \{x_k\} \in E\right\}, \\ E^\beta &= \left\{\{A_k\} \subseteq \mathcal{L}(X,Y) : \sum_k \|A_k x_k\| < \infty \quad \text{for all} \quad \{x_k\} \in E\right\}, \end{aligned}$$

where $\mathcal{L}(X,Y)$ is the set of all linear operator from a normed space X into a normed space Y.

There have been several works on the notions of α -dual and β -dual defined by Maddox mentioned above (see [6], [7], [8], and [10] for some references). Grosse-Erdmann [6] investigated some topological and sequence structure of scalar-valued sequence spaces of Maddox. In 2002, Suantai and Sanhan [21] provided some general properties of β -dual of the vector-valued sequence spaces of Maddox, and gave characterizations of β -dual of the sequence spaces $l(p)$ when $p_k > 1$ for all $k \in \mathbb{N}$. In 2012, Rakbud and Suantai [18] gave a general theorem on duality for a class of Banach-valued function spaces which is a generalization of the classical sequence space l_p for $1 \le p < \infty$. The β -dual of Banach-space-valued difference sequence spaces $E(\Delta) = \{\{x_k\} : \{\Delta x_k\} \in E\}$ where $E = \{l_\infty, c, c_0\}$ and $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$, was studied by Bhardwaj and Gupta [2].

For any Banach space X, an X -valued sequence space E is called a BK -space if it is a Banach space and the k -th coordinate mapping $p_k : E \to X$, $p_k(x) = x_k$, is continuous for all $k \in \mathbb{N}$. In 2013, Faroughi, Osgooei and Rahimi provided some properties of α -dual and β -dual of a BK -space. Furthermore, the concepts of α -dual and β -dual of a BK -space were used by the same authors in their other works to define some new spaces (see [4], [5], and [15]).

Let $1 \le p, q < \infty$. An infinite scalar matrix $A = [a_{jk}]$ is said to define a linear operator from l_p into l_q if for every $\{x_k\}$ in l_p, the $\sum_k a_{jk}x_k$ converges for all j and the sequence $Ax = \{\sum_k a_{jk}x_k\}$ is in l_q. If a matrix A defines a linear operator from l_p into l_q, we call the operator $x \mapsto Ax$ the linear operator defined by A. By the uniform boundedness principle, the linear operator defined by A is bounded. Let $\mathcal{B}(l_p, l_q)$ be the set of all bounded linear operator from l_p into l_q. Then $\mathcal{B}(l_p, l_q)$ is the Banach space, and it is isometrically isomorphic to the space of matrices defining a linear operator from l_p into l_q endowed with the operator norm on $\mathcal{B}(l_p, l_q)$.

For any two matrices $A = [a_{jk}]$ and $C = [c_{jk}]$ of the same size, the Schur product of A and C is the matrix $A \bullet C$ given by $A \bullet C = [a_{jk}c_{jk}]$. Schur [19] showed that the Banach space $\mathcal{B}(l_2)$ is a commutative Banach algebra under the operator norm and the Schur product multiplication. The Banach space $\mathcal{B}(l_p, l_q)$ under the Schur product operation is a Banach algebra proven by Bennett [1]. Let $1 \le r < \infty$, for a fixed Banach algebra $\mathbf{B}$ with identity, Chaisuriya and Ong [3] considered the space of matrices

$$S^r_{p,q}(\mathbf{B}) = \left\{A : A^{[r]} \in \mathcal{B}(l_p, l_q)\right\}$$

where $A^{[r]} = [\|a_{jk}\|^r]$. They obtained that it is a Banach algebra under the absolute Schur r -norm defined by $\|A\|_{p,q,r} = \|A^{[r]}\|^{\frac{1}{r}}_{p,q}$, and also proved that $S^2_{p,q}(\mathbb{C})$ contains $\mathcal{B}(l_p, l_q)$ as an ideal. In 2001, Livshits, Ong and Wang [9] studied the duality in the absolute Schur algebra $S^r_{2,2}(\mathbb{C})$ by a way analogous to Dixmiers theorem and Schattens theorem. After that, Rakbud and Chaisuriya [17] generalized the results of Livshits, Ong and Wang [9] to the absolute Schur algebra $S^2_{\Lambda,\Sigma}(\mathbf{B})$, where $\Lambda, \Sigma \in \{l_p : 1 \le p < \infty\} \cup \{c_0\}$, which was examined by the same authors in 2005 (see [16]).

In this work, we extend the definition of the set $S^r_{p,q}(\mathbf{B})$ defined by Chaisuriya and Ong [3] from the fixed real number r, which is greater than or equal to 1, to a fixed bounded matrix $R = [r_{jk}]$ of scalars, which is greater than or equal to 1. Hence our setting becomes

$$S^R_{p,q}(\mathbf{B}) = \Big\{A : A^{[R]} \in \mathcal{B}(l_p, l_q)\Big\}$$

where $A^{[R]} = [\|a_{jk}\|^{r_{jk}}]$. Our goal is to define a paranorm on the vector space $S^R_{p,q}(\mathbf{B})$ and investigate the properties of this paranormed space, including its existence, completeness, and duality.

2. The paranormed vector space over a Banach algebra

In this section, the two versions of the Minkowski's inequality that Maddox demonstrated in [12] are mentioned to achieve our results.

Theorem 2.1. *Let $p \geq 1$, $a_1, a_2, \cdots, a_n \geq 0$ and $b_1, b_2, \cdots, b_n \geq 0$. Then*

$$\left(\sum_{k=1}^{n}(a_k + b_k)^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} a_k^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} b_k^p\right)^{\frac{1}{p}}.$$

Theorem 2.2. *Let $0 < p \leq 1$, $a_1, a_2, \cdots, a_n \geq 0$ and $b_1, b_2, \cdots, b_n \geq 0$. Then*

$$\sum_{k=1}^{n}(a_k + b_k)^p \leq \sum_{k=1}^{n} a_k^p + \sum_{k=1}^{n} b_k^p.$$

In the following theorems, we investigate some elementary properties of $S^R_{p,q}(\mathbf{B})$.

Theorem 2.3. *$S^R_{p,q}(\mathbf{B})$ is a linear space.*

Proof. Let $A = [a_{jk}]$ and $B = [b_{jk}]$ be matrices in $S^{[R]}_{p,q}(\mathbf{B})$. Then $A^{[R]}$ and $B^{[R]} \in \mathcal{B}(l_p, l_q)$. Let $M = \max(1, \sup_{j,k} r_{jk})$. For any fixed positive integers J and K, and any fixed unit vector $x = \{x_k\} \in l_p$, by Theorem 2.1 and Theorem 2.2,

$$\begin{aligned}
&\sum_{j=1}^{J}\left[\sum_{k=1}^{K}\|a_{jk}+b_{jk}\|^{r_{jk}}|x_k|\right]^q\\
&\le \sum_{j=1}^{J}\left[\sum_{k=1}^{K}\left(\|a_{jk}\|^{\frac{r_{jk}}{M}}|x_k|^{\frac{1}{M}}+\|b_{jk}\|^{\frac{r_{jk}}{M}}|x_k|^{\frac{1}{M}}\right)^M\right]^q\\
&\le \sum_{j=1}^{J}\left[\left(\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}|x_k|\right)^{\frac{1}{M}}+\left(\sum_{k=1}^{K}\|b_{jk}\|^{r_{jk}}|x_k|\right)^{\frac{1}{M}}\right]^{qM}\\
&\le \left\{\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}|x_k|\right)^q\right]^{\frac{1}{qM}}+\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{K}\|b_{jk}\|^{r_{jk}}|x_k|\right)^q\right]^{\frac{1}{qM}}\right\}^{qM}\\
&\le \left\{\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}|x_k|\right)^q\right]^{\frac{1}{q}}+\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{K}\|b_{jk}\|^{r_{jk}}|x_k|\right)^q\right]^{\frac{1}{q}}\right\}^{qM}\\
&\le \left(\|(A^{[R]})(|x|)\|_q+\|(B^{[R]})(|x|)\|_q\right)^{qM}\\
&\le \left(\|A^{[R]}\|_{p,q}\|x\|_p+\|B^{[R]}\|_{p,q}\|x\|_p\right)^{qM}\\
&\le \left(\|A^{[R]}\|_{p,q}+\|B^{[R]}\|_{p,q}\right)^{qM}.
\end{aligned}$$

Since J and K are arbitrary, we have

$$\begin{aligned}
\|(A+B)^{[R]}x\|_q^q &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\|a_{jk}+b_{jk}\|^{r_{jk}}x_k\right|^q\\
&\le \sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\|a_{jk}+b_{jk}\|^{r_{jk}}|x_k|\right]^q\\
&\le \left(\|A^{[R]}\|_{p,q}+\|B^{[R]}\|_{p,q}\right)^{qM}.
\end{aligned}$$

This implies that $\|(A+B)^{[R]}x\|_q \le \left(\|A^{[R]}\|_{p,q}+\|B^{[R]}\|_{p,q}\right)^M$. Since $A^{[R]}$ and $B^{[R]} \in \mathcal{B}(l_p, l_q)$, $\|(A+B)^{[R]}\|_{p,q} < \infty$. So $A+B \in S^R_{p,q}(\mathbf{B})$. Next, we let $\alpha \in \mathbf{B}$. For any fixed unit vector $x = \{x_k\} \in l_p$, we obtain

$$\begin{aligned}\left\|(\alpha A)^{[R]}x\right\|_q &\leq \left\{\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\|\alpha a_{jk}\|^{r_{jk}}|x_k|\right]^q\right\}^{\frac{1}{q}} \\ &\leq \left\{\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\|\alpha\|^M\|a_{jk}\|^{r_{jk}}|x_k|\right]^q\right\}^{\frac{1}{q}} \\ &\leq \|\alpha\|^M\|(A^{[R]})|x|\|_q \\ &\leq \|\alpha\|^M\|A^{[R]}\|_{p,q}\end{aligned}$$

Since $A^{[R]} \in \mathcal{B}(l_p, l_q)$, $\|(\alpha A)^{[R]}\|_{p,q} < \infty$. Then $\alpha A \in S^R_{p,q}(\mathbf{B})$. By the termwise sum and any scalar multiple of any matrices in $S^R_{p,q}(\mathbf{B})$, for any matrices $A, B, C \in S^R_{p,q}(\mathbf{B})$ and any scalars $\alpha, \beta \in \mathbf{B}$,

1. $(A+B)^{[R]} = (B+A)^{[R]}$,
2. $(A+(B+C))^{[R]} = ((A+B)+C)^{[R]}$,
3. there exists $\underline{0} \in S^{[R]}_{p,q}(\mathbf{B})$ such that $(A+\underline{0})^{[R]} = A^{[R]}$,
4. there is $-A \in S^R_{p,q}(\mathbf{B})$ such that $(A+(-A))^{[R]} = \underline{0}$,
5. $(\alpha(\beta A))^{[R]} = ((\alpha\beta)A)^{[R]}$,
6. $((\alpha+\beta)A)^{[R]} = (\alpha A + \beta A)^{[R]}$,
7. $(\alpha(A+B))^{[R]} = (\alpha A + \alpha B)^{[R]}$,
8. since the identity $1 \in \mathbf{B}$, $(1A)^{[R]} = A^{[R]}$.

This completes the proof. □

We define $G : S^R_{p,q}(\mathbf{B}) \to \mathbb{R}^+$ by

$$G(A) := \|A^{[R]}\|_{p,q}^{\frac{1}{M}}$$

where $M = \max(1, \sup_{j,k} r_{jk})$.

Theorem 2.4. $S^R_{p,q}(\mathbf{B})$ *equipped with* $G(\cdot)$ *is a paranormed space.*

Proof. It is obvious that $G(\underline{0}) = 0$. Since $\|A^{[R]}\|_{p,q} = \|(-A)^{[R]}\|_{p,q}$, $G(A) = G(-A)$ for all $A \in S^R_{p,q}(\mathbf{B})$. Next, let $A, B \in S^R_{p,q}(\mathbf{B})$. To show $G(A+B) \leq G(A) + G(B)$, we use the same technique of the proof in Theorem 2.3. For any fixed positive integers J and K, and any unit vector $x = \{x_k\} \in l_p$,

$$\left\{\sum_{j=1}^{J}\left[\sum_{k=1}^{K}\|a_{jk}+b_{jk}\|^{r_{jk}}|x_k|\right]^q\right\}^{\frac{1}{M}}$$
$$= \left\{\sum_{j=1}^{J}\left[\sum_{k=1}^{K}\left(\|a_{jk}+b_{jk}\|^{\frac{r_{jk}}{M}}|x_k|^{\frac{1}{M}}\right)^M\right]^q\right\}^{\frac{1}{M}}$$
$$\leq \left\{\sum_{j=1}^{J}\left[\sum_{k=1}^{K}\left(\|a_{jk}\|^{\frac{r_{jk}}{M}}|x_k|^{\frac{1}{M}}+\|b_{jk}\|^{\frac{r_{jk}}{M}}|x_k|^{\frac{1}{M}}\right)^M\right]^q\right\}^{\frac{1}{M}}$$
$$\leq \left\{\sum_{j=1}^{J}\left[\left(\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}|x_k|\right)^{\frac{1}{M}}+\left(\sum_{k=1}^{K}\|b_{jk}\|^{r_{jk}}|x_k|\right)^{\frac{1}{M}}\right]^{qM}\right\}^{\frac{1}{M}}$$
$$\leq \left\{\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}|x_k|\right)^q\right]^{\frac{1}{qM}}+\left[\sum_{j=1}^{J}\left(\sum_{k=1}^{K}\|b_{jk}\|^{r_{jk}}|x_k|\right)^q\right]^{\frac{1}{qM}}\right\}^q$$
$$\leq \left\{\|(A^{[R]})|x|\|_q^{\frac{1}{M}}+\|(B^{[R]})|x|\|_q^{\frac{1}{M}}\right\}^q$$
$$\leq \left(\|A^{[R]}\|_{p,q}^{\frac{1}{M}}+\|B^{[R]}\|_{p,q}^{\frac{1}{M}}\right)^q.$$

Therefore

$$\|(A+B)^{[R]}x\|_q^{\frac{q}{M}} = \left\{\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\|a_{jk}+b_{jk}\|^{r_{jk}}x_k\right|^q\right\}^{\frac{1}{M}}$$
$$\leq \left\{\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\|a_{jk}+b_{jk}\|^{r_{jk}}|x_k|\right]^q\right\}^{\frac{1}{M}}$$
$$\leq \left(\|A^{[R]}\|_{p,q}^{\frac{1}{M}}+\|B^{[R]}\|_{p,q}^{\frac{1}{M}}\right)^q$$

which implies that $\|(A+B)^{[R]}\|_{p,q}^{\frac{1}{M}} \leq \|A^{[R]}\|_{p,q}^{\frac{1}{M}}+\|B^{[R]}\|_{p,q}^{\frac{1}{M}}$. Thus $G(A+B) \leq G(A)+G(B)$. Finally, we assume that $\{\alpha_n\}$ is a sequence in $\mathbf{B}$ such that $\|\alpha_n-\alpha\| \to 0$ and $\{A_n = [a_{jk}^{(n)}]\}$ is a sequence in $S_{p,q}^R(\mathbf{B})$ with $G(A_n-A) \to 0$ as $n \to \infty$. We claim that $\left\{\|(A_n)^{[R]}\|_{p,q}^{\frac{1}{M}}\right\}$ is bounded. Since $G(A_n-A) \to 0$ as $n \to \infty$, there are $T \in \mathbb{N}$ and $K > 0$ such that for all $n \geq T$,

$$G(A_n - A) < K. \tag{2.1}$$

Because of finiteness of $G(A)$, there is $L > 0$ such that $G(A) < L$. By (2.1), we see that for all $n \geq T$,

$$\begin{aligned}\|(A_n)^{[R]}\|_{p,q}^{\frac{1}{M}} &= G(A_n)\\ &= G(A_n - A + A)\\ &\leq G(A_n - A) + G(A)\\ &\leq K + L.\end{aligned}$$

So we get the claim. Next, we want to show that $G(\alpha_n A_n - \alpha A) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$. Since $\{A_n\}$ is a sequence in $S^R_{p,q}(\mathbf{B})$, by the boundedness of $\left\{\|(A_n)^{[R]}\|_{p,q}^{\frac{1}{M}}\right\}$, there exists $J > 0$ such that

$$\|(A_n)^{[R]}\|_{p,q}^{\frac{1}{M}} < J \quad \text{for all } n \in \mathbb{N}.$$

Because $\|\alpha_n - \alpha\| \to 0$ as $n \to \infty$, there is $Q \in \mathbb{N}$ such that for all $n \geq Q$,

$$\|\alpha_n - \alpha\| < \left(\frac{\varepsilon}{2J}\right)^{\frac{1}{q}}.$$

By the assumption that $G(A_n - A) \to 0$ as $n \to \infty$, there is $P \in \mathbb{N}$ such that for all $n \geq P$,

$$G(A_n - A) < \frac{\varepsilon}{2\|\alpha\|^q}.$$

That is

$$\|(A_n - A)^{[R]}\|_{p,q}^{\frac{1}{M}} < \frac{\varepsilon}{2\|\alpha\|^q}.$$

Choose $N = \max(Q, P)$ and let $n \geq N$,

$$\begin{aligned}&\|(\alpha_n A_n - \alpha A)^{[R]}\|_{p,q}^{\frac{1}{M}}\\ &\quad= \|(\alpha_n A_n - \alpha A_n + \alpha A_n - \alpha A)^{[R]}\|_{p,q}^{\frac{1}{M}}\\ &\quad\leq \|(\alpha_n A_n - \alpha A_n)^{[R]}\|_{p,q}^{\frac{1}{M}} + \|(\alpha A_n - \alpha A)^{[R]}\|_{p,q}^{\frac{1}{M}}\\ &\quad\leq \left(\|\alpha_n - \alpha\|^{Mq}\right)^{\frac{1}{M}} \|(A_n)^{[R]}\|_{p,q}^{\frac{1}{M}} + \left(\|\alpha\|^{Mq}\right)^{\frac{1}{M}} \|(A_n - A)^{[R]}\|_{p,q}^{\frac{1}{M}}\\ &\quad< \|\alpha_n - \alpha\|^q J + \|\alpha\|^q \left(\frac{\varepsilon}{2\|\alpha\|^q}\right)\\ &\quad= \left[\left(\frac{\varepsilon}{2J}\right)^{\frac{1}{q}}\right]^q J + \frac{\varepsilon}{2}\\ &\quad= \varepsilon.\end{aligned}$$

Hence we have the theorem. □

Theorem 2.5. $S^R_{p,q}(\mathbf{B})$ *is a complete paranormed space.*

Proof. Let $\{A_n = [a^{(n)}_{jk}]\}$ be a Cauchy sequence in $S^{[R]}_{p,q}(\mathbf{B})$. We will show that there is $A \in S^{[R]}_{p,q}(\mathbf{B})$ such that $G(A_n - A) \to 0$ as $n \to \infty$. Since $\{A_n\}$ is a Cauchy sequence in $S^{[R]}_{p,q}(\mathbf{B})$,

$$G(A_n - A_m) \to 0 \quad \text{as } m, n \to \infty. \tag{2.2}$$

For a fixed j and k, we have

$$\begin{aligned} \|a^{(n)}_{jk} - a^{(m)}_{jk}\|^{r_{jk}}|x_k| &\le \left(\sum_{j=1}^{\infty}\left[\sum_{k=1}^{\infty}\|a^{(n)}_{jk} - a^{(m)}_{jk}\|^{r_{jk}}|x_k|\right]^q\right)^{\frac{1}{qM}} \\ &= \|(A_n - A_m)^{[R]}\|^{\frac{1}{M}}_{p,q}. \end{aligned} \tag{2.3}$$

From (2.2) and (2.3), the sequence $\{a^{(n)}_{jk}\}$ is a Cauchy sequence in $\mathbf{B}$. Since $\mathbf{B}$ is a Banach algebra, there is $a_{jk} \in \mathbf{B}$ such that

$$\|a^{(n)}_{jk} - a_{jk}\| \to 0 \quad \text{as } n \to \infty$$

for any j and k. Let $A = [a_{jk}]$. To show that $A \in S^{[R]}_{p,q}(\mathbf{B})$. Let $x = \{x_k\} \in l_p$ with $\|x\|_p \le 1$. Since $\{A_n\}$ is a Cauchy sequence in $S^{[R]}_{p,q}(\mathbf{B})$, there exists $N_1 \in \mathbb{N}$ such that for all $m, n \ge N_1$,

$$\|(A_n - A_m)^{[R]}x\|^{\frac{1}{M}}_q = G(A_n - A_m) < 1.$$

For a fixed J and K, we have

$$\sum_{j=1}^{J}\left[\sum_{k=1}^{K}\|a^{(n)}_{jk} - a^{(m)}_{jk}\|^{r_{jk}}|x_k|\right]^q < 1 \quad \text{for all } m, n \ge N_1.$$

Thus by taking the limit on $m \to \infty$, we get

$$\sum_{j=1}^{J}\left[\sum_{k=1}^{K}\|a^{(n)}_{jk} - a_{jk}\|^{r_{jk}}|x_k|\right]^q < 1 \quad \text{for all } n \ge N_1.$$

Consider $A_{N_1} = [a^{(N_1)}_{jk}] \in S^{[R]}_{p,q}(\mathbf{B})$. Since $(A_{N_1})^{[R]} \in B(l_p, l_q)$, there is $T > 0$ such that

$$\|(A_{N_1})^{[R]}x\|^{\frac{1}{M}}_q < T.$$

Therefore

$$\left(\sum_{j=1}^{J}\left|\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}x_k\right|^q\right)^{\frac{1}{qM}} \le \left(\|(A-A_{N_1}+A_{N_1})^{[R]}x\|_q\right)^{\frac{1}{M}}$$
$$\le \|(A-A_{N_1})^{[R]}x\|_q^{\frac{1}{M}} + \|(A_{N_1})^{[R]}x\|_q^{\frac{1}{M}}$$
$$\le 1+T.$$

Then

$$\sum_{j=1}^{J}\left|\sum_{k=1}^{K}\|a_{jk}\|^{r_{jk}}x_k\right|^q \le (1+T)^{qM}.$$

This implies that

$$\|A^{[R]}x\|_q = \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\|a_{jk}\|^{r_{jk}}x_k\right|^q\right)^{\frac{1}{q}} \le (1+T)^M.$$

So $A^{[R]} \in B(l_p, l_q)$ and then $A \in S_{p,q}^{[R]}(\mathbf{B})$. Next, we will prove that $G(A_n - A) \to 0$ as $n \to \infty$. Let $\varepsilon > 0$ and $x = \{x_k\} \in l_p$ with $\|x\|_p \le 1$. Since $\{A_n\}$ is a Cauchy sequence in $S_{p,q}^{[R]}(\mathbf{B})$, there exists $N_2 \in \mathbb{N}$ such that

$$\|(A_n - A_m)^{[R]}x\|_q^{\frac{1}{M}} = G(A_n - A_m) < \frac{\varepsilon}{2} \quad \text{for all } m, n \ge N_2.$$

For a fixed J and K, we have

$$\left(\sum_{j=1}^{J}\left|\sum_{k=1}^{K}\|a_{jk}^{(n)} - a_{jk}^{(m)}\|^{r_{jk}}x_k\right|^q\right)^{\frac{1}{qM}} < \frac{\varepsilon}{2} \quad \text{for all } m, n \ge N_2.$$

Thus for all J and K, and $n \ge N_2$, by taking the limit as $m \to \infty$, we get

$$\left(\sum_{j=1}^{J}\left|\sum_{k=1}^{K}\|a_{jk}^{(n)} - a_{jk}\|^{r_{jk}}x_k\right|^q\right)^{\frac{1}{qM}} < \frac{\varepsilon}{2}.$$

By taking the limits as $K \to \infty$ and then $J \to \infty$,

$$\|(A_n - A)^{[R]}x\|_q^{\frac{1}{M}} = \left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\|a_{jk}^{(n)} - a_{jk}\|^{r_{jk}}x_k\right|^q\right)^{\frac{1}{qM}} < \frac{\varepsilon}{2},$$

as asserted. □

3. Duality of matrix spaces of infinite matrices

Let E be a vector subspace of the vector space of all infinite matrices over a Banach algebra $\mathbf{B}$. We call E a matrix space.
For any $A = [a_{jk}] \in E$, we define a partial sum of $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}$ by the finite sum

$$s_{mn} = \sum_{j=1}^{m}\sum_{k=1}^{n} a_{jk}$$

for all $m, n \in \mathbb{N}$.

Definition 3.1. The double series $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}$ is said to converge if $\sum_{k=1}^{\infty} a_{jk}$ converges for all $j \in \mathbb{N}$, and $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}$ converges.

Definition 3.2. A matrix space E is said to be normal if $A = [a_{jk}] \in E$ whenever $\|a_{jk}\| \leq \|b_{jk}\|$ for all $j, k \in \mathbb{N}$ and $B = [b_{jk}] \in E$.

Define

$$E^{\alpha} = \left\{ A = [a_{jk}] : \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}b_{jk}\| < \infty \quad \text{for all } B = [b_{jk}] \in E \right\},$$

and

$$E^{\beta} = \left\{ A = [a_{jk}] : \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}b_{jk} \quad \text{converges for all } B = [b_{jk}] \in E \right\}.$$

Theorem 3.1. *Let E, E_1, E_2 be matrix spaces.*

1. $E^{\alpha} \subseteq E^{\beta}$.
2. *If $E_1 \subseteq E_2$, then $E_2^{\beta} \subseteq E_1^{\beta}$.*
3. *If $E = E_1 + E_2$, then $E^{\beta} = E_1^{\beta} \cap E_2^{\beta}$.*
4. *If E is normal, then $E^{\alpha} = E^{\beta}$.*

Proof. By using the properties of absolutely convergent double series, the proof is completed. □

Subsequently, we present characterizations of the β -dual of the paranormed space $S_{p,q}^{R}(\mathbf{B})$.

Theorem 3.2. *Let $R = [r_{jk}]$ be a bounded matrix of scalars with $r_{jk} > 1$ for all $j, k \in \mathbb{N}$. Then*

$$\left(S_{p,q}^{R}(\mathbf{B})\right)^{\beta} = S^{Q}(\mathbf{B})$$

where $S^Q(\mathbf{B}) = \{A = [a_{jk}] : \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}} < \infty$ *for some* $L \in \mathbb{N}\}$ *and* $Q = [q_{jk}]$ *is a bounded matrix of scalars such that* $\frac{1}{r_{jk}} + \frac{1}{q_{jk}} = 1$ *for all* $j, k \in \mathbb{N}$.

Proof. Suppose that $A \in S^Q(\mathbf{B})$. Then $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}} < \infty$ for some $L \in \mathbb{N}$. We will show that $A \in S^R_{p,q}(\mathbf{B})$. Let $B = [b_{jk}] \in S^R_{p,q}(\mathbf{B})$. Then $\sum_{j=1}^{\infty} |\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} x_k|^q < \infty$ for any $x = (x_k) \in l_p$. This implies that $\|B^{[R]}\|_{p,q} < \infty$. For any positive integer j, and any unit vector $x = \{x_k\} \in l_p$, by using Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{k=1}^{\infty} \|a_{jk}b_{jk}\| &\le \sum_{k=1}^{\infty} \|a_{jk}\| L^{-\frac{1}{r_{jk}}} L^{\frac{1}{r_{jk}}} \|b_{jk}\| |x_k|^{-\frac{1}{r_{jk}}} |x_k|^{\frac{1}{r_{jk}}} \\
&\le \left[\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} (L|x_k|)^{-\frac{q_{jk}}{r_{jk}}}\right]^{\frac{1}{q_{jk}}} \left[\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} L|x_k|\right]^{\frac{1}{r_{jk}}} \\
&= \left[\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} (L|x_k|)^{-(q_{jk}-1)}\right]^{\frac{1}{q_{jk}}} \left[\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} L|x_k|\right]^{\frac{1}{r_{jk}}} \\
&\le \left[L\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}}\right]^{\frac{1}{q_{jk}}} \left[L\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} |x_k|\right]^{\frac{1}{r_{jk}}} \\
&\le L\left[\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}}\right] \left[\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} |x_k|\right]. \qquad (3.1)
\end{aligned}$$

For each $j \in \mathbb{N}$, $\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}} \le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}} < \infty$ and $\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} |x_k| \le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} |x_k| \le \|B^{[R]}\|_{p,q} < \infty$. Thus $\sum_{k=1}^{\infty} a_{jk}b_{jk}$ converges for all $j \in \mathbb{N}$. From (3.1), we get

$$\begin{aligned}
\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}b_{jk}\| &\le \sum_{j=1}^{\infty} L\left[\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}}\right] \left[\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} |x_k|\right] \\
&\le L\left[\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}}\right] \left[\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|b_{jk}\|^{r_{jk}} |x_k|\right] \\
&< \infty.
\end{aligned}$$

Therefore $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}b_{jk}$ converges. Consequently, $A \in \left(S^R_{p,q}(\mathbf{B})\right)^\beta$.

Next, we assume that $A \in \left(S^R_{p,q}(\mathbf{B})\right)^\beta$. For each $B = [b_{jk}] \in S^R_{p,q}(\mathbf{B})$, we choose a scalar matrix $T = [t_{jk}]$ such that $\|t_{jk}\| = 1$ and $\|a_{jk}t_{jk}b_{jk}\| = a_{jk}t_{jk}b_{jk}$ for all $j, k \in \mathbb{N}$. Note that $\|t_{jk}b_{jk}\| = \|b_{jk}\|$ for all $j, k \in \mathbb{N}$. By normality of $S^R_{p,q}(\mathbf{B})$ and $B \in S^R_{p,q}(\mathbf{B})$, $[t_{jk}b_{jk}] \in S^R_{p,q}(\mathbf{B})$. Since $A \in \left(S^R_{p,q}(\mathbf{B})\right)^\beta$, $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}t_{jk}b_{jk}$

converges. Then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}t_{jk}b_{jk} < \infty.$$

Because $\|a_{jk}t_{jk}b_{jk}\| = a_{jk}t_{jk}b_{jk}$ for all $j, k \in \mathbb{N}$,

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}b_{jk}\| = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}t_{jk}b_{jk}\| = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk}t_{jk}b_{jk} < \infty. \tag{3.2}$$

To prove that $A \in S^Q(\mathbf{B})$. Suppose that $A \notin S^Q(\mathbf{B})$. Then

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}} = \infty \quad \text{for all } L \in \mathbb{N}.$$

It follows that for all $J, K \in \mathbb{N}$,

$$\sum_{j>J}\sum_{k>K} \|a_{jk}\|^{q_{jk}} L^{-q_{jk}} = \infty \quad \text{for all } L \in \mathbb{N}. \tag{3.3}$$

By (3.3), we let $L_1 = 1$. Then there are $j_1, k_1 \in \mathbb{N}$ such that

$$\sum_{j \le j_1}\sum_{k \le k_1} \|a_{jk}\|^{q_{jk}} L_1^{-q_{jk}} > 1.$$

By (3.3), we can choose $L_2 > L_1$, $j_2 > j_1$ and $k_2 > k_1$ with $L_2 > 2^2$ such that

$$\sum_{j_1 < j \le j_2}\sum_{k_1 < k \le k_2} \|a_{jk}\|^{q_{jk}} L_2^{-q_{jk}} > 1.$$

Continuing this process, we can choose sequences of positive integers $\{L_i\}$, $\{j_i\}$ and $\{k_i\}$ with $1 = k_0 < k_1 < k_2 < \ldots$, $1 = j_0 < j_1 < j_2 < \ldots$ and $L_1 < L_2 < \ldots$ such that $L_i > 2^i$ and

$$\sum_{j_{i-1} < j \le j_i}\sum_{k_{i-1} < k \le k_i} \|a_{jk}\|^{q_{jk}} L_i^{-q_{jk}} > 1 \quad \text{for all } i \in \mathbb{N}.$$

For each $i \in \mathbb{N}$, we let $a_i = \sum_{j_{i-1} < j \le j_i}\sum_{k_{i-1} < k \le k_i} \|a_{jk}\|^{q_{jk}} L_i^{-q_{jk}}$ where $k_0 = j_0 = 1$. Define $C = [c_{jk}]$ where $c_{jk} = a_i^{-1} L_i^{-q_{jk}} \|a_{jk}\|^{q_{jk}-1}$ if $j_{i-1} < j \le j_i$ and $k_{i-1} < k \le k_i$. For any fixed positive integers i and any unit vector $x = \{x_k\} \in l_p$, by the fact that $r_{jk}q_{jk} = r_{jk} + q_{jk}$ and $r_{jk}(q_{jk} - 1) = q_{jk}$ for all $j, k \in \mathbb{N}$,

$$
\begin{aligned}
&\sum_{j_{i-1}<j\le j_i} \left[\sum_{k_{i-1}<k\le k_i} \|c_{jk}\|^{r_{jk}} |x_k| \right]^q \\
&= \sum_{j_{i-1}<j\le j_i} \left[\sum_{k_{i-1}<k\le k_i} a_i^{-r_{jk}} L_i^{-(r_{jk}+q_{jk})} \|a_{jk}\|^{q_{jk}} |x_k| \right]^q \\
&\le \sum_{j_{i-1}<j\le j_i} \left\{ \left[\sum_{k_{i-1}<k\le k_i} \left(a_i^{-r_{jk}} L_i^{-(r_{jk}+q_{jk})} \|a_{jk}\|^{q_{jk}} \right)^q \right]^{\frac{1}{q}} \left[\sum_{k_{i-1}<k\le k_i} |x_k|^p \right]^{\frac{1}{p}} \right\}^q \\
&\le \sum_{j_{i-1}<j\le j_i} \left[\sum_{k_{i-1}<k\le k_i} a_i^{-1} L_i^{-1} L_i^{-q_{jk}} \|a_{jk}\|^{q_{jk}} \right]^q \\
&= a_i^{-q} L_i^{-q} \sum_{j_{i-1}<j\le j_i} \left[\sum_{k_{i-1}<k\le k_i} L_i^{-q_{jk}} \|a_{jk}\|^{q_{jk}} \right]^q \\
&\le a_i^{-1} L_i^{-1} a_i \\
&< \frac{1}{2^i}.
\end{aligned}
$$

By taking the limit as $i \to \infty$, we have

$$\sum_{j=1}^{\infty} \left[\sum_{k=1}^{\infty} \|c_{jk}\|^{r_{jk}} |x_k| \right]^q \le \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Thus $C = [c_{jk}] \in S_{p,q}^R(\mathbf{B})$. For each $i \in \mathbb{N}$,

$$
\begin{aligned}
\sum_{j_{i-1}<j\le j_i} \sum_{k_{i-1}<k\le k_i} \|a_{jk}c_{jk}\| &= \sum_{j_{i-1}<j\le j_i} \sum_{k_{i-1}<k\le k_i} a_i^{-1} L_i^{-q_{jk}} \|a_{jk}\|^{q_{jk}} \\
&= a_i^{-1} \sum_{j_{i-1}<j\le j_i} \sum_{k_{i-1}<k\le k_i} L_i^{-q_{jk}} \|a_{jk}\|^{q_{jk}} \\
&= a_i^{-1} a_i \\
&= 1.
\end{aligned}
$$

By taking the limit as $i \to \infty$, we get

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \|a_{jk}c_{jk}\| = \sum_{i=1}^{\infty} 1 = \infty$$

which contradicts (3.2). Hence $A \in S^Q(\mathbf{B})$ and the proof is finished. □

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References

[1] G. Bennett, *Schur multipliers*, Duke Math. J., **44(3)**(1977), 603–639.

[2] V. K. Bhardwaj and S. Gupta, *On the β -Dual of Banach-Space-Valued Difference Sequence Spaces*, Ukr. Math. J., **65(8)**(2013), 1145–1151.

[3] P. Chaisuriya and S. C. Ong, *Absolute Schur algebras and unbounded matrices*, SIAM J. Matrix Anal. Appl., **20(3)**(1999), 596–605.

[4] M. H. Faroughi, E. Osgooei and A. Rahimi, (X_d, X_d^*) *-Bessel multipliers in Banach spaces*, Banach J. Math. Anal., **7(2)**(2013), 146–161.

[5] M. H. Faroughi, E. Osgooei and A. Rahimi, *Some properties of* (X_d, X_d^*) *and* (l_∞, X_d, X_d^*) *-Bessel multipliers*, Azerbaijan J. Math., **3(2)**(2013), 70–78.

[6] K. G. Grosse-Erdmann, *The structure of the sequence spaces of Maddox*, Can. J. Math., **44(2)**(1992), 298–307.

[7] C. G. Lascarides, *A study of certain sequence spaces of Maddox and a generalization of a theorem of Iyer*, Pacific J. Math., **38(2)**(1971), 487–500.

[8] C. G. Lascarides and I. Maddox, *Matrix transformations between some classes of sequences*, Proc. Cambridge Philos. Soc., **68(1)**(1970), 99–104.

[9] L. Livshits, S. C. Ong and S. W. Wang, *Banach space duality of absolute Schur algebras*, Integral Equations Operator Theory, **41(3)**(2001), 343–359.

[10] I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser., **18(1)**(1967), 345–355.

[11] I. J. Maddox, *Paranormed sequence spaces generated by infinite matrices.* Proc. Cambridge Philos. Soc., **64(2)**(1968), 335–340.

[12] I. J. Maddox, *Elements of functional analysis*, Cambridge University Press(1970).

[13] I. J. Maddox, *Infinite matrices of operators*, Springer-Verlag, Berlin-Heidelberg, New York(1980).

[14] H. Nakano, *Modulared sequence spaces*, Proc. Japan Acad., **27(9)**(1951), 508–512.

[15] E. Osgooei, *G-vector-valued sequence space frames*, Kyungpook Math. J., **56(3)**(2016), 793–806.

[16] J. Rakbud and P. Chaisuriya, *Classes of infinite matrices over Banach algebras*, J. Anal. Appl., **3**(2005), 31–46.

[17] J. Rakbud and P. Chaisuriya, *Schattens theorem on absolute Schur algebras*, J. Korean Math. Soc., **45(2)**(2008), 313–329.

[18] J. Rakbud and S. Suantai, *Duality theorem for Banach-valued function spaces*, Int. J. Math. Anal., **6(24)**(2012), 1179–1192.

[19] J. Schur, *Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen*, J. für die Reine und Angew. Math., **140**(1911), 1–28.

[20] S. Simons, *The sequence spaces $l(p_v)$ and $m(p_v)$*, Proc. London Math. Soc., **3(1)**(1965), 422–436.

[21] S. Suantai and W. Sanhan, *On β -dual of vector-valued sequence spaces of Maddox*, Int. J. Math. Math. Sci., **30(7)**(2002), 383–392.