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Weakly Right IQNN Rings

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Abstract. In this article we look at the property of a 2 by 2 full matrix ring over the ring of integers, of being *weakly right IQNN*. This generalisation of the property of being right IQNN arises from products of idempotents and nilpotents. We shown that it is, indeed, a proper generalization of right IQNN. We consider the property of beign weakly right IQNN in relation to several kinds of factorizations of a free algebra in two indeterminates over the ring of integers modulo 2.

1. Prerequisites

Throughout this article every ring is an associative ring with identity unless otherwise stated. Let R be a ring. $I(R)$ is used to denote the set of all idempotents of R, and $I(R)' = I(R)\backslash\{0, 1\}$. We use $N(R)$, and $N^*(R)$ to denote the set of all nilpotent elements, and upper nilradical (i.e., the sum of all nil ideals) of R,

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respectively. It is evident that $N^*(R) \subseteq N(R)$. A nilpotent element is also called a *nilpotent* for simplicity. For $n \geq 2$, denote the full and upper triangular matrix rings over R by $Mat_n(R)$ and $T_n(R)$ respectively. Let $\mathbb{Z}$, $\mathbb{Z}_n$, and $\mathbb{Q}$ denote the ring of integers, ring of integers modulo n, and the field of rational numbers, respectively. For $m, n \in \mathbb{Z}$, $gcd(m, n)$ is the greatest common divisor of m, n. The characteristic of R is denoted by $ch(R)$.

Following Kwak et al. [5, Definition 1.2], a ring R is called *right idempotent-quasi-normalizing on nilpotents*, abbreviated to *right IQNN*, provided that $I(R)'$ is empty, or else for every pair $(e, a) \in I(R)' \times N(R)$ there exists $(b, f) \in N(R) \times I(R)'$ such that $ea = bf$. A *left IQNN* ring is defined symmetrically. A ring is *IQNN* if it is both right and left IQNN. Abelian rings, in which every idempotent is central, are clearly IQNN but the converse is not true, as is shown in [5]. The following facts have essential role in this article.

Define the sets

$$\begin{array}{llll} E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) & E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) & E_3 = \left(\begin{smallmatrix} 1 & t \\ 0 & 0 \end{smallmatrix}\right) & \\ E_4 = \left(\begin{smallmatrix} 1 & 0 \\ u & 0 \end{smallmatrix}\right) & E_5 = \left(\begin{smallmatrix} 0 & t \\ 0 & 1 \end{smallmatrix}\right) & E_6 = \left(\begin{smallmatrix} 0 & 0 \\ u & 1 \end{smallmatrix}\right) & t \neq 0, u \neq 0 \\ B_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) & B_2 = \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right) & B_3 = \left(\begin{smallmatrix} 0 & 0 \\ u & 0 \end{smallmatrix}\right) & \end{array}$$

where our notation here means, for example, that $E_3 = \{\left(\begin{smallmatrix} 1 & t \\ 0 & 0 \end{smallmatrix}\right) \mid t \neq 0\}$, and define

$$\begin{array}{ll} E_7 = \left(\begin{smallmatrix} s & t \\ u & 1-s \end{smallmatrix}\right) & s \notin \{0,1\} \text{ and } s(1-s) = tu \\ B_4 = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) & a \neq 0, b \neq 0, c \neq 0, \text{ and } a^2 = -bc. \end{array}$$

The following is from [5, Lemma 2.3(2, 3)].

Lemma 1.1. *If F is a commutative domain and $R = Mat_2(F)$, then*

$$I(R)' = E_1 \cup E_2 \cup \cdots \cup E_7 \quad \text{and} \quad N(R) = B_1 \cup \cdots \cup B_4.$$

The following is from [1, Lemma 2.1(2)].

Lemma 1.2. *Let F be a commutative domain, $R = Mat_2(F)$, and K be the quotient field of F. In $Mat_2(K)$, we have*

(i) $E_7B_4 = \begin{pmatrix} sa+tc & \frac{b}{a}(sa+tc) \\ \frac{u}{s}(sa+tc) & \frac{bu}{as}(sa+tc) \end{pmatrix}$, $B_4E_7 = \begin{pmatrix} as+bu & \frac{t}{s}(as+bu) \\ \frac{c}{a}(as+bu) & \frac{ct}{as}(as+bu) \end{pmatrix}$;

(ii) $E_4B_4 = \begin{pmatrix} a & b \\ ua & ub \end{pmatrix} = \begin{pmatrix} a & -\frac{a}{c}a \\ ua & -\frac{ua}{c}a \end{pmatrix}$, $E_5B_4 = \begin{pmatrix} tc & -ta \\ c & -a \end{pmatrix} = \begin{pmatrix} -\frac{ta}{b}a & -ta \\ -\frac{a}{b}a & -a \end{pmatrix}$;

(iii) $B_4E_3 = \begin{pmatrix} a & ta \\ c & tc \end{pmatrix} = \begin{pmatrix} a & ta \\ -\frac{a}{b}a & -\frac{ta}{b}a \end{pmatrix}$, $B_4E_6 = \begin{pmatrix} ub & b \\ -ua & -a \end{pmatrix} = \begin{pmatrix} -\frac{ua}{c}a & -\frac{a}{c}a \\ -ua & -a \end{pmatrix}$, *from which it follows that if*

$$M \in E_7B_4 \cup B_4E_7 \cup E_4B_4 \cup E_5B_4 \cup B_4E_3 \cup B_4E_6,$$

we have that if M is nonzero, then every entry of M is nonzero.

In the following, we see a practical application of Lemma 1.2 that may provide useful information to the studies related to products of idempotents and nilpotents.

Remark 1.3. Let $F = \mathbb{Z}$ and $R = Mat_2(F)$.

(1) Let p, q be any nonzero integers. Let $C = \left(\begin{smallmatrix} 0 & p \\ 0 & q \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then, by Lemma 1.1, we have the cases that $A = B_2 = \left(\begin{smallmatrix} 0 & v \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$, and

$$E = E_7 = \begin{pmatrix} p' & m \\ q' & 1-p' \end{pmatrix} \in I(R)', \text{ where } p'(1-p') = q'm \neq 0$$

or

$$E = E_4 = \begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix} \in I(R)', \text{ where } u \neq 0.$$

That is, EA is $\left(\begin{smallmatrix} 0 & p'v \\ 0 & q'v \end{smallmatrix}\right)$ with $p = p'v$ and $q = q'v$, or $\left(\begin{smallmatrix} 0 & v \\ 0 & uv \end{smallmatrix}\right)$ with $p = v$ and $q = uv$. We will find $B \in N(R)$ and $E' \in I(R)'$ such that the left ideal RBE' of R contains EA.

`Case 1.` Suppose that p and q do not divide each other, and $gcd(p', q') \neq 1$. Evidently $|p'|, |q'| \geq 2$. Letting $p' = p''v_1$ and $q' = q''v_1$ with $gcd(p', q') = v_1$ (then $gcd(p'', q'') = 1$), we also have $|p''|, |q''| \geq 2$ since p and q do not divide each other.

Let $p'' = p_1^{u_1} \cdots p_f^{u_f}$ and $q'' = q_1^{v_1} \cdots q_g^{v_g}$, with $u_i, v_j \geq 1$, we the prime number decompositions of p'' and q'' respectively. Since $p''(1-p') = q''m$ and $gcd(p'', q'') = 1$, q'' must divide $1 - p'$. Letting $1 - p' = q''m'$, we have $p' + q''m' = 1$ and this implies $gcd(p', q'') = 1$ (hence $gcd(v_1, q'') = 1$).

Since q'' divides $1 - p'$ as above, we have that $-\frac{1-p'}{q''} \in \mathbb{Z}$ and

$$BE' = \begin{pmatrix} -q''p' & p'^2 \\ -q''^2 & q''p' \end{pmatrix} \begin{pmatrix} 0 & -\frac{1-p'}{q''} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & p' \\ 0 & q'' \end{pmatrix},$$

noting $B = \left(\begin{smallmatrix} -q''p' & p'^2 \\ -q''^2 & q''p' \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & -\frac{1-p'}{q''} \\ 0 & 1 \end{smallmatrix}\right) \in I(R)'$. From this, we also have

$$\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} BE' = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} -q''p' & p'^2 \\ -q''^2 & q''p' \end{pmatrix} \begin{pmatrix} 0 & -\frac{1-p'}{q''} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & p'v \\ 0 & q''v \end{pmatrix} = \begin{pmatrix} 0 & p \\ 0 & q''v \end{pmatrix}$$

and

$$\left(\begin{smallmatrix} vv_1 & 0 \\ 0 & vv_1 \end{smallmatrix}\right) BE' = \left(\begin{smallmatrix} vv_1 & 0 \\ 0 & vv_1 \end{smallmatrix}\right) \left(\begin{smallmatrix} -q''p' & p'^2 \\ -q''^2 & q''p' \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -\frac{1-p'}{q''} \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & p'vv_1 \\ 0 & q''vv_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & pv_1 \\ 0 & q \end{smallmatrix}\right),$$

noting $\left(\begin{smallmatrix} v & 0\\ 0 & v\end{smallmatrix}\right)\left(\begin{smallmatrix} -q''p' & p'^2\\ -q''^2 & q''p'\end{smallmatrix}\right), \left(\begin{smallmatrix} vv_1 & 0\\ 0 & vv_1\end{smallmatrix}\right)\left(\begin{smallmatrix} -q''p' & p'^2\\ -q''^2 & q''p'\end{smallmatrix}\right) \in N(R)$.

Thus RBE' contains the matrices

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}0 & p\\ 0 & q''v\end{pmatrix} = \begin{pmatrix}0 & p\\ 0 & 0\end{pmatrix} \text{ and } \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & pv_1\\ 0 & q\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & q\end{pmatrix},$$

entailing $EA \in RBE'$.

`Case 2.` The results in this case are obtained by the argument of [1, Lemma 2.1(3)].

(i) Suppose that p divides q. Then $BE' = \left(\begin{smallmatrix} -q & p\\ -\frac{q^2}{p} & q\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & p\\ 0 & q\end{smallmatrix}\right) = EA$, where $B = \left(\begin{smallmatrix} -q & p\\ -\frac{q^2}{p} & q\end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & 0\\ 0 & 1\end{smallmatrix}\right) \in I(R)'$.

(ii) Suppose that q divides p. Then $\left(\begin{smallmatrix} p & -\frac{p^2}{q}\\ q & -p\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1+\frac{p}{q}\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & p\\ 0 & q\end{smallmatrix}\right) = EA$, where $B = \left(\begin{smallmatrix} p & -\frac{p^2}{q}\\ q & -p\end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & 1+\frac{p}{q}\\ 0 & 1\end{smallmatrix}\right) \in I(R)'$.

(iii) Suppose that p and q do not divide each other and $gcd(p,q) = 1$. Then we get $BE' = \left(\begin{smallmatrix} -qp & p^2\\ -q^2 & qp\end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -\frac{1-p}{q}\\ 0 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & p\\ 0 & q\end{smallmatrix}\right) = EA$, where $B = \left(\begin{smallmatrix} -qp & p^2\\ -q^2 & qp\end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & -\frac{1-p}{q}\\ 0 & 1\end{smallmatrix}\right) \in I(R)'$.

Thus there exist $B \in N(R)$ and $E' \in I(R)'$ such that $EA \in RBE'$ in any case of (i), (ii) and (iii).

(2) Let p, q be any nonzero integers. Let $C = \left(\begin{smallmatrix} q & 0\\ p & 0\end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then, by Lemma 1.1, we have the cases that $0 \neq A = \left(\begin{smallmatrix} 0 & 0\\ s & 0\end{smallmatrix}\right) \in N(R)$, and

$$E = E_7 = \begin{pmatrix}1-p' & q'\\ m & p'\end{pmatrix} \in I(R)' \text{ (where } p'(1-p') = q'm \neq 0)$$

or

$$E = E_5 = \begin{pmatrix}0 & t\\ 0 & 1\end{pmatrix} \in I(R)' \text{ (where } t \neq 0);$$

that is, $EA = \left(\begin{smallmatrix} q's & 0\\ p's & 0\end{smallmatrix}\right)$ with $p = p's$ and $q = q's$, or $EA = \left(\begin{smallmatrix} st & 0\\ s & 0\end{smallmatrix}\right)$ with $p = s$ and $q = st$.

We will find $B \in N(R)$ and $E' \in I(R)'$ such that the left ideal RBE' of R contains EA.

`Case 1.` Suppose that p and q do not divide each other, and $gcd(p', q') \neq 1$.

By applying the argument and using the notation of (1), we have

$$BE' = \begin{pmatrix}q''p' & -q''^2\\ p'^2 & -q''p'\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1-p'}{q''} & 0\end{pmatrix} = \begin{pmatrix}q'' & 0\\ p' & 0\end{pmatrix},$$

noting $B = \left(\begin{smallmatrix} q''p' & -q''^2\\ p'^2 & -q''p'\end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & 0\\ -\frac{1-p'}{q''} & 0\end{smallmatrix}\right) \in I(R)'$. From this, we also have

$$\begin{pmatrix}v & 0\\ 0 & v\end{pmatrix}\begin{pmatrix}q''p' & -q''^2\\ p'^2 & -q''p'\end{pmatrix}\begin{pmatrix}1 & 0\\ -\frac{1-p'}{q''} & 0\end{pmatrix} = \begin{pmatrix}q''v & 0\\ p'v & 0\end{pmatrix} = \begin{pmatrix}q''v & 0\\ p & 0\end{pmatrix}$$

and

$$\begin{pmatrix} vv_1 & 0 \\ 0 & vv_1 \end{pmatrix}\begin{pmatrix} q''p' & -q''^2 \\ p'^2 & -q''p' \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{1-p'}{q''} & 0 \end{pmatrix} = \begin{pmatrix} q''vv_1 & 0 \\ p'vv_1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ pv_1 & 0 \end{pmatrix},$$

noting $\left(\begin{smallmatrix} v & 0 \\ 0 & v \end{smallmatrix}\right)\left(\begin{smallmatrix} q''p' & -q''^2 \\ p'^2 & -q''p' \end{smallmatrix}\right), \left(\begin{smallmatrix} vv_1 & 0 \\ 0 & vv_1 \end{smallmatrix}\right)\left(\begin{smallmatrix} q''p' & -q''^2 \\ p'^2 & -q''p' \end{smallmatrix}\right) \in N(R)$.

Thus RBE' contains the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} q & 0 \\ pv_1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q''v & 0 \\ p & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix},$$

entailing $EA \in RBE'$.

`Case 2.` The results in this case are obtained by the argument of [1, Lemma 2.1(3)].

(i) Suppose that p divides q. Then $BE' = \left(\begin{smallmatrix} q & -\frac{q^2}{p} \\ p & -q \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} q & 0 \\ p & 0 \end{smallmatrix}\right) = EA$, where $B = \left(\begin{smallmatrix} q & -\frac{q^2}{p} \\ p & -q \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$.

(ii) Suppose that q divides p. Then $BE' = \left(\begin{smallmatrix} -p & q \\ -\frac{p^2}{q} & p \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 1+\frac{p}{q} & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} q & 0 \\ p & 0 \end{smallmatrix}\right) = EA$, where $B = \left(\begin{smallmatrix} -p & q \\ -\frac{p^2}{q} & p \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & 0 \\ 1+\frac{p}{q} & 0 \end{smallmatrix}\right) \in I(R)'$.

(iii) Suppose that p and q do not divide each other and $gcd(p,q) = 1$. Then we get $BE' = \left(\begin{smallmatrix} qp & -q^2 \\ p^2 & -qp \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -\frac{1-p}{q} & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} q & 0 \\ p & 0 \end{smallmatrix}\right) = EA$, where $B = \left(\begin{smallmatrix} qp & -q^2 \\ p^2 & -qp \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & 0 \\ -\frac{1-p}{q} & 0 \end{smallmatrix}\right) \in I(R)'$.

Thus there exist $B \in N(R)$ and $E' \in I(R)'$ such that $EA \in RBE'$ in any case of (i), (ii) and (iii).

(3) Let p, q be any nonzero integers. Let $C = \left(\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then we have the cases that E is $E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$ or $E_3 = \left(\begin{smallmatrix} 1 & t \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$, and $A = B_4 = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in N(R)$; that is, EA is $\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right)$ with $p = a, q = b$, or $\left(\begin{smallmatrix} a+tc & b-ta \\ 0 & 0 \end{smallmatrix}\right)$ with $p = a + tc, q = b - ta$.

We will find $B \in N(R)$ and $E' \in I(R)'$ such that the right ideal $BE'R$ of R contains EA.

`Case 1.` Suppose that q divides p.

We have

$$\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \frac{p}{q} & 1 \end{pmatrix} = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = EA,$$

noting $B = \left(\begin{smallmatrix} 0 & q \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & 0 \\ \frac{p}{q} & 1 \end{smallmatrix}\right) \in I(R)'$.

Based on Case 1, we can assume that $|q| \geq 2$ in the cases below.

`Case 2.` Suppose that p divides q.

We have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1-q & \frac{q(1-q)}{p} \\ p & q \end{pmatrix} = \begin{pmatrix} p & q \\ 0 & 0 \end{pmatrix} = EA,$$

noting $B = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1-q & \frac{q(1-q)}{p} \\ p & q \end{smallmatrix}\right) \in I(R)'$.

`Case 3.` Suppose that p and q do not divide each other.

Let $p = p'k$ and $q = q'k$ with $gcd(p,q) = k$. Take $B = \left(\begin{smallmatrix} 0 & k \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(R)'$. Then $BE'R$ contains $BE' = \left(\begin{smallmatrix} 0 & k \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & k \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} k & k \\ 0 & 0 \end{smallmatrix}\right)$ and, consequently, contains $\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Thus $BE'R$ contains $\left(\begin{smallmatrix} k & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} p' & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & k \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & q' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & q \\ 0 & 0 \end{smallmatrix}\right)$, and hence contains $\left(\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right)$.

(4) Let p, q be any nonzero integers. Let $C = \left(\begin{smallmatrix} 0 & 0 \\ q & p \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then we have the cases that E is $E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(R)'$ or $E_6 = \left(\begin{smallmatrix} 0 & 0 \\ u & 1 \end{smallmatrix}\right) \in I(R)'$, and $A = B_4 = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in N(R)$; that is, EA is $\left(\begin{smallmatrix} 0 & 0 \\ c & -a \end{smallmatrix}\right)$ with $q = c, p = -a$, or $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $q = ua + c, q = ub - a$.

We will find $B \in N(R)$ and $E' \in I(R)'$ such that the right ideal $BE'R$ of R contains EA.

`Case 1.` Suppose that q divides p.

We have

$$\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix}\begin{pmatrix} 1 & \frac{p}{q} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ q & p \end{pmatrix} = EA,$$

noting $B = \left(\begin{smallmatrix} 0 & 0 \\ q & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & \frac{p}{q} \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$.

Based on Case 1, we assume $|q| \geq 2$ in the cases below.

`Case 2.` Suppose that p divides q.

We have

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q & p \\ \frac{q(1-q)}{p} & 1-q \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ q & p \end{pmatrix} = EA,$$

noting $B = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} q & p \\ \frac{q(1-q)}{p} & 1-q \end{smallmatrix}\right) \in I(R)'$.

`Case 3.` Suppose that p and q do not divide each other.

Let $p = p'k$ and $q = q'k$ with $gcd(p,q) = k$. Take $B = \left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$. Then $BE'R$ contains $BE' = \left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ k & k \end{smallmatrix}\right)$ and, consequently, contains $\left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)$.

Thus $BE'R$ contains $\left(\begin{smallmatrix} 0 & 0 \\ k & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} q' & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ q & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & k \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 0 & p' \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & p \end{smallmatrix}\right)$, and hence contains $\left(\begin{smallmatrix} 0 & 0 \\ q & p \end{smallmatrix}\right)$.

(5) Let p be any nonzero integer.

Let $C = \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then we have the case that $E = E_3 = \left(\begin{smallmatrix} 1 & t \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$ and $A = B_3 = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right) \in N(R)$; that is, $EA = \left(\begin{smallmatrix} ct & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $p = ct$. Take $B = \left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(R)'$. Then $BE'R$ contains $\left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} p & p \\ 0 & 0 \end{smallmatrix}\right)$ and, consequently, contains $EA = \left(\begin{smallmatrix} p & 0 \\ 0 & 0 \end{smallmatrix}\right)$.

Let $C = \left(\begin{smallmatrix} 0 & 0 \\ 0 & p \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then we have the case that $E = E_6 = \left(\begin{smallmatrix} 0 & 0 \\ u & 1 \end{smallmatrix}\right) \in I(R)'$ and $A = B_2 = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$;

that is, $EA = \left(\begin{smallmatrix} 0 & 0 \\ 0 & bu \end{smallmatrix}\right)$ with $p = bu$. Take $B = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$. Then $BE'R$ contains $\left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ p & p \end{smallmatrix}\right)$ and, consequently, contains $EA = \left(\begin{smallmatrix} 0 & 0 \\ 0 & p \end{smallmatrix}\right)$.

Let $C = \left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then we have the case that $E = E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$ and $A = B_2 = \left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$. Take $B = \left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(R)'$. Then $BE'R$ contains $EA = \left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right)$.

Let $C = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then we have the case that $E = E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(R)'$ and $A = B_3 = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right) \in N(R)$. Take $B = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right) \in N(R)$ and $E' = E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in I(R)'$. Then $BE'R$ contains $EA = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix}\right)$.

(6) Let p, q, r, s be nonzero integers, and let $C = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in R$ be such that $C = EA$ for some $E \in I(R)'$ and $A \in N(R)$. Then, by Lemma 1.1, BE' must be one of B_4E_3, B_4E_6, or B_4E_7.

By (1) and (2), there exist $C_i \in N(R)$ and $F_i \in I(R)'$ such that $\left(\begin{smallmatrix} 0 & q \\ 0 & s \end{smallmatrix}\right) \in RC_1F_1$ and $\left(\begin{smallmatrix} p & 0 \\ r & 0 \end{smallmatrix}\right) \in RC_2F_2$, from which we obtain $C \in RC_1F_1 + RC_2F_2$.

By (3) and (4), there exist $C_i' \in N(R)$ and $F_i' \in I(R)'$ such that $\left(\begin{smallmatrix} p & q \\ 0 & 0 \end{smallmatrix}\right) \in C_1'F_1'R$ and $\left(\begin{smallmatrix} 0 & 0 \\ r & s \end{smallmatrix}\right) \in C_2'F_2'R$, from which we obtain $C \in C_1'F_1'R + C_2'F_2'R$.

Similar arguments are available to the cases of $\left(\begin{smallmatrix} p & q \\ r & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} p & q \\ 0 & s \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} p & 0 \\ r & s \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & q \\ r & s \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} p & 0 \\ 0 & s \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & q \\ r & 0 \end{smallmatrix}\right)$.

$Mat_2(\mathbb{Z})$ is shown to be not right IQNN by [1, Theorem 2.3(1)]. In the next section, we introduce a closely related property that $Mat_2(\mathbb{Z})$ satisfies, based on Remark 1.3.

2. Weakly Right IQNN Rings

Motivated by the arguments of Remark 1.3., we consider the following new ring property as a generalization of right IQNN ring.

Definition 2.1. A ring R is said to be *weakly right IQNN* provided that $I(R)'$ is empty, or else every pair $(e, a) \in I(R)' \times N(R)$ satisfies one of the following:

(i) There exist $b \in N(R)$ and $f \in I(R)'$ such that $ea \in bfR$;

(ii) There exist $b_i \in N(R)$ and $f_i \in I(R)'$ $(i = 1, 2)$ such that $ea \in b_1f_1R + b_2f_2R$.

R is called *weakly left IQNN* provided that $I(R)'$ is empty, or else every pair $(e, a) \in I(R)' \times N(R)$ satisfies one of the following:

(i) There exist $b' \in N(R)$ and $f' \in I(R)'$ such that $ae \in Rf'b'$;

(ii) There exist $b_i' \in N(R)$ and $f_i' \in I(R)'$ $(i = 1, 2)$ such that $ae \in Rf_1'b_1' + Rf_2'b_2'$.

A ring is *weakly IQNN* if it is both weakly right IQNN and weakly left IQNN.

Right IQNN rings are clearly weakly right IQNN, but not conversely as we see in the arguments below.

Theorem 2.2. *$Mat_2(A)$ is weakly IQNN over any ring A.*

Proof. Let $R = Mat_1(A)$. We apply the argument of Remark 1.3. Let $0 \neq M = \left(\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right) \in R$. Take

$$E_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in I(R)'$$

and

$$B_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in N(R).$$

Then we have

$$M = B_1E_1 \begin{pmatrix} 0 & 0 \\ p & q \end{pmatrix} + B_2E_2 \begin{pmatrix} r & s \\ 0 & 0 \end{pmatrix} \in B_1E_1R + B_2E_2R.$$

Thus R is weakly right IQNN. The proof for the case of weakly left IQNN can be done symmetrically. □

$Mat_2(\mathbb{Z})$ is not right IQNN as mentioned above, and can be shown to be not left IQNN by a symmetrical method of the proof of [1, Theorem 2.3(1)]. Thus the concept of weakly right (resp., left) IQNN is a proper generalization of right (resp., left) IQNN.

In the following, we see another kind of weakly right IQNN rings but not right IQNN.

Example 2.3. Let $K = \mathbb{Z}_2$ and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K.

(1) We use the ring of the ring of [4, Example 2.3(2)]. Let I be the ideal of A generated by $a^2 - a, b^2, ab$ and set $R = A/I$ and identify the elements in A with their images in R_1 for simplicity. Then $a^2 = a$ and $ab = 0 = b^2$.

By applying the arguments of [4, Example 2.3(1)] and [5, Example 2.6], we have the following:

(i) every element $r \in R$ is of the form $r = \alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ba$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in K$;

(ii) $I(R)' = \{1 + a + \gamma ba, a + \gamma' ba \mid \gamma, \gamma' \in K\}$ and $N(R) = \{\alpha ba + \beta b \mid \alpha, \beta \in K\}$, that is an ideal of R (i.e., $N(R) = N^*(R)$);

(iii) $S_1 = \{en \mid e \in I(R)', n \in N(R)\} = N(R)$ and $S_2 = \{n'e' \mid n' \in N(R),\ e' \in I(R)'\} = \{b + ba, \eta ba \mid \delta, \eta \in K\}$. So $S_1 \supsetneq S_2$.

Since $S_1 \supsetneq S_2$, R is (weakly) left IQNN. Next consider $e = 1 + a \in I(R)'$ and $b \in N(R)$. Then $eb = b$. Since $b \notin S_2$, R is not right IQNN. But

$$b = (b + ba) + ba = (b + ba)(1 + a) + (ba)a \in c_1e_1R + c_2e_2R,$$

where $e_1 = 1+a, e_2 = a \in I(R)'$ and $c_1 = b+ba, c_2 = ba \in N(R)$. Thus R is weakly right IQNN.

(2) Let J be the ideal of A generated by $a^2 - a, b^2, ba$. and set $R' = A/J$. Then R' is the opposite ring of R of (1). Then a similar argument shows that R' is weakly IQNN but not left IQNN.

(3) Let I' be the ideal of A generated by $a^2 - a, b^2, ab - b$ and set $R = A/I'$ and identify the elements in A with their images in R for simplicity. Then $a^2 = a$, $ab = b$, and $b^2 = 0$ in R. From this relation we obtain the following:

(i) Every element $r \in R$ is of the form $r = \alpha_0 + \alpha_1 a + \alpha_2 b + \alpha_3 ba$, where $\alpha_i \in K$;

(ii) $I(R)' = \{1 + a + \gamma b + \gamma ba, a + \gamma' b + \gamma' ba \mid \gamma, \gamma' \in K\}$ and $N(R) = \{\alpha b + \alpha' ba \mid \alpha, \alpha' \in K\}$, that is an ideal of R;

(iii) $S_1 = \{en \mid e \in I(R)', n \in N(R)\} = N(R)$ and $S_2 = \{n'e' \mid n' \in N(R),\ e' \in I(R)'\} = \{\alpha ba, b + ba\}$. So $S_1 \supsetneq S_2$.

Since $S_1 \supsetneq S_2$, R is (weakly) left IQNN. Next consider $e = a \in I(R)'$ and $b \in N(R)$. Then $eb = ab = b$. But $b \notin S_2$ and so R is not right IQNN. But

$$b = (b + ba) + ba = (b + ba)(1 + a) + (ba)a \in c_1e_1R + c_2e_2R,$$

where $e_1 = 1+a, e_2 = a \in I(R)'$ and $c_1 = b+ba, c_2 = ba \in N(R)$. Thus R is weakly right IQNN.

(4) Let J' be the ideal of A generated by $a^2 - a, b^2, ba - b$. and set $R' = A/J'$. Then R' is the opposite ring of R of (3). Then a similar argument shows that R' is weakly IQNN but not left IQNN.

The non-Abelian rings of Example 2.3 are all weakly IQNN. Next we provide a method by which one can construct non-Abelian rings that are neither weakly right nor weakly left IQNN.

Example 2.4. We use the ring of [3, Example 1.2(2)]. Let $K = \mathbb{Z}_2$ and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by $a^2 - a, b^2$ and set $R = A/I$. Identify the elements in A with their images in R for simplicity. Then $a^2 = a$ and $b^2 = 0$. By help of the argument of [3, Example 1.2(2)], we can express $r \in R$ and $c \in N(R)$ by

$$r = k_0 + k_1a + k_2b + af_1a + af_2b + bf_3a + bf_4b \text{ and } c = kb + bfb$$

where $k, k_i \in K$ and $f, f_j \in R$ for all j.

Let $e = a \in I(R)'$ and $c = b \in N(R)$. Assume that $ec = ab = c'e'r$ for some $c' = kb + bfb \in N(R)$, $e' \in I(R)'$ and $r \in R$. Since $ab \neq 0$, $c' = kb + bfb = b(k + fb) \neq 0$ and this yields

$$ab = b(k + fb)e'r \text{ and } 0 \neq bab = bb(k + fb)e'r = 0,$$

a contradiction. Next assume that $ec = ab = c_1e_1r_1 + c_2e_2r_2$ for some $0 \neq c_i = k_ib + bf_ib \in N(R)$, $e_i \in I(R)'$ and $r_i \in R$ $(i = 1, 2)$. This yields

$$ab = b((k_1 + f_1b)e_1r_1 + (k_2 + f_2b)e_2r_2)$$

and

$$0 \neq bab = bb((k_1 + f_1 b)e_1 r_1 + (k_2 + f_2 b)e_2 r_2) = 0,$$

a contradiction. Thus R is not weakly right IQNN. It is also shown by a symmetrical argument that R is not weakly left IQNN.

Next we consider two kinds of rings R over which $T_2(R)$ may be weakly right IQNN.

Proposition 2.5. *Let R be a ring.*

(1) *If $N(R) = N^*(R)$ then $T_2(R)$ is weakly right IQNN.*

(2) *If $I(R) = \{0, 1\}$ then $T_2(R)$ is weakly right IQNN.*

Proof. Write $T = T_2(R)$. Note that

$$I(T)' = \left\{ \begin{pmatrix} e & g \\ 0 & f \end{pmatrix} \in T \mid e, f \in I(R), (e, f) \notin \{(0, 0), (1, 1)\}, eg + gf = g \right\}$$

and

$$N(T) = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in T \mid a, b \in N(R) \text{ and } c \in R \right\}.$$

(1) Assume $N(R) = N^*(R)$. Let $E = \left(\begin{smallmatrix} e & g \\ 0 & f \end{smallmatrix}\right) \in I(T)'$ and $A = \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) \in N(T)$. Then $EA = \left(\begin{smallmatrix} ea & ec+gb \\ 0 & fb \end{smallmatrix}\right) \in N(T)$, i.e., $ea, fb \in N(R)$, by assumption. Take $E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B_1 = \left(\begin{smallmatrix} ea & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $B_2 = \left(\begin{smallmatrix} 0 & ec+gb \\ 0 & fb \end{smallmatrix}\right)$. Then $E_i \in I(T)'$ and $B_i \in N(T)$ such that $EA = B_1E_1 + B_2E_2 \in B_1E_1T + B_2E_2T$. Thus T is weakly right IQNN.

(2) Assume $I(R) = \{0, 1\}$. Then

$$I(T)' = \left\{ \begin{pmatrix} e & g \\ 0 & f \end{pmatrix} \in T \mid (e, f) \in \{(1, 0), (0, 1)\}, g \in R \right\}.$$

Let $E = \left(\begin{smallmatrix} e & g \\ 0 & f \end{smallmatrix}\right) \in I(T)'$ and $A = \left(\begin{smallmatrix} a & c \\ 0 & b \end{smallmatrix}\right) \in N(T)$. Then $EA = \left(\begin{smallmatrix} ea & ec+gb \\ 0 & fb \end{smallmatrix}\right) \in N(T)$ since $e, f \in \{0, 1\}$; in fact, ea is zero or a, and fb is also zero or b. Take $E_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $E_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B_1 = \left(\begin{smallmatrix} ea & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $B_2 = \left(\begin{smallmatrix} 0 & ec+gb \\ 0 & fb \end{smallmatrix}\right)$. Then $E_i \in I(T)'$ and $B_i \in N(T)$. Since $EA = B_1E_1 + B_2E_2 \in B_1E_1T + B_2E_2T$. Thus T is weakly right IQNN. □

In the following argument we see a condition under which the weakly IQNN property is right-left symmetric. Let R be a ring. An involution on a ring R is a function $* : R \to R$ which satisfies the properties that $(x + y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $1^* = 1$, and $(x^*)^* = x$ for all $x, y \in R$. It is easily checked that $0^* = 0$, $a \in N(R)$ implies $a^* \in N(R)$, and $e^* \in I(R)'$ for $e \in I(R)'$. We use these facts without referring.

Proposition 2.6. *Let R be a ring with an involution $*$. Then R is weakly right IQNN if and only if R is weakly left IQNN.*

Proof. Assume that $I(R)'$ is nonempty. Suppose that R is weakly right IQNN. Let $a \in N(R)$ and $e \in I(R)'$. Then $a^* \in N(R)$ and $e^* \in I(R)'$. Since R is weakly right IQNN, we have the following four cases. We proceed our argument on a case-by-case computation.

(i) There exist $b \in N(R)$, $f \in I(R)'$ and $s \in R$ such that $e^*a^* = bfs$. This implies that

$$ae = ((ae)^*)^* = (e^*a^*)^* = (bfs)^* = s^*f^*b^* \in Rf^*b^*.$$

(ii) There exist $b_i \in N(R)$, $f_i \in I(R)'$ and $s_i \in R$ $(i = 1, 2)$ such that $e^*a^* = b_1f_1s_1 + b_2f_2s_2$. This implies that

$$ae = ((ae)^*)^* = (e^*a^*)^* = (b_1f_1s_1 + b_2f_2s_2)^* = s_1^*f_1^*b_1^* + s_2^*f_2^*b_2^* \in Rf_1^*b_1^* + Rf_2^*b_2^*.$$

Since $b^*, b_i^* \in N(R)$ and $f^*, f_i^* \in I(R)'$, we now conclude that R is weakly left IQNN by the results (i) and (ii).

Conversely suppose that R is weakly left IQNN. Then we have the following cases.

(iii) There exist $b' \in N(R)$, $f' \in I(R)'$ and $r \in R$ such that $a^*e^* = rf'b'$. This implies that

$$ea = ((ea)^*)^* = (a^*e^*)^* = (rf'b')^* = b'^*f'^*r^* \in b'^*f'^*R.$$

(iv) There exist $b_i' \in N(R)$, $f_i' \in I(R)'$ and $r_i \in R$ $(i = 1, 2)$ such that $a^*e^* = r_1f_1'b_1' + r_2f_2'b_2'$. This implies that

$$ea = ((ea)^*)^* = (a^*e^*)^* = (r_1f_1'b_1' + r_2f_2'b_2')^* = b_1'^*f_1'^*r_1{}^* + b_2'^*f_2'^*r_2^* \in b_1'^*f_1'^*R+$$

$b_2'^*f_2'^*R$.

Since $b'^*, b'^*_i \in N(R)$ and $f'^*, f'^*_i \in I(R)'$, we now conclude that R is weakly right IQNN by the results (iii) and (iv). □

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