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On Some Skew Constants in Banach Spaces

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Abstract. We introduce the constants $E[t, X]$, $C_{\mathrm{NJ}}[X]$ and $J[t, X]$ to describe the asymmetry of the norm. They can be seen as the skew version of the Gao's parameter, von Neumann-Jordan constant and Milman's moduli, respectively. We establish basic properties of these constants, relating them other well known constants, and use these properties to calculate the constants for specific spaces. We then use these constants to study Hilbert spaces, uniformly non-square spaces and their normal structures. With the Banach-Mazur distance, we use them to study isomorphic Banach spaces.

1. Introduction

The geometric theory of Banach spaces is an important branch of functional analysis. It has important applications in many mathematical fields, such as approximation theory, fixed point theory. Since Clarkson [7] put forward the concept of the modulus of convexity in 1936, geometric constants have evolved as a useful way to compare the geometric properties of Banach spaces. Quantifying geometric properties with numbers is an easy intuitive way to understand these properties on a given Banach spaces. Some of the more prominant geometric constants to date are the modulus of smoothness $\rho_X(t)$ proposed by Day [15], the James constant $J(X)$ proposed by Gao and Lau [12], and the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$

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proposed by Jordan and von Neumann [22]. There are, however, many other geometric constants worth noting. For readers interested in this field, we recommend the references mentioned in this paper.

Fitzpatrick and Reznick [26] introduced the skewness $s(X)$ of a Banach space X, which describes the asymmetry of the norm:

$$s(X) = \sup\left\{\lim_{t\to 0^+} \frac{\|x+ty\| - \|y+tx\|}{t} \;:\; x, y \in S_X\right\}.$$

They showed that $s(X) = 0$ if and only if X is a Hilbert space, and calculated the values of $s(X)$ for L_p spaces where $1 \le p \le \infty$. Moreover, they showed that the uniform non-squareness of X can be described in terms of $s(X)$. Further, based on the work in [26], Baronti and Papini [19] established the relationship between $s(X)$ and $\rho_X(1)$, and Mitani et al. [13] established the relationship between $s(X)$ and $J(X)$. All the results mentioned above illustrate that a geometric constant which describes the asymmetry of norm is worth studying. In this paper, we introduce three constants, $E[t, X]$, $C_{\mathrm{NJ}}[X]$ and $J[t, X]$, all of which describe the asymmetry of the norm.

In Section 2, we recall some necessary concepts.

In Section 3, we introduce the constant $E[t, X]$. Some basic properties of $E[t, X]$ are given and used to calculate the values of $E[t, X]$ for some specific spaces. The relation between $E[t, X]$ and $C'_{\mathrm{NJ}}(X)$ is established. Also, $E[t, X]$ is used to study some geometric properties of Banach spaces. Moreover, we discuss the relation of the values of $E[t, X]$ for two isomorphic Banach spaces in terms of Banach-Mazur distance.

In Section 4, we introduce the constant $C_{\mathrm{NJ}}[X]$ and use it to study Hilbert spaces and uniformly non-square spaces. We apply results from Section 3 to establish properties of $C_{\mathrm{NJ}}[X]$.

In Section 5, we consider the constant $J[t, X]$, relate it to $J[t, X^*]$, and use it to study the uniformly non-square spaces.

2. Preliminaries

Throughout the paper, let X be a real Banach space with $\dim X \ge 2$. The unit ball and the unit sphere of X are denoted by B_X and S_X, respectively. We now recall some concepts that we need in this paper.

Definition 2.1. ([23]) A Banach space X is said to be uniformly non-square, if there exists $\delta \in (0,1)$ such that if $x,\, y \in S_X$ then

$$\left\|\frac{x+y}{2}\right\| \leq 1-\delta \quad \text{or} \quad \left\|\frac{x-y}{2}\right\| \leq 1-\delta.$$

Definition 2.2. ([17]) A Banach space X is said to have (weak) normal structure, if for every (weakly compact) closed bounded convex subset K in X that contains more than one point, there exists a point $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| \ : \ y \in K\} < d(K) = \sup\{\|x-y\| \ : \ x, y \in K\}.$$

Remark 2.3. For the reflexive Banach spaces X, the normal structure and weak normal structure coincide.

The above concepts are closely related to the fixed point property, since Kirk [27] proved that every reflexive Banach space with normal structure has the fixed point property in 1965.

Definition 2.4. ([14]) For isomorphic Banach spaces X and Y, the Banach-Mazur distance between X and Y, denoted by $d(X,Y)$, is defined to be the infimum of $\|T\|\|T^{-1}\|$ taken over all isomorphisms T from X onto Y, that is,

$$d(X,Y) = \inf\{\|T\|\|T^{-1}\| \ : \ T : X \to Y \text{ being an isomorphism}\}.$$

3. The Constant $E[t,X]$

Based on the Pythagorean theorem, Gao introduced the following two quadratic parameters called Gao's parameters in [9] and [10], respectively,

$$E(X) = \sup\{\|x+y\|^2 + \|x-y\|^2 \ : \ x, y \in S_X\}.$$

$$E_\epsilon(X) = \sup\{\|x+\epsilon y\|^2 + \|x-\epsilon y\|^2 \ : \ x, y \in S_X\}, \quad 0 \leq \epsilon \leq 1.$$

He showed that both uniformly non-square spaces and normal structures are closely related to them. More results on $E(X)$ and $E_\epsilon(X)$ can be found in [2].

As mentioned in the Introduction, in this section we consider the following constant,

$$E[t,X] = \sup\{\|x+ty\|^2 + \|tx-y\|^2 \ : \ x, y \in S_X\}, \quad t \in \mathbb{R},$$

which can be regard as the skew version of $E_\epsilon(X)$.

3.1. Some basic conclusions about $E[t,X]$

In this section, we will give some basic conclusions about $E[t,X]$, and calculate the values of $E[t,X]$ for some specific spaces. First, we give the bounds of $E[t,X]$.

Proposition 3.1.1. *Let X be a Banach space. Then*

$$2(1+t^2) \le E[t,X] \le 2(1+t)^2, \quad t \in \mathbb{R}.$$

Proof. By taking $x, y \in S_X$ such that $x = y$, we can easily get $E[t,X] \ge 2(1+t^2)$. In addition, by the triangle inequality, it is obvious that $E[t,X] \le 2(1+t)^2$. □

Next, we give the some equivalent forms of $E[t,X]$, which will be used in our subsequent discussion.

Proposition 3.1.2. *Let X be a Banach space. Then*

(1) $E[t,X] = \sup\{\|x+ty\|^2 + \|tx-y\|^2 \ : \ x, y \in B_X\}, \quad t \in \mathbb{R}.$

(2) If X is a reflexive Banach space, then

$$E[t,X] = \sup\{\|x+ty\|^2 + \|tx-y\|^2 \ : \ x, y \in ext(B_X)\}, \quad t \in \mathbb{R}.$$

(3) $E[t,X] = \sup\{E[t,X_0] \ : \ X_0 \subset X, \ dim(X_0) = 2\}, \quad t \in \mathbb{R}.$

Proof. (1) First, for any $x, y \in B_X$, we have

$$x = \frac{1-\|x\|}{2}\left(-\frac{x}{\|x\|}\right) + \left(1 - \frac{1-\|x\|}{2}\right)\frac{x}{\|x\|},$$

$$y = \frac{1-\|y\|}{2}\left(-\frac{y}{\|y\|}\right) + \left(1 - \frac{1-\|y\|}{2}\right)\frac{y}{\|y\|}.$$

For convenience, we denote $\frac{1-\|x\|}{2}, 1-\frac{1-\|x\|}{2}, \frac{1-\|y\|}{2}, 1-\frac{1-\|y\|}{2}, -\frac{x}{\|x\|}, \frac{x}{\|x\|}, -\frac{y}{\|y\|}, \frac{y}{\|y\|}$ by $a_1, a_2, b_1, b_2, x_1, x_2, y_1, y_2$, respectively. Thus

$$x = a_1x_1 + a_2x_2, \quad y = b_1y_1 + b_2y_2.$$

It is obvious that $x_1, x_2, y_1, y_2 \in S_X$ and $a_1, a_2, b_1, b_2 \in [0,1]$ such that $a_1 + a_2 = 1, b_1 + b_2 = 1$.

Let $t \in \mathbb{R}$. Since $f(x) = x^2$ is a convex function on $[0, \infty)$, for any $x, y \in B_X$

we have

$$
\begin{aligned}
&\|x+ty\|^2+\|tx-y\|^2\\
=&\|\sum_{i=1}^{2}a_ix_i+t(\sum_{i=1}^{2}a_iy)\|^2+\|t(\sum_{i=1}^{2}a_ix_i)-(\sum_{i=1}^{2}a_iy)\|^2\\
=&\|\sum_{i=1}^{2}a_i(x_i+ty)\|^2+\|\sum_{i=1}^{2}a_i(tx_i-y)\|^2\\
\leq&(\sum_{i=1}^{2}a_i\|x_i+ty\|)^2+(\sum_{i=1}^{2}a_i\|tx_i-y\|)^2\\
\leq&\sum_{i=1}^{2}a_i\|x_i+ty\|^2+\sum_{i=1}^{2}a_i\|tx_i-y\|^2\\
=&\sum_{i=1}^{2}a_i\|\sum_{j=1}^{2}b_jx_i+t(\sum_{j=1}^{2}b_jy_j)\|^2+\sum_{i=1}^{2}a_i\|t(\sum_{j=1}^{2}b_jx_i)-(\sum_{j=1}^{2}b_jy_j)\|^2\\
=&\sum_{i=1}^{2}a_i\|\sum_{j=1}^{2}b_j(x_i+ty_j)\|^2+\sum_{i=1}^{2}a_i\|\sum_{j=1}^{2}b_j(tx_i-y_j)\|^2\\
\leq&\sum_{i=1}^{2}a_i(\sum_{j=1}^{2}b_j\|x_i+ty_j\|)^2+\sum_{i=1}^{2}a_i(\sum_{j=1}^{2}b_j\|tx_i-y_j\|)^2\\
\leq&\sum_{i=1}^{2}a_i\sum_{j=1}^{2}b_j\|x_i+ty_j\|^2+\sum_{i=1}^{2}a_i\sum_{j=1}^{2}b_j\|tx_i-y_j\|^2\\
=&\sum_{i=1}^{2}a_i\sum_{j=1}^{2}b_j(\|x_i+ty_j\|^2+\|tx_i-y_j\|^2)\\
\leq&\max\{\|x_i+ty_j\|^2+\|tx_i-y_j\|^2\ :\ i=1,2,\quad j=1,2\}\\
\leq&\sup\{\|x+ty\|^2+\|tx-y\|^2\ :\ x,y\in S_X\}.
\end{aligned}
$$

This shows that

$$E[t,X]\geq\sup\left\{\|x+ty\|^2+\|tx-y\|^2:x,y\in B_X\right\},\quad t\in\mathbb{R}.$$

In addition, it is obvious that

$$E[t,X]\leq\sup\left\{\|x+ty\|^2+\|tx-y\|^2:x,y\in B_X\right\},\quad t\in\mathbb{R}.$$

(2) Since X is a reflexive Banach space, then, according to Krein-Milman theorem, we know that $B_X=\overline{\text{co}}(ext(B_X))$. Then, by (1), we obtain

$$E[t,X]=\sup\{\|x+ty\|^2+\|tx-y\|^2\ :\ x,y\in\overline{\text{co}}(ext(B_X))\},\quad t\in\mathbb{R}.$$

Further, according to the continuity of norm, we can easily know

$$E[t,X]=\sup\{\|x+ty\|^2+\|tx-y\|^2 \ : \ x,y\in \mathrm{co}(ext(B_X))\},\quad t\in\mathbb{R}. \tag{3.1.1}$$

Now, for any $x,y\in \mathrm{co}(ext(B_X))$, there exist $\{x_i\}_{i=1}^{n_x},\{y_j\}_{j=1}^{n_y}\in ext(B_X)$ and $\{a_i\}_{i=1}^{n_x},\{b_j\}_{j=1}^{n_y}\in[0,1]$ such that

$$x=\sum_{i=1}^{n_x}a_ix_i,\quad y=\sum_{j=1}^{n_y}b_jy_j,\quad \sum_{i=1}^{n_x}a_i=1,\quad \sum_{j=1}^{n_y}b_j=1.$$

Moreover, since $f(x)=x^2$ is a convex function on $[0,\infty)$, then, for any $t\in\mathbb{R}$ and any $x,y\in \mathrm{co}(ext(B_X))$, we obtain

$$\begin{aligned}
&\|x+ty\|^2+\|tx-y\|^2\\
=&\|\sum_{i=1}^{n_x}a_ix_i+t(\sum_{i=1}^{n_x}a_iy)\|^2+\|t(\sum_{i=1}^{n_x}a_ix_i)-(\sum_{i=1}^{n_x}a_iy)\|^2\\
=&\|\sum_{i=1}^{n_x}a_i(x_i+ty)\|^2+\|\sum_{i=1}^{n_x}a_i(tx_i-y)\|^2\\
\le&(\sum_{i=1}^{n_x}a_i\|x_i+ty\|)^2+(\sum_{i=1}^{n_x}a_i\|tx_i-y\|)^2\\
\le&\sum_{i=1}^{n_x}a_i\|x_i+ty\|^2+\sum_{i=1}^{n_x}a_i\|tx_i-y\|^2\\
=&\sum_{i=1}^{n_x}a_i\|\sum_{j=1}^{n_y}b_jx_i+t(\sum_{j=1}^{n_y}b_jy_j)\|^2+\sum_{i=1}^{n_x}a_i\|t(\sum_{j=1}^{n_y}b_jx_i)-(\sum_{j=1}^{n_y}b_jy_j)\|^2\\
=&\sum_{i=1}^{n_x}a_i\|\sum_{j=1}^{n_y}b_j(x_i+ty_j)\|^2+\sum_{i=1}^{n_x}a_i\|\sum_{j=1}^{n_y}b_j(tx_i-y_j)\|^2\\
\le&\sum_{i=1}^{n_x}a_i(\sum_{j=1}^{n_y}b_j\|x_i+ty_j\|)^2+\sum_{i=1}^{n_x}a_i(\sum_{j=1}^{n_y}b_j\|tx_i-y_j\|)^2\\
\le&\sum_{i=1}^{n_x}a_i\sum_{j=1}^{n_y}b_j\|x_i+ty_j\|^2+\sum_{i=1}^{n_x}a_i\sum_{j=1}^{n_y}b_j\|tx_i-y_j\|^2\\
=&\sum_{i=1}^{n_x}a_i\sum_{j=1}^{n_y}b_j(\|x_i+ty_j\|^2+\|tx_i-y_j\|^2)\\
\le&\max\{\|x_i+ty_j\|^2+\|tx_i-y_j\|^2 \ : \ i=1,\cdots,n_x,\quad j=1,\cdots,n_y\}\\
\le&\sup\{\|x+ty\|^2+\|tx-y\|^2 \ : \ x,y\in ext(B_X)\}.
\end{aligned}$$

Thus, by (3.1.1), we have

$$E[t,X]\le\sup\{\|x+ty\|^2+\|tx-y\|^2 \ : \ x,y\in ext(B_X)\},\quad t\in\mathbb{R}.$$

Further, it is clear that

$$E[t,X] \geq \sup\{\|x+ty\|^2 + \|tx-y\|^2 \ : \ x,y \in ext(B_X)\}, \quad t \in \mathbb{R}.$$

(3) Let $t \in \mathbb{R}$. First, it is clear that

$$E[t,X] \geq \sup\{E[t,X_0] \ : \ X_0 \subset X, dim(X_0) = 2\}.$$

Second, for any $\varepsilon > 0$, there exist $x, y \in S_X$ such that

$$E[t,X] - \varepsilon \leq \|x+ty\|^2 + \|tx-y\|^2.$$

Let X_0 be a two-dimensional subspace of X that contains x and y, we have

$$E[t,X]-\varepsilon \leq \|x+ty\|^2+\|tx-y\|^2 \leq E[t,X_0] \leq \sup\{E[t,X_0] \ : \ X_0 \subset X, \ dim(X_0) = 2\}.$$

Let $\varepsilon \to 0$, we can obtain

$$E[t,X] \leq \sup\{E[t,X_0] \ : \ X_0 \subset X, \ dim(X_0) = 2\}.$$

This completes the proof. □

Now, we will use the above conclusion to calculate the values of $E[t,X]$ for some spaces.

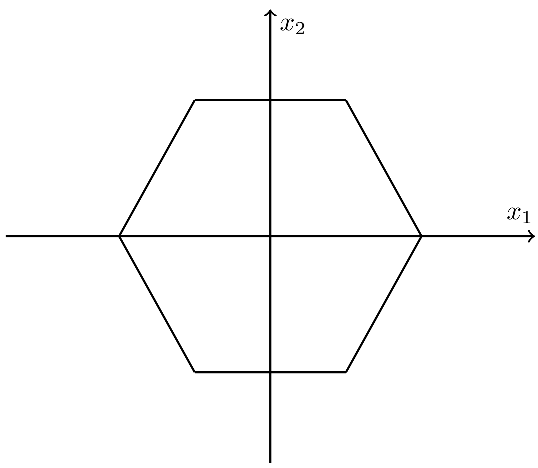
Example 3.1.3. Let X be the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1,x_2)\| = \max\left\{\left|x_1 + \frac{1}{\sqrt{3}}x_2\right|, \left|x_1 - \frac{1}{\sqrt{3}}x_2\right|, \frac{2}{\sqrt{3}}|x_2|\right\}.$$

Then

$$E[t,X] = \begin{cases} (1+|t|)^2 + 1 & |t| \leq 1, \\ (1+|t|)^2 + t^2 & |t| \geq 1. \end{cases}$$

Proof. First, notice that the unit ball of the this norm is a regular hexagon (see [14]).



By the above figure and some simple calculations, we can get that

$$ext(B_X) = \left\{(1,0), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), (-1,0), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\right\}.$$

Now, because finite-dimensional Banach spaces must be reflexive spaces, according to Proposition 3.1.2 (2), we can get

$$E[t, X] = \sup\{\|x + ty\|^2 + \|tx - y\|^2 \ : \ x, y \in ext(B_X)\}, \quad t \in \mathbb{R}.$$

Thus, by some simple calculations, it is not difficult for us to get

$$E[t, X] = \begin{cases} (1 + |t|)^2 + 1 & |t| \leq 1, \\ (1 + |t|)^2 + t^2 & |t| \geq 1. \end{cases}$$

□

EXample 3.1.4. Let X be the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_1 & (x_1x_2 \leq 0)\,, \\ \|(x_1, x_2)\|_\infty & (x_1x_2 \geq 0)\,. \end{cases}$$

Then

$$E[t, X] = \begin{cases} (1 + |t|)^2 + 1 & |t| \leq 1, \\ (1 + |t|)^2 + t^2 & |t| \geq 1. \end{cases}$$

Proof. First, because finite-dimensional Banach spaces must be reflexive spaces, by Proposition 3.1.2 (2) we get

$$E[t, X] = \sup\{\|x + ty\|^2 + \|tx - y\|^2 \ : \ x, y \in ext(B_X)\}, \quad t \in \mathbb{R}.$$

Now, since

$$ext(B_X) = \{(1,0), (1,1), (0,1), (-1,0), (-1,-1), (0,-1)\},$$

by some simple calculations, it is not difficult for us to get

$$E[t, X] = \begin{cases} (1 + |t|)^2 + 1 & |t| \leq 1, \\ (1 + |t|)^2 + t^2 & |t| \geq 1. \end{cases}$$

□

Example 3.1.5. Let X be the space $\mathbb{R}^2$ with the norm defined by

$$\|(x_1, x_2)\| = \begin{cases} \|(x_1, x_2)\|_1 & (x_1x_2 \leq 0)\,, \\ \|(x_1, x_2)\|_2 & (x_1x_2 \geq 0)\,. \end{cases}$$

Then

$$E[t,X]=\begin{cases} 2(1+t^2)+2t & t\geq 0,\\ 2(1+t^2)-2t & t\leq 0.\end{cases}$$

Proof. First, we will prove that

$$\|x+ty\|^2+\|tx-y\|^2\leq 2(1+t^2)+2t,\quad t\geq 0, \tag{3.1.2}$$

holds for any $x=(x_1,x_2), y=(y_1,y_2)\in ext(B_X)$.

Let $t\geq 0$. Then, for any $x=(x_1,x_2), y=(y_1,y_2)\in ext(B_X)$, we consider the following four cases:

Case 1: $x_1,x_2,y_1,y_2\geq 0$.

Case 1a: $(tx_1-y_1)(tx_2-y_2)\geq 0$. Then

$$\begin{aligned}\|x+ty\|^2+\|tx-y\|^2 &= \|(x_1+ty_1,x_2+ty_2)\|_2^2+\|(tx_1-y_1,tx_2-y_2)\|_2^2\\ &= 2(1+t^2).\end{aligned}$$

Case 1b: $tx_1-y_1\geq 0,\ tx_2-y_2\leq 0$. Then

$$\begin{aligned}\|x+ty\|^2+\|tx-y\|^2 &= \|(x_1+ty_1,x_2+ty_2)\|_2^2+\|(tx_1-y_1,tx_2-y_2)\|_1^2\\ &= 2(1+t^2)+2tx_1y_2+2tx_2y_1-2y_1y_2-2t^2x_1x_2\\ &\leq 2(1+t^2)+2tx_1y_2+2tx_2y_1\\ &\leq 2(1+t^2)+2t((x_1^2+x_2^2)^{\frac{1}{2}}(y_1^2+y_2^2)^{\frac{1}{2}})\\ &= 2(1+t^2)+2t.\end{aligned}$$

Case 1c: $tx_1-y_1\leq 0,\ tx_2-y_2\geq 0$. Similar to the proof of Case 1b, we can get

$$\|x+ty\|^2+\|tx-y\|^2\leq 2(1+t^2)+2t.$$

Thus, we obtain (3.1.2) holds for any $x=(x_1,x_2), y=(y_1,y_2)\in ext(B_X)$ with $x_1,x_2,y_1,y_2\geq 0$.

Case 2: $x_1,x_2\geq 0, y_1,y_2\leq 0$. Let $z=-y$. Then, by Case 1, we obtain

$$\|x+ty\|^2+\|tx-y\|^2=\|x-tz\|^2+\|tx+z\|^2=\|z+tx\|^2+\|tz-x\|^2\leq 2(1+t^2)+2t.$$

Case 3: $x_1,x_2\leq 0, y_1,y_2\geq 0$. Similar to the proof of Case 2, we omit it.

Case 4: $x_1,x_2\leq 0, y_1,y_2\leq 0$. Let $z=-y$ and $w=-x$. Then, by Case 1, we obtain

$$\|x+ty\|^2+\|tx-y\|^2=\|-w-tz\|^2+\|-tw+z\|^2=\|w+tz\|^2+\|tw-z\|^2\leq 2(1+t^2)+2t.$$

Consequently, we prove (3.1.2) holds.

Now, according to Proposition 3.1.2 (2) and the fact that finite-dimensional Banach spaces must be reflexive spaces, we can get

$$E[t,X] \leq 2(1+t^2)+2t, \quad t \geq 0.$$

Further, put $x=(1,0)$ and $y=(0,1)$, we can get

$$E[t,X] \geq \|x+ty\|^2 + \|tx-y\|^2 = \|(1,t)\|_2^2 + \|(t,-1)\|_1^2 = 2(1+t^2)+2t, \quad t \geq 0,$$

which shows that $E[t,X] = 2(1+t^2)+2t, \quad t \geq 0$. Since, it is obvious that $E[t,X]=E[-t,X], \quad t \in \mathbb{R}$, we can also get $E[t,X] = 2(1+t^2)-2t, \quad t \leq 0$. □

3.2. The relation between $C'_{NJ}(X)$ and $E[t,X]$

Alonso et al. [8] introduced the following constants $C'_{\mathrm{NJ}}(X)$ in 2008, according to the characterization of Hilbert spaces called the rhombus law given by Day [16],

$$C'_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{4} \ : \ x,y \in S_X \right\}.$$

This constant is closely related to some geometric properties of Banach spaces and some other constants (see [8]), and also plays an important role in the study of Tingley's problem (see [24]).

It is clear that $4C'_{\mathrm{NJ}}(X) = E[1,X]$. Thus, it is natural for us to ask whether there is a relationship between $C'_{\mathrm{NJ}}(X)$ and $E[t,X], \ t \neq 1$. In order to answer this question, we need to give the following result first.

Proposition 3.2.1. *Let X be a Banach space. The following statements hold.*
(1) $E[t,X]$ is a convex function of t on $\mathbb{R}$.
(2) $E[t,X]$ is a continuous function of t on $\mathbb{R}$.
(3) $E[t,X]$ is an even function of t on $\mathbb{R}$.
(4) $E[t,X]$ is non-decreasing on $[0,+\infty)$ and non-increasing on $(-\infty,0]$.

Proof. (1) Let $t_1,t_2 \in \mathbb{R}, \lambda \in (0,1)$. Then, for any $x,y \in S_X$, we can deduce that

$$\begin{aligned}
&\|x+(\lambda t_1+(1-\lambda)t_2)\,y\|^2 + \|(\lambda t_1+(1-\lambda)t_2)\,x-y\|^2 \\
\leqslant &(\lambda\|x+t_1y\|+(1-\lambda)\|x+t_2y\|)^2 + (\lambda\|t_1x-y\|+(1-\lambda)\|t_2x-y\|)^2 \\
\leqslant &\lambda\left(\|x+t_1y\|^2+\|t_1x-y\|^2\right) + (1-\lambda)\left(\|x+t_2y\|^2+\|t_2x-y\|^2\right) \\
\leqslant &\lambda E[t_1,X]+(1-\lambda)E[t_2,X],
\end{aligned}$$

which implies that

$$E[\lambda t_1+(1-\lambda)t_2,X] \leqslant \lambda E[t_1,X]+(1-\lambda)E[t_2,X].$$

(2) By (1), we can obtain $E[t,X]$ is continuous on $\mathbb{R}$.

(3) By the definition of $E[t,X]$, it is obvious that $E[t,X] = E[-t,X]$ holds for any $t \in \mathbb{R}$.

(4) Let $t_2 > t_1 \geq 0$. Then, from (1) and (3), we have

$$E[t_1,X] = E\left[\frac{t_2+t_1}{2t_2}t_2 + \frac{t_2-t_1}{2t_2}(-t_2), X\right] \leq E[t_2,X]. \tag{3.2.1}$$

which deduces that $E[t,X]$ is non-decreasing on $[0,+\infty)$. Then, from (3), we can obtain $E[t,X]$ non-increasing on $(-\infty,0]$. This completes the proof. $\square$

From the above proposition, we know that we only need to consider $E[t,X]$ on $t \in [0,\infty]$, since $E[t,X]$ is an even function. Also, it is obvious that $E[0,X] = 2$ for any Banach spaces. Thus, in the following discussion, we only consider $E[t,X]$ with $t \in (0,\infty)$. Now, we use the above proposition to give the relation between $C'_{NJ}(X)$ and $E[t,X]$.

Proposition 3.2.2. *Let X be a Banach space. Then*

$$4tC'_{NJ}(X) \leq E[t,X] \leq 4tC'_{NJ}(X) + 2\max\{t,1\}|t-1|, \quad t > 0.$$

Proof. Let $t > 0$. First, notice that, for any $x, y \in S_X$, we have

$$\frac{\left\|x + \frac{1}{t}y\right\|^2 + \left\|\frac{1}{t}x - y\right\|^2}{1 + \frac{1}{t^2}} = \frac{\|tx+y\|^2 + \|x-ty\|^2}{t^2+1}.$$

From the above equality, we can imply that

$$\frac{E[t,X]}{1+t^2} = \frac{E\left[\left(\frac{1}{t}\right), X\right]}{1+\frac{1}{t^2}}. \tag{3.2.2}$$

Now, since $E[t,X]$ is convex function and (3.2.2), we obtain

$$\begin{aligned}
E[1,X] &= E\left[\left(\frac{1}{1+t}\cdot t + \left(1 - \frac{1}{1+t}\right)\cdot\frac{1}{t}\right), X\right] \\
&\leq \frac{1}{1+t}E[t,X] + \left(1 - \frac{1}{1+t}\right)E\left[\frac{1}{t}, X\right] \\
&= \frac{1+t^2}{1+t}\frac{E[t,X]}{1+t^2} + \frac{1+t^2}{t(1+t)}\frac{E\left[\frac{1}{t},X\right]}{1+\frac{1}{t^2}} \\
&= \frac{1+t^2}{1+t}\frac{E[t,X]}{1+t^2} + \frac{1+t^2}{t(1+t)}\frac{E[t,X]}{1+t^2} \\
&= \frac{1}{t}E[t,X],
\end{aligned}$$

which implies that $4tC'_{NJ}(X) \leq E[t,X]$.

For the other inequality, we divide the proof into the following two cases.

Case 1 : $0 < t < 1$. Using $E[t, X]$ is a convex function again, we have

$$\begin{aligned} E[t, X] &= E[(t \cdot 1 + (1-t) \cdot 0), X] \\ &\leq tE[1, X] + (1-t)E[0, X] \\ &= 4tC'_{NJ}(X) + 2(1-t), \end{aligned}$$

and hence $E[t, X] \leq 4tC'_{NJ}(X) + 2\max\{t, 1\}|t-1|$.

Case 2. : $t \geq 1$. According to (3.2.2) and Case 1, we have

$$\frac{E[t, X]}{1+t^2} = \frac{E\left[\frac{1}{t}, X\right]}{1+\frac{1}{t^2}} \leq \frac{4t}{1+t^2}C'_{NJ}(X) + \frac{2t(t-1)}{1+t^2},$$

which means that

$$E[t, X] \leq 4tC'_{NJ}(X) + 2\max\{t, 1\}|t-1|.$$

This completes the proof. □

3.3. The relations between $E[t, X]$ and some geometric properties of Banach spaces

Next, we will discuss the relations between $E[t, X]$ and some geometric properties of Banach spaces. First, we will use the following lemma to characterize Hilbert spaces by $E[t, X]$.

Lemma 3.3.1. ([4]) *A normed space X is an inner product space if and only if for all $x, y \in S_X$ there exist $\lambda, \mu \neq 0$ such that*

$$\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2 \approx 2(\lambda^2 + \mu^2), \tag{3.3.1}$$

where $\approx$ means either $\leq$ or $\geq$.

Theorem 3.3.2. *Let X be a Banach space. Then the following statements are equivalent.*

(1) $E[t, X] = 2(1+t^2)$ for all $t \in (0, \infty)$.

(2) $E[t, X] = 2(1+t^2)$ for some $t \in (0, \infty)$.

(3) X is a Hilbert space.

Proof. (1)⇒(2). Obvious.

(2)⇒(3). Since $E[t, X] = 2(1+t^2)$ holds for some $t \in (0, \infty)$, we can deduce that there exists $t_0 \in (0, \infty)$ such that

$$\|x + t_0 y\|^2 + \|t_0 x - y\|^2 \leqslant 2\left(1 + t_0^2\right), \quad x, y \in S_X.$$

Then, from Lemma 3.3.1, we know that X is a Hilbert space.

(3)⇒(1). Since X is a Hilbert space, then, for any $t \in (0, \infty)$ and $x, y \in S_X$, we obtain

$$\|x+ty\|^2+\|tx-y\|^2 = \|x\|^2+2t\langle x,y\rangle+t^2\|y\|+t^2\|x\|^2-2t\langle x,y\rangle+\|y\|^2 = 2(1+t^2).$$

This shows that $E[t, X] = 2(1 + t^2)$ for all $t \in (0, \infty)$. □

Remark 3.3.3. The above conclusion also shows that the lower bound of $E[t, X]$ given in Proposition 3.1.1 is sharp.

Second, we will give the relation between $E[t, X]$ and uniformly non-square spaces.

Theorem 3.3.4. *Let X be a Banach space. Then the following statements are equivalent.*
(1) $E[t, X] < 2(1 + t)^2$ for all $t \in (0, \infty)$.
(2) $E[t, X] < 2(1 + t)^2$ for some $t \in (0, \infty)$.
(3) X is uniformly non-square.

Proof. It is clear that we only need to show that the following statements are equivalent
(i) $E[t, X] = 2(1 + t)^2$ for all $t \in (0, \infty)$.
(ii) $E[t, X] = 2(1 + t)^2$ for some $t \in (0, \infty)$.
(iii) X is not uniformly non-square.
(i) ⇒ (ii). Obvious.
(ii) ⇒ (iii). Since $E[t, X] = 2(1 + t)^2$ for some $t \in (0, \infty)$, we can deduce that there exists $t_0 \in (0, \infty)$ and $x_n, y_n \in S_X$ such that,

$$\|x_n + t_0 y_n\|^2 + \|t_0 x_n - y_n\|^2 \to 2(1 + t_0)^2.$$

Notice that $\|x_n + t_0 y_n\|^2 \le (1 + t_0)^2$ and $\|t_0 x_n - y_n\|^2 \le (1 + t_0)^2$, thus we obtain

$$\|x_n + t_0 y_n\|^2 \to (1 + t_0)^2, \quad \|t_0 x_n - y_n\|^2 \to (1 + t_0)^2.$$

Further, we obtain

$$\|x_n + t_0 y_n\| \to 1 + t_0, \quad \|t_0 x_n - y_n\| \to 1 + t_0,$$

which can deduces that

$$\|x_n + y_n\| \to 2, \quad \|x_n - y_n\| \to 2.$$

So, X is not uniformly non-square.
(iii) ⇒ (i). Since X is not uniformly non-square , there exist $x_n, y_n \in S_X$ such that

$$\|x_n + y_n\| \to 2, \quad \|x_n - y_n\| \to 2.$$

Thus, for all $t \in (0,\infty)$, we obtain

$$\|x_n + ty_n\| \to 1+t, \quad \|tx_n - y_n\| \to 1+t.$$

Furthermore, from Proposition 3.1.1, then we obtain

$$2(1+t)^2 \geq E[t,X] \geq \|x_n + ty_n\|^2 + \|tx_n - y_n\|^2, \quad t \in (0,\infty)$$

Let $n \to \infty$, we obtain $E[t,X] = 2(1+t)^2$ for all $t \in (0,\infty)$. □

Remark 3.3.5. The above conclusion also shows that the upper bound of $E[t,X]$ given in Proposition 3.1.1 is sharp.

Finally, we use $E[t,X]$ to give a sufficient condition for normal structure by the following lemma.

Definition 3.3.6. ([11]) Let X be a Banach space, a hexagon H in X is called a normal hexagon if the length of each side of H is 1 and each pair of opposite sides are parallel.

Lemma 3.3.7. ([11]) *Let X be a Banach space without weak normal structure, then for any ϵ, $0 < \epsilon < 1$, and $x_1 \in S_X$, there exists an inscribed normal hexagon with vertices $x_1, x_2, x_3, -x_1, -x_2$ and $-x_3 \in S_X$ satisfying*
(1) $x_1 = x_2 - x_3$.
(2) $\|(x_1+x_2)/2\|, \|(x_3+(-x_1))/2\| > 1-\epsilon$.

Theorem 3.3.8. *Let X be a Banach space. Then*
(1) Let $t \in [\frac{2}{3}, 1]$. If $E[t,X] < 5t^2 - 2t + 2$, then X has normal structure.
(2) Let $t \in [1, \frac{3}{2}]$. If $E[t,X] < 2t^2 - 2t + 5$, then X has normal structure.

Proof. (1) Let $t \in [\frac{2}{3}, 1]$. According to Theorem 3.3.4, we know that X is uniformly non-square when $E[t,X] < 5t^2 - 2t + 2$. Hence X is reflexive (see [23]), and normal structure and weak normal structure coincide. So, we only need to show that X has weak normal structure.

Now, suppose conversely that X does not have weak normal structure. By Lemma 3.3.7, for any $\epsilon \in (0,1)$, there exist $x_1, x_2, x_3 \in S_X$ such that

$$x_1 = x_2 - x_3, \quad \|(x_1+x_2)/2\| > 1-\epsilon, \quad \|(x_3+(-x_1))/2\| > 1-\epsilon.$$

Now, we have

$$\begin{aligned} \|x_1 + tx_2\| &= \|(x_1+x_2) - (1-t)x_2\| \\ &\geq \|x_1+x_2\| - \|(1-t)x_2\| \\ &\geq 2 - 2\epsilon - (1-t) \\ &= 1 + t - 2\epsilon, \end{aligned}$$

and

$$\begin{aligned}\|tx_1 - x_2\| &= \|tx_1 + tx_2 - tx_2 - x_2\| \\ &= \|tx_1 + tx_2 - t(x_1 + x_3) - x_2\| \\ &= \|t(x_1 - x_3) + (t-1)x_2 - tx_1\| \\ &\geq \|t(x_1 - x_3)\| - \|(t-1)x_2 - tx_1\| \\ &\geq (2-2\epsilon)t - (1 - t + t) \\ &= (2-2\epsilon)t - 1.\end{aligned}$$

Thus, we obtain

$$E[t,X] \geq \|x_1 + tx_2\|^2 + \|tx_1 - x_2\|^2 \geq (1 + t - 2\epsilon)^2 + ((2-2\epsilon)t - 1)^2.$$

Let $\epsilon \to 0$, we have

$$E[t,X] \geq (1+t)^2 + (2t-1)^2 = 5t^2 - 2t + 2.$$

This contradicts $E[t,X] < 5t^2 - 2t + 2$.

(2) Let $t \in [1, \frac{3}{2}]$. By (3.2.2), we have $E\left[\frac{1}{t}, X\right] = \frac{1}{t^2}E[t,X]$. Further, since $E[t,X] < 2t^2 - 2t + 5$, we get

$$E\left[\frac{1}{t}, X\right] = \frac{1}{t^2}E[t,X] < \frac{1}{t^2}(2t^2 - 2t + 5) = 5\frac{1}{t^2} - 2\frac{1}{t} + 2,$$

which implies that X has normal structure by (1). □

Remark 3.3.9. Through simple calculations, we get that $2(1+t^2) \leq 5t^2 - 2t + 2$ if and only if $t \in [\frac{2}{3}, 1]$. The reason we take $t \in [\frac{2}{3}, 1]$ in (1) is to ensure that there exists space X satisfying $E[t,X] < 5t^2 - 2t + 2$ by Proposition 3.1.1. This is also why we take $t \in [1, \frac{3}{2}]$ in (2).

3.4. BanachMazur distance and stability

In this section, we will use the Banach-Mazur distance to discuss the relation of the values of $E[t,X]$ for two isomorphic Banach spaces. This relationship plays a decisive role in our subsequent discussion of the relationship between $E[t,X]$ and $E[t,X^{**}]$.

Theorem 3.4.1. *If X and Y are isomorphic Banach spaces, then,*

$$\frac{E[t,Y]}{d(X,Y)^2} \leq E[t,X] \leq d(X,Y)^2 E[t,Y], \quad t > 0.$$

Proof. Let $x, y \in S_X$ and $t > 0$. For each $\varepsilon > 0$, there exists an isomorphism T from X onto Y such that $\|T^{-1}\|\|T\| \leq (1+\varepsilon)d(X,Y)$. Set

$$\overline{x} = \frac{Tx}{\|T\|}, \quad \overline{y} = \frac{Ty}{\|T\|},$$

then $\overline{x}, \overline{y} \in B_Y$. Now, according to Proposition 3.1.2 (1), we obtian

$$\begin{aligned}\|x+ty\|^2+\|tx-y\|^2 &= \|T^{-1}(Tx+tTy)\|^2+\|T^{-1}(Ttx+Ty)\|^2\\ &= \|T^{-1}(\|T\|\overline{x}+t\|T\|\overline{y})\|^2+\|T^{-1}(\|T\|t\overline{x}+\|T\|\overline{y})\|^2\\ &= \|T^{-1}\|^2\|T\|^2(\|\overline{x}+t\overline{y}\|^2+\|t\overline{x}-\overline{y}\|^2)\\ &\le (1+\varepsilon)^2 d(X,Y)^2 E[t,Y].\end{aligned}$$

This means that $E[t,X] \le (1+\varepsilon)^2 d(X,Y)^2 E[t,Y]$. Let $\varepsilon \to 0$, we obtain

$$E[t,X] \le d(X,Y)^2 E[t,Y].$$

The other inequality follows by interchanging X and Y. □

Corollary 3.4.2. *Let X be a Banach space and let $X_1 = (X, \|\cdot\|_1)$, where $\|\cdot\|_1$ is an equivalent norm on X satisfying, for $a, b > 0$,*

$$a\|x\| \le \|x\|_1 \le b\|x\|, \quad x \in X$$

Then

$$\frac{a^2}{b^2}E[t,X] \le E[t,X_1] \le \frac{b^2}{a^2}E[t,X], \quad t > 0.$$

Proof. This follows from Theorem 3.4.1 and the fact that $d(X, X_1) \le \frac{b}{a}$. □

Finally, we will use Theorem 3.4.1 to show that $E[t,X] = E[t,X^{**}]$. To do this, we need to recall the definition of finite representability. A Banach space X is finitely representable in a Banach space Y if, for every $\varepsilon > 0$ and for every finite-dimensional subspace X_0 of X, there exists a finite-dimensional subspace Y_0 of Y with $dim(X_0) = dim(Y_0)$ such that $d(X_0, Y_0) \le 1 + \varepsilon$.

Theorem 3.4.3. *Let X and Y be Banach spaces. The following statements hold.*
(1) If X is finitely representable in Y, then $E[t,X] \le E[t,Y]$, $t > 0$.
(2) $E[t,X] = E[t,X^{**}]$, $t > 0$.

Proof. (1) Let $t > 0$ and X_0 be a two-dimensional subspace of X. For any $\varepsilon > 0$, since X is finitely representable in Y, there exists a two-dimensional subspace Y_0 of Y such that $d(X_0, Y_0) \le 1 + \varepsilon$. Applying Theorem 3.4.1 to the pair of X_0 and Y_0, we obtain

$$E[t,X_0] \le (1+\varepsilon)E[t,Y_0] \le (1+\varepsilon)E[t,Y].$$

Let $\varepsilon \to 0$, we obtain $E[t,X_0] \le E[t,Y]$. Further, by Proposition 3.1.2 (3), we obtain $E[t,X] \le E[t,Y]$.

(2) For any Banach spaces X, by the principle of local reflexivity, X^{**} is always finitely representable in X. Then, by (1),

$$E[t,X] \ge E[t,X^{**}], \quad t > 0.$$

On the other hand, X is isometric to a subspace of X^{**}, therefore,

$$E[t, X] \leq E[t, X^{**}], \quad t > 0.$$

This completes the proof. □

4. Skew Von Neumann-Jordan Constant $C_{\mathrm{NJ}}[X]$

In 1935, Jordan and von Neumann [22] pointed out that for any Banach spaces X there is an unique smallest positive constant C between one and two with the property that if x and y are elements of X, not both equal to the zero element, then the following relation holds:

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C.$$

In [22], Jordan and von Neumann also demonstrated that $C = 1$ is a necessary and sufficient condition for X to be a Hilbert space. This is actually a famous characterization of Hilbert spaces called the parallelogram law. And then in 1937, Clarkson [6] gave a precise evaluation of this constant C for the Lebesgue spaces L_p and l_p, for all $p \geq 1$. It was these two articles that attracted the attention of scholars to this constant C and began to study it. Later, scholars always define the constant C in the following equivalent way, and call this constant C the von Neumann-Jordan constant, denoted by $C_{\mathrm{NJ}}(X)$,

$$C_{\mathrm{NJ}}(X) = \sup \left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \ : \ x, y \in X, \ (x, y) \neq (0, 0) \right\}.$$

After a large number of studies by scholars, it is found that $C_{\mathrm{NJ}}(X)$ can not only be used to characterize Hilbert spaces, but also can be used to characterize uniformly non-square spaces (see [28]) and superreflexive spaces (see [20]), and even has the close relation with uniformly normal structure (see [25]). For more results on $C_{\mathrm{NJ}}(X)$, we recommend [1, 18] to interested readers.

As we mentioned above, the constant $C_{\mathrm{NJ}}(X)$ is closely related to the parallelogram law. Note that Lemma 3.3.1 that we used earlier in proving Theorem 3.3.2 is actually a generalization of the parallelogram law. Therefore, we naturally want to define a new constant based on it. To this end, we need to explain the following two facts first. The first thing is that, by Lemma 3.3.1 and let $t = \frac{\mu}{\lambda}$, X is a Hilbert space if and only if for all $x, y \in S_X$ there exists $t \neq 0$ such that

$$\|x + ty\|^2 + \|tx - y\|^2 \approx 2(1 + t^2), \tag{4.1}$$

where $\approx$ means either $\leq$ or $\geq$. The second thing is that, according to the definition of $C_{\mathrm{NJ}}(X)$, $C_{\mathrm{NJ}}(X)$ has the following equivalent form:

$$C_{\mathrm{NJ}}(X) = \sup \left\{ \frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} \ : \ x, y \in S_X, \ 0 \leq t \leq 1 \right\}. \tag{4.2}$$

Therefore, based on (4.1) and (4.2), we consider the following constant

$$C_{\mathrm{NJ}}[X] = \sup\left\{\frac{\|x+ty\|^2+\|tx-y\|^2}{2(1+t^2)}\ :\ x,y\in S_X,\ 0\le t\le 1\right\},$$

which can be regard as the skew version of von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$. Moreover, it is clearly that

$$C_{\mathrm{NJ}}[X] = \sup\left\{\frac{E[t,X]}{2\,(1+t^2)}\ :\ 0\le t\le 1\right\}.$$

First, the bounds of $C_{\mathrm{NJ}}[X]$ are shown as below.

Proposition 4.1. *Let X be a Banach space. Then $1 \le C_{\mathrm{NJ}}[X] \le 2$.*

Proof. First, Let $x, y \in S_X$ such that $x = y$, we can easily get $C_{\mathrm{NJ}}[X] \ge 1$. Second, for any $t \in [0,1]$ and $x, y \in S_X$, we have

$$\begin{aligned}&\frac{\|x+ty\|^2+\|tx-y\|^2}{2(1+t^2)}\\ \le&\frac{(1+t)^2+(1+t)^2}{2(1+t^2)} = 1+\frac{2t}{1+t^2}\\ \le&1+\sup\left\{\frac{2t}{1+t^2}\ :\ 0\le t\le 1\right\} = 2,\end{aligned}$$

which means $C_{\mathrm{NJ}}[X] \le 2$. □

From the definition of $C_{\mathrm{NJ}}[X]$ and the results shown in Example 3.1.3, Example 3.1.4 and Example 3.1.5, we can obtain the following results easily by some simple calculations.

Example 4.2.

(1) Let X be the space $\mathbb{R}^2$ with the norm shown in Example 3.1.3. Then $C_{\mathrm{NJ}}[X] = \frac{3+\sqrt{5}}{4}$.

(2) Let X be the space $\mathbb{R}^2$ with the norm shown in Example 3.1.4. Then $C_{\mathrm{NJ}}[X] = \frac{3+\sqrt{5}}{4}$.

(3) Let X be the space $\mathbb{R}^2$ with the norm shown in Example 3.1.5. Then $C_{\mathrm{NJ}}[X] = \frac{3}{2}$.

From the Theorem 3.3.2 and Theorem 3.3.4, we know that $E[t,X]$ can be used to characterize the Hilbert spaces and uniformly non-square spaces. Moreover, form the definition of $C_{\mathrm{NJ}}[X]$, one can know that $C_{\mathrm{NJ}}[X]$ is closely related to $E[t,X]$. Thus, it is natural to ask whether $C_{\mathrm{NJ}}[X]$ can also be used to characterize the Hilbert spaces and uniformly non-square spaces. The answer is yes.

Theorem 4.3. *Let X be a Banach space. Then X is a Hilbert space if and only if $C_{\mathrm{NJ}}[X] = 1$.*

Proof. If $C_{\mathrm{NJ}}[X]=1$, then, according to $C_{\mathrm{NJ}}[X]=\sup\left\{\frac{E[t,X]}{2(1+t^2)}\ :\ 0\le t\le 1\right\}$, we have

$$E[t,X]\le 2\left(1+t^2\right),\quad 0\le t\le 1.$$

Further, form Proposition 3.1.1, we can obtain

$$E[t,X]=2\left(1+t^2\right),\quad 0\le t\le 1,$$

which implies X is a Hilbert space by Theorem 3.3.2.

On the contrary, if X is a Hilbert space, then according to Theorem 3.3.2 and $C_{\mathrm{NJ}}[X]=\sup\left\{\frac{E[t,X]}{2(1+t^2)}\ :\ 0\le t\le 1\right\}$, we can easily get $C_{\mathrm{NJ}}[X]=1$. □

Remark 4.4. The above conclusion also shows that the lower bound of $C_{\mathrm{NJ}}[X]$ given in Proposition 4.1 is sharp.

Theorem 4.5. *Let X be a Banach space. Then the following statements are equivalent.*

(1) $C_{\mathrm{NJ}}[X]<2$.

(2) X is uniformly non-square.

Proof. (1)⇒ (2). If X is not uniformly non-square, then, by Theorem 3.3.4, we obtain

$$E[t,X]=2(1+t)^2,\quad t\in[0,1].$$

Then, we have

$$C_{\mathrm{NJ}}[X]=\sup\left\{\frac{E[t,X]}{2\left(1+t^2\right)}\ :\ 0\le t\le 1\right\}=\sup\left\{\frac{(1+t)^2}{1+t^2}\ :\ 0\le t\le 1\right\}=2.$$

This contradicts $C_{\mathrm{NJ}}[X]<2$.

(2) ⇒ (1). Since X is uniformly non-square, then there exists $\delta>0$ such that, for any $x,y\in S_X$,

$$\min\left\{\left\|\frac{x+y}{2}\right\|,\left\|\frac{x-y}{2}\right\|\right\}\le 1-\delta.$$

Without loss of generality, for any $x,y\in S_X$, we always assume that

$$\min\left\{\left\|\frac{x+y}{2}\right\|,\left\|\frac{x-y}{2}\right\|\right\}=\left\|\frac{x-y}{2}\right\|.$$

Now, Let $t\in[0,1]$. Then, for any $x,y\in S_X$, we have

$$\begin{aligned}\|tx-y\|&=\left\|2t\left(\frac{x-y}{2}\right)-(1-t)\,y\right\|\\&\le 2t\left\|\frac{x-y}{2}\right\|+(1-t)\,\|y\|\\&\le 2t(1-\delta)+1-t\\&=1+t-2t\delta.\end{aligned}$$

Further, for any $x, y \in S_X$, we have

$$\begin{aligned}\frac{\|x+ty\|^2+\|tx-y\|^2}{2(1+t^2)} &\leq \frac{(1+t)^2+\|tx-y\|^2}{2(1+t^2)}\\ &\leq 1+\frac{\|tx-y\|^2}{2(1+t^2)}\\ &\leq 1+\frac{(1+t-2t\delta)^2}{2(1+t^2)}\\ &\leq 1+\sup\left\{\frac{(1+t-2t\delta)^2}{2(1+t^2)} \ : \ 0\leq t\leq 1\right\}\\ &= 1+\frac{(2-2\delta)^2}{4},\end{aligned}$$

which implies that $C_{\mathrm{NJ}}[X] \leq 1 + \frac{(2-2\delta)^2}{4} < 2$. □

Remark 4.6. The above conclusion also shows that the upper bound of $C_{\mathrm{NJ}}[X]$ given in Proposition 4.1 is sharp.

Finally, we need to point out that since

$$C_{\mathrm{NJ}}[X] = \sup\left\{\frac{E[t,X]}{2\,(1+t^2)} \ : \ 0\leq t\leq 1\right\},$$

we can get the following conclusions easily through Theorem 3.4.1 and Theorem 3.4.3.

Proposition 4.7. *Let X and Y be Banach spaces. Then the following statements hold.*

(1) If X and Y are isomorphic Banach spaces, then

$$\frac{C_{\mathrm{NJ}}[Y]}{d(X,Y)^2} \leq C_{\mathrm{NJ}}[X] \leq d(X,Y)^2 C_{\mathrm{NJ}}[Y].$$

(2) $C_{\mathrm{NJ}}[X] = C_{\mathrm{NJ}}[X^{**}]$.

5. Skew James Constant $J[t,X]$

In 1990, in order to simplify the Schäffer's girth and perimeter of the unit spheres, Gao and Lau [12] introduced the following constant called James constant

$$J(X) = \sup\{\min\{\|x+y\|, \|x-y\|\} \ : \ x,y\in S_X\},$$

and they proved that $J(X) < 2$ if and only if X is uniformly non-square. After a lot of researchs, it is found that James constant has good properties. For example, it can be used to characterize Hilbert spaces (see [21]), and has a very beautiful relationship with $C_{\mathrm{NJ}}(X)$, that is, $C_{\mathrm{NJ}}(X) \le J(X)$ (see [29]). For more results on $J(X)$ can be found in [5, 18].

In 2007, He and Cui [3] discuss the following constant called Milman's moduli which can be regard as a generalization of James constant.

$$J(t,X) = \sup\{\min\{\|x+ty\|, \|x-ty\|\} \ : \ x,y \in S_X\}, \quad t > 0.$$

Based on the motivation mentioned in the Introduction, we consider the following constant

$$J[t,X] = \sup\{\min\{\|x+ty\|, \|tx-y\|\} \ : \ x,y \in S_X\}, \quad t > 0,$$

which can be regard as the skew version of $J(t,X)$.

The value of $J[t,X]$ for Hilbert space is shown below.

Proposition 5.1. *If X is a Hilbert space, then $J[t,X] = \sqrt{t^2+1}$ holds for all $t > 0$.*

Proof. Since X be a Hilbert space, then for any $x,y \in S_X$ and any $t > 0$, we have

$$\begin{aligned} & \|x+ty\|^2 + \|tx-y\|^2 \\ =& \|x\|^2 + t^2\|y\|^2 + t^2\|x\|^2 + \|y\|^2 \\ =& 2t^2 + 2. \end{aligned}$$

Thus, for any $x,y \in S_X$ and any $t > 0$, we have

$$\min\{\|x+ty\|^2, \|tx-y\|^2\} \le \frac{\|x+ty\|^2 + \|tx-y\|^2}{2} = t^2 + 1,$$

which means

$$J[t,X] \le \sqrt{t^2+1}, \quad t > 0.$$

On the other hand, it is obvious that there exist $x_0, y_0 \in S_X$ such that $x_0 \perp y_0$. Thus, for any $t > 0$, we obtain

$$\begin{aligned} J[t,X] &\ge \min\{\|x_0+ty_0\|, \|tx_0-y_0\|\} \\ &= \min\{\sqrt{\|x_0\|^2 + 2t\langle x_0,y_0\rangle + t^2\|y_0\|^2}, \sqrt{t^2\|x_0\|^2 - 2t\langle x_0,y_0\rangle + \|y_0\|^2}\} \\ &= \sqrt{t^2+1}. \end{aligned}$$

This completes the proof. □

Next, we will use Hahn-Banach theorem to establish the relation between $J[t,X]$ and $J[t,X^*]$.

Proposition 5.2. *Let X be a Banach space. Then*

$$2J[t,X] - (1+t) \le J[t,X^*] \le \frac{1}{2}(J[t,X] + 1 + t), \quad t > 0.$$

Proof. Let $t > 0$. First, for any $\varepsilon > 0$, there exist $x, y \in S_X$ such that

$$\min\{\|x + ty\|, \|tx - y\|\} \geq J[t, X] - \varepsilon.$$

In addition, according to Hahn-Banach theorem, there exist $f, g \in S_{X^*}$ such that

$$f(x + ty) = \|x + ty\|, \quad g(tx - y) = \|tx - y\|.$$

Then, we have

$$\begin{aligned} J[t, X^*] &\geq \min\{\|f + tg\|, \|tf - g\|\} \\ &= \|f + tg\| + \|tf - g\| - \max\{\|f + tg\|, \|tf - g\|\} \\ &\geq \|f + tg\| + \|tf - g\| - (1 + t) \\ &\geq (f + tg)(x) + (tf - g)(y) - (1 + t) \\ &= f(x + ty) + g(tx - y) - (1 + t) \\ &= \|x + ty\| + \|tx - y\| - (1 + t) \\ &\geq 2\min\{\|x + ty\|, \|tx - y\|\} - (1 + t) \\ &\geq 2(J[t, X] - \varepsilon) - (1 + t). \end{aligned}$$

Let $\varepsilon \to 0$, we have

$$J[t, X^*] \geq 2J[t, X] - (1 + t).$$

Second, let $f, g \in S_{X^*}$, then for any $\varepsilon > 0$, there exist $x, y \in S_X$ such that

$$(f + tg)(x) > \|f + tg\| - \varepsilon, \quad (tf - g)(y) > \|tf - g\| - \varepsilon.$$

Thus, for any $f, g \in S_{X^*}$, we have

$$\begin{aligned} \min\{\|f + tg\|, \|tf - g\|\} &\leq \frac{1}{2}(\|f + tg\| + \|tf - g\|) \\ &< \frac{1}{2}((f + tg)(x) + (tf - g)(y) + 2\varepsilon) \\ &= \frac{1}{2}(f(x + ty) + g(tx - y) + 2\varepsilon) \\ &\leq \frac{1}{2}(\|x + ty\| + \|tx - y\| + 2\varepsilon) \\ &\leq \frac{1}{2}(\min\{\|x + ty\|, \|tx - y\|\} + 1 + t + 2\varepsilon) \\ &\leq \frac{1}{2}(J[t, X] + 1 + t + 2\varepsilon), \end{aligned}$$

which shows that

$$J[t, X^*] \leq \frac{1}{2}(J[t, X] + 1 + t + 2\varepsilon), \quad \varepsilon > 0.$$

Let $\varepsilon \to 0$, we have

$$J[t, X^*] \le \frac{1}{2}(J[t, X] + 1 + t).$$

□

Next, we will establish the relation between $J(X)$ and $J[t, X]$, which will help us to give the relation between $J[t, X]$ and uniformly non-square spaces.

Proposition 5.3. *Let X be a Banach space. Then*

$$\max\{1, t\}J(X) - |1 - t| \le J[t, X] \le \min\{1, t\}J(X) + |1 - t|.$$

Proof. For $x, y \in S_X$ and $t > 0$, we have

$$\begin{aligned}\min\{\|x + y\|, \|x - y\|\} &= \min\{\|x + ty + (1 - t)y\|, \|tx - y + (1 - t)x\|\} \\ &\le \min\{\|x + ty\| + |1 - t|, \|tx - y\| + |1 - t|\} \\ &= \min\{\|x + ty\|, \|tx - y\|\} + |1 - t| \\ &\le J[t, X] + |1 - t|,\end{aligned}$$

and

$$\begin{aligned}t \min\{\|x + y\|, \|x - y\|\} &= \min\{\|tx + ty\|, \|tx - ty\|\} \\ &\le \min\{\|x + ty\| + |1 - t|, \|tx - y\| + |1 - t|\} \\ &= \min\{\|x + ty\|, \|tx - y\|\} + |1 - t| \\ &\le J[t, X] + |1 - t|.\end{aligned}$$

This shows that $\max\{1, t\}J(X) - |1 - t| \le J[t, X]$.

In addition, for $x, y \in S_X$ and $t > 0$, we also have

$$\begin{aligned}\min\{\|x + ty\|, \|tx - y\|\} &= \min\{\|x + y + (t - 1)y\|, \|x - y + (t - 1)x\|\} \\ &\le \min\{\|x + y\| + |1 - t|, \|x - y\| + |1 - t|\} \\ &= \min\{\|x + y\|, \|x - y\|\} + |1 - t| \\ &\le J(X) + |1 - t|,\end{aligned}$$

and

$$\begin{aligned}\min\{\|x + ty\|, \|tx - y\||\} &= \min\{\|tx + ty + (1 - t)x\|, \|tx - ty + (t - 1)y\|\} \\ &\le \min\{\|tx + ty\| + |1 - t|, \|tx - ty\| + |1 - t|\} \\ &= t \min\{\|x + y\|, \|x - y\|\} + |1 - t| \\ &\le tJ(X) + |1 - t|.\end{aligned}$$

This shows that $J[t, X] \le \min\{1, t\}J(X) + |1 - t|$. □

Now, from Proposition 5.3 and the fact that X is uniformly non-square if and only if $J(X) < 2$, we can obtain following result easily.

Corollary 5.4. *Let X be a Banach space. Then the following statements are equivalent.*

(1) X is uniformly non-square.

(2) $J[t, X] < t + 1$ holds for all $t > 0$.

(3) $J[t, X] < t + 1$ holds for some $t > 0$.

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