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Infinite Families of Congruences for Partition Functions $\overline{\mathscr{E}\mathscr{O}}(n)$ and $\mathscr{E}\mathscr{O}_e(n)$

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Abstract. In 2018, Andrews introduced the partition functions $\mathscr{E}\mathscr{O}(n)$ and $\overline{\mathscr{E}\mathscr{O}}(n)$. The first of these denotes the number of partitions of n in which every even part is less than each odd part, and the second counts the number of partitions enumerated by the first in which only the largest even part appears an odd number of times. In 2021, Pore and Fathima introduced a new partition function $\mathscr{E}\mathscr{O}_e(n)$ which counts the number of partitions of n which are enumerated by $\overline{\mathscr{E}\mathscr{O}}(n)$ together with the partitions enumerated by $\mathscr{E}\mathscr{O}(n)$ where all parts are odd and the number of parts is even. They also proved some particular congruences for $\overline{\mathscr{E}\mathscr{O}}(n)$ and $\mathscr{E}\mathscr{O}_e(n)$. In this paper, we establish infinitely many families of congruences modulo 2, 4, 5 and 8 for $\overline{\mathscr{E}\mathscr{O}}(n)$ and modulo 4 for $\mathscr{E}\mathscr{O}_e(n)$. For example, if $p \geq 5$ is a prime with Legendre symbol $\left(\frac{-3}{p}\right) = -1$, then for all integers $n \geq 0$ and $\alpha \geq 0$, we have

$$\overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha+1}(pn+j) + \frac{19 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod 8; \qquad 1 \leq j \leq (p-1).$$

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum equals n. The number of partitions of a non-negative integer n is usually denoted by $p(n)$ (with $p(0) = 1$). For example, $p(4) = 5$ with the relevant partitions 4, 3+1, 2+2, 2+1+1 and 1+1+1+1. The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}, \tag{1.1}$$

where, for any complex number a,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n), \quad |q| < 1. \tag{1.2}$$

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We will use the notation, for any positive integer k,

$$f_k := (q^k; q^k)_\infty. \tag{1.3}$$

Andrews [2] introduced the partition function $\mathscr{EO}(n)$ which counts the number of partitions of n in which every even part is less than each odd part. For example, $\mathscr{EO}(4) = 4$ with the relevant partitions 4, 3+1, 2+2 and 1+1+1+1. The generating function for $\mathscr{EO}(n)$ [2] is given by

$$\sum_{n=0}^{\infty} \mathscr{EO}(n)q^n = \frac{1}{(1-q)(q^2;q^2)_\infty}. \tag{1.4}$$

Andrews [2] also introduced another partition function $\overline{\mathscr{EO}}(n)$ which counts the number of partitions enumerated by $\mathscr{EO}(n)$ in which only the largest even part appears an odd number of times. For example $\overline{\mathscr{EO}}(4) = 3$, with the relevant partitions 4, 3+1 and 1+1+1+1. The generating function of $\overline{\mathscr{EO}}(n)$ [2] is given by

$$\sum_{n=0}^{\infty} \overline{\mathscr{EO}}(n)q^n = \frac{f_4^3}{f_2^2}. \tag{1.5}$$

Andrews [2, p. 434, (1.6)] established that

$$\overline{\mathscr{EO}}(10n+8) \equiv 0 \pmod{5}.$$

Recently, Pore and Fathima [8] proved some particular congruences for $\overline{\mathscr{EO}}(n)$ modulo 2,4,10 and 20. For example, they proved that

$$\overline{\mathscr{EO}}(4n+2) \equiv 0 \pmod{4}, \quad \overline{\mathscr{EO}}(20n+18) \equiv 0 \pmod{10}, \quad \overline{\mathscr{EO}}(40n+38) \equiv 0 \pmod{20}.$$

Pore and Fathima [8] also defined a new partition function $\mathscr{EO}_e(n)$ which counts the number of partitions of n which are enumerated by $\overline{\mathscr{EO}}(n)$ together with the partitions enumerated by $\mathscr{EO}(n)$ where all parts are odd and the number of parts is even, i.e. $\mathscr{EO}_e(n)$ denotes the number of partitions enumerated by $\mathscr{EO}(n)$ in which only the largest even part appears an odd number of times except when parts are odd and number of parts is even. For example, $\mathscr{EO}_e(4) = 3$ with the relevant partitions 4, 3+1 and 1+1+1+1. The generating function of $\mathscr{EO}_e(n)$ [8] is given by

$$\sum_{n=0}^{\infty} \mathscr{EO}_e(n)q^n = \frac{f_4^2}{f_2^2}. \tag{1.6}$$

Pore and Fathima [8, Corollary 5.2] proved that

$$\mathscr{EO}_e(2n+1) = 0 \quad \text{and} \quad \mathscr{EO}_e(4n+2) \equiv 0 \pmod{2}.$$

Motivated by the above work, in Section 3 of this paper, we prove infinite families of congruences modulo 2, 4, 5 and 8 for $\overline{\mathscr{EO}}(n)$. In Section 4, we prove infinite families of congruences modulo 2 and 4 for $\mathscr{EO}_e(n)$. To prove our results, we employ some known q-series identities which are listed in Section 2.

2. Some q-series Identities

Lemma 2.1. ([3, p. 39, Entry 24(ii)]) *We have*

$$f_1^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2}. \tag{2.1}$$

Lemma 2.2. We have

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.2}$$

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8}. \tag{2.3}$$

The identity (2.2) is the 2-dissection of $\phi(q)$ [6, (1.9.4)]. The equation (2.3) can be obtained from the equation (2.2) by replacing q by $-q$.

Lemma 2.3. ([1, Lemma 2.3]) *For any prime $p \geq 3$, we have*

$$f_1^3 = \sum_{\substack{k=0 \\ k \neq (p-1)/2}}^{(p-1)} (-1)^k q^{k(k+1)/2} \sum_{n=0}^{\infty} (-1)^n (2pn + 2k + 1) q^{pn \cdot (pn+2k+1)/2}$$

$$+ p(-1)^{(p-1)/2} q^{(p^2-1)/8} f_{p^2}^3. \tag{2.4}$$

Furthermore, if $k \neq \dfrac{(p-1)}{2}$, $\quad 0 \leq k \leq p-1$, *then*

$$\frac{(k^2+k)}{2} \not\equiv \frac{(p^2-1)}{8} \pmod{p}.$$

Lemma 2.4. ([4, Theorem 2.2]) *For any prime $p \geq 5$, we have*

$$f_1 = \sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)$$

$$+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2}, \tag{2.5}$$

where

$$\frac{\pm p - 1}{6} = \begin{cases} \dfrac{(p-1)}{6}, & if\ p \equiv 1 \pmod{6}, \\ \dfrac{(-p-1)}{6}, & if\ p \equiv -1 \pmod{6}. \end{cases}$$

Furthermore, if

$$\frac{-(p-1)}{2} \leq k \leq \frac{(p-1)}{2} \quad and \quad k \neq \frac{(\pm p-1)}{6},$$

then

$$\frac{(3k^2+k)}{2} \not\equiv \frac{(p^2-1)}{24} \pmod{p}.$$

Lemma 2.5. ([7]) *We have*

$$f_1 = f_{25}\left(R\left(q^5\right) - q - \frac{q^2}{R\left(q^5\right)}\right), \tag{2.6}$$

where R(q) is the Roger-Ramanujan continued fraction defined by

$$R(q) := \cfrac{q^{1/5}}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}, \quad |q|<1.$$

Hirschhorn and Hunt [5, Lemma 2.2] proved that, if R is a series in powers of q^5, then

$$\eta = q^{-1}R - 1 - qR^{-1}, \tag{2.7}$$

where

$$\eta = \frac{f_1}{qf_{25}}. \tag{2.8}$$

Hirschhorn and Hunt [5] showed that

$$H_5(\eta^3) = 5, \tag{2.9}$$

where H_5 is an operator which acts on a series of positive and negative powers of a single variable and simply picks out the term in which the power is congruent to 0 modulo 5.

In addition to above q-series identities, we will be using following congruence properties which follow from binomial theorem and (1.2)*: For positive integers k and m,*

$$f_k^{2m} \equiv f_{2k}^m \pmod{2}, \tag{2.10}$$

$$f_k^{4m} \equiv f_{2k}^{2m} \pmod{4}, \tag{2.11}$$

$$f_1^5 \equiv f_5 \pmod{5}, \tag{2.12}$$

3. Congruences for $\overline{\mathscr{EO}}(n)$

Theorem 3.1. *Let $p \geq 5$ be a prime with $\left(\frac{-3}{p}\right) = -1$ and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{\mathscr{EO}}\left(8\cdot p^{2\alpha}n + \frac{19\cdot p^{2\alpha}-1}{3}\right)q^n \equiv 4f_{16}f_1^3 \pmod{8}, \tag{3.1}$$

$$\overline{\mathscr{EO}}\left(8\cdot p^{2\alpha+1}(pn+j) + \frac{19\cdot p^{2\alpha+2}-1}{3}\right) \equiv 0 \pmod{8}, \tag{3.2}$$

where, here and throughout the paper $(\frac{\cdot}{\cdot})$ denotes the Legendre symbol.

Proof. From (1.5), we note that

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(n)q^n = \frac{f_4^3}{f_2^2}. \tag{3.3}$$

Extracting the terms involving q^{2n} and using (2.2), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(2n)q^n = \frac{f_8^5}{f_2^2 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^2 f_8}. \tag{3.4}$$

Extracting the terms involving q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(4n+2)q^n = 2\frac{f_2^2 f_8^2}{f_1^2 f_4}. \tag{3.5}$$

Using (2.2) in (3.5), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(4n+2)q^n = 2\frac{f_8^7}{f_2^3 f_4 f_{16}^2} + 4q\frac{f_4 f_8 f_{16}^2}{f_2^3}. \tag{3.6}$$

Extracting the terms involving q^{2n+1} from (3.6), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(8n+6)q^n = 4\frac{f_2 f_4 f_8^2}{f_1^3}. \tag{3.7}$$

Using (2.10) in (3.7), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(8n+6)q^n \equiv 4f_{16}f_1^3 \pmod{8}. \tag{3.8}$$

Congruence (3.8) is the $\alpha = 0$ case of (3.1). Suppose that congruence (3.1) is true for all $\alpha \geq 0$. Utilizing (2.4) and (2.5) in (3.1), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2\alpha}n + \frac{19\cdot p^{2\alpha}-1}{3}\right)q^n \equiv \\
&\quad 4\Bigg\{ \sum_{\substack{k=-(p-1)/2 \\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{16(3k^2+k)/2} f\left(-q^{16(3p^2+(6k+1)p)/2}, -q^{16(3p^2-(6k+1)p)/2}\right) \\
&\qquad + (-1)^{(\pm p-1)/6} q^{16(p^2-1)/24} f_{16p^2} \Bigg\} \\
&\quad \times \Bigg\{ \sum_{\substack{m=0 \\ m\neq(p-1)/2}}^{(p-1)} (-1)^m q^{m(m+1)/2} \sum_{n=0}^{\infty} (-1)^n (2pn+2m+1) q^{pn\cdot(pn+2m+1)/2} \\
&\qquad + p(-1)^{(p-1)/2} q^{(p^2-1)/8} f_{p^2}^3 \Bigg\} \pmod{8}.
\end{aligned} \tag{3.9}$$

Consider the congruence

$$\frac{16(3k^2+k)}{2}+\frac{(m^2+m)}{2}\equiv\frac{19(p^2-1)}{24}\pmod{p},$$

which is equal to

$$(24k+4)^2+3(2m+1)^2\equiv 0\pmod{p}.$$

For $\left(\frac{-3}{p}\right)=-1$, the above congruence has only solution $m=\frac{p-1}{2}$ and $k=\frac{\pm p-1}{6}$. Therefore, extracting the terms involving $q^{pn+19(p^2-1)/24}$ from both sides of (3.9), dividing throughout by $q^{19(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2\alpha+1}n+\frac{19\cdot p^{2\alpha+2}-1}{3}\right)q^n\equiv 4f_{16p}f_p^3\pmod{8}. \tag{3.10}$$

Extracting the terms involving q^{pn} from (3.10) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2(\alpha+1)}n+\frac{19\cdot p^{2(\alpha+1)}-1}{3}\right)q^n\equiv 4f_{16}f_1^3\pmod{8}, \tag{3.11}$$

which is the $\alpha+1$ case of (3.1). Thus, by the principle of mathematical induction, we arrive at (3.1). Extracting the coefficients of terms involving q^{pn+j} for $1\le j\le p-1$, from both sides of (3.10), we complete the proof of (3.2). □

Theorem 3.2. *Let $p\ge 5$ be a prime with $\left(\frac{-3}{p}\right)=-1$ and $1\le j\le (p-1)$. Then for all integers $n\ge 0$ and $\alpha\ge 0$, we have*

$$\sum_{n=0}^{\infty}\overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2\alpha}n+\frac{7\cdot p^{2\alpha}-1}{3}\right)q^n\equiv 2f_4f_1^3\pmod{4}, \tag{3.12}$$

$$\overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2\alpha+1}(pn+j)+\frac{7\cdot p^{2\alpha+2}-1}{3}\right)\equiv 0\pmod{4}. \tag{3.13}$$

Proof. From (3.5), we note that

$$\sum_{n=0}^{\infty}\overline{\mathscr{E}\mathscr{O}}(4n+2)q^n=2\frac{f_2^2f_8^2}{f_1^2f_4}. \tag{3.14}$$

Employing (2.10) in (3.14), we have

$$\sum_{n=0}^{\infty}\overline{\mathscr{E}\mathscr{O}}(4n+2)q^n\equiv 2f_2^7\pmod{4}. \tag{3.15}$$

Extracting the terms involving q^{2n} from both side of (3.15), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathscr{E}\mathscr{O}}(8n+2)q^n\equiv 2f_1^7\pmod{4}. \tag{3.16}$$

Employing (2.10) in (3.16), we obtain

$$\sum_{n=0}^{\infty}\overline{\mathscr{EO}}(8n+2)q^n \equiv 2f_4f_1^3 \pmod 4. \tag{3.17}$$

Congruence (3.17) is the $\alpha=0$ case of (3.12). Suppose that congruence (3.12) is true for all $\alpha \geq 0$. Utilizing (2.4) and (2.5) in (3.12), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{\mathscr{EO}}\left(8\cdot p^{2\alpha}n+\frac{7\cdot p^{2\alpha}-1}{3}\right)q^n &\equiv \\ & 2\Bigg\{\sum_{\substack{k=-(p-1)/2\\ k\neq(\pm p-1)/6}}^{(p-1)/2}(-1)^k q^{4(3k^2+k)/2} f\left(-q^{4(3p^2+(6k+1)p)/2}, -q^{4(3p^2-(6k+1)p)/2}\right) \\ &\quad +(-1)^{(\pm p-1)/6}q^{4(p^2-1)/24}f_{4p^2}\Bigg\} \\ &\times\Bigg\{\sum_{\substack{m=0\\ m\neq(p-1)/2}}^{(p-1)}(-1)^m q^{m(m+1)/2}\sum_{n=0}^{\infty}(-1)^n(2pn+2m+1)q^{pn\cdot(pn+2m+1)/2} \\ &\quad +p(-1)^{(p-1)/2}q^{(p^2-1)/8}f_{p^2}^3\Bigg\} \pmod 4. \end{aligned} \tag{3.18}$$

Consider the congruence

$$\frac{4(3k^2+k)}{2}+\frac{(m^2+m)}{2}\equiv\frac{7(p^2-1)}{24} \pmod p,$$

which is equal to

$$(12k+2)^2+3(2m+1)^2\equiv 0 \pmod p.$$

For $\left(\frac{-3}{p}\right)=-1$, the above congruence has only solution $m=\frac{p-1}{2}$ and $k=\frac{\pm p-1}{6}$. Therefore, extracting the terms involving $q^{pn+7(p^2-1)/24}$ from both sides of (3.18), dividing throughout by $q^{7(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathscr{EO}}\left(8\cdot p^{2\alpha+1}n+\frac{7\cdot p^{2\alpha+2}-1}{3}\right)q^n \equiv 2f_{4p}f_p^3 \pmod 4. \tag{3.19}$$

Extracting the terms involving q^{pn} from (3.19) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty}\overline{\mathscr{EO}}\left(8\cdot p^{2(\alpha+1)}n+\frac{7\cdot p^{2(\alpha+1)}-1}{3}\right)q^n \equiv 2f_4f_1^3 \pmod 4, \tag{3.20}$$

which is the $\alpha+1$ case of (3.12). Thus, by the principle of mathematical induction, we arrive at (3.12). Extracting the coefficients of terms involving q^{pn+j} for $1\leq j\leq p-1$, from both sides of (3.19), we complete the proof of (3.13). □

Theorem 3.3. *Let $p \geq 5$ be a prime with $\left(\frac{-3}{p}\right) = -1$ and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha} n + \frac{13 \cdot p^{2\alpha} - 1}{3}\right) q^n \equiv 2 f_1 f_4^3 \pmod 4, \tag{3.21}$$

$$\overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha+1}(pn + j) + \frac{13 \cdot p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod 4. \tag{3.22}$$

Proof. Extracting the terms involving q^{2n} from (3.4) and using (2.11), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(4n) q^n \equiv \frac{f_4}{f_1^2} \pmod 4. \tag{3.23}$$

Employing (2.2) in (3.23) and extracting the terms involving q^{2n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(8n+4) q^n \equiv 2 \frac{f_2^3 f_8^2}{f_1^5 f_4} \pmod 4. \tag{3.24}$$

Using (2.10) in (3.24), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(8n+4) q^n \equiv 2 f_1 f_4^3 \pmod 4. \tag{3.25}$$

Congruence (3.25) is the $\alpha = 0$ case of (3.21). Suppose that congruence (3.21) is true for all $\alpha \geq 0$. Utilizing (2.4) and (2.5) in (3.21), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha} n + \frac{13 \cdot p^{2\alpha} - 1}{3}\right) q^n \equiv \\
2\Bigg\{ \sum_{\substack{k=-(p-1)/2 \\ k \neq (\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) \\
+ (-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2} \Bigg\} \\
\times \Bigg\{ \sum_{\substack{m=0 \\ m \neq (p-1)/6}}^{(p-1)} (-1)^m q^{2m(m+1)} \sum_{n=0}^{\infty} (-1)^n (2pn + 2m + 1) q^{2pn \cdot (pn+2m+1)} \\
+ p(-1)^{(p-1)/2} q^{(p^2-1)/2} f_{p^2}^3 \Bigg\} \pmod 4.
\end{aligned} \tag{3.26}$$

Consider the congruence

$$\frac{(3k^2+k)}{2} + 2m(m+1) \equiv \frac{13(p^2-1)}{24} \pmod p,$$

which is equal to

$$(6k+1)^2 + 3(4m+2)^2 \equiv 0 \pmod{p}.$$

For $\left(\frac{-3}{p}\right) = -1$, the above congruence has only solution $m = \frac{p-1}{2}$ and $k = \left(\frac{\pm p - 1}{6}\right)$. Therefore, extracting the terms involving $q^{pn+13(p^2-1)/24}$ from both sides of (3.26), dividing throughout by $q^{13(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha+1} n + \frac{13 \cdot p^{2\alpha+2} - 1}{3}\right) q^n \equiv 2 f_p f_{4p}^3 \pmod{4}. \tag{3.27}$$

Extracting the terms involving q^{pn} from (3.27) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2(\alpha+1)} n + \frac{13 \cdot p^{2(\alpha+1)} - 1}{3}\right) q^n \equiv 2 f_1 f_4^3 \pmod{4}, \tag{3.28}$$

which is the $\alpha + 1$ case of (3.21). Thus, by the principle of mathematical induction, we arrive at (3.21). Extracting the coefficients of terms involving q^{pn+j} for $1 \le j \le p-1$, from both sides of (3.27), we complete the proof of (3.22). □

Theorem 3.4. *Let $p \ge 5$ be a prime and $1 \le j \le p-1$. Then for all integers $\alpha, n \ge 0$, we have*

$$\overline{\mathscr{E}\mathscr{O}}(8n+k) \equiv 0 \pmod{2}, \quad \text{for} \quad 1 \le k \le 7. \tag{3.29}$$

$$\sum_{n \ge 0} \overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha} n + \frac{p^{2\alpha} - 1}{3}\right) q^n \equiv f_1 \pmod{2}, \tag{3.30}$$

$$\overline{\mathscr{E}\mathscr{O}}\left(8 \cdot p^{2\alpha+1}(pn+j) + \frac{p^{2\alpha+2} - 1}{3}\right) \equiv 0 \pmod{2}. \tag{3.31}$$

Proof. From (1.5), we note that

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(n) q^n = \frac{f_4^3}{f_2^2}. \tag{3.32}$$

Employing (2.10) in (3.32), we have

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(n) q^n \equiv f_8 \pmod{2}. \tag{3.33}$$

Extracting the terms involving q^{8n+k}, we obtain the proof of (3.29). Again extracting the terms involving q^{8n} and then replacing q^8 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(8n) q^n \equiv f_1 \pmod{2}. \tag{3.34}$$

Congruence (3.34) is the $\alpha = 0$ case of (3.30). Suppose that congruence (3.30) is true for all integer $\alpha \geq 0$. Employing (2.5) in (3.30), we obtain

$$\sum_{n\geq 0} \overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2\alpha}n + \frac{p^{2\alpha}-1}{3}\right) q^n \equiv$$

$$\left\{ \sum_{\substack{k=-(p-1)/2 \\ k\neq(\pm p-1)/6}}^{(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right)\right.$$

$$\left. +(-1)^{(\pm p-1)/6} q^{(p^2-1)/24} f_{p^2} \right\} \pmod{2}. \tag{3.35}$$

Extracting the term involving $q^{pn+(p^2-1)/24}$ from both sides of (3.35), dividing throughout by $q^{(p^2-1)/24}$ and then replacing q^p by q, we obtain

$$\sum_{n\geq 0} \overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2\alpha+1}n + \frac{p^{2\alpha+2}-1}{3}\right) q^n \equiv f_p \pmod{2}. \tag{3.36}$$

Extracting the terms involving q^{pn} from (3.36) and replacing q^p by q, we obtain

$$\sum_{n\geq 0} \overline{\mathscr{E}\mathscr{O}}\left(8\cdot p^{2(\alpha+1)}n + \frac{p^{2(\alpha+1)}-1}{3}\right) q^n \equiv f_1 \pmod{2}, \tag{3.37}$$

which is the $\alpha + 1$ case of (3.30). Thus, by the principle of mathematical induction, we arrive at (3.30). Extracting the coefficients of terms involving q^{pn+j} for $1 \leq j \leq p-1$, from both sides of (3.36), we complete the proof of (3.31). □

Theorem 3.5. *For any non-negative integers n and k, where $\ell = k(k+1)$ and $k \equiv 2 \pmod{5}$, we have*

$$\overline{\mathscr{E}\mathscr{O}}(10n + 6 + 2\ell) \equiv 0 \pmod{5}. \tag{3.38}$$

Proof. Extracting q^{2n} from (3.3) and then employing (2.12), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(2n) q^n \equiv \frac{f_2^3 f_1^3}{f_5} \pmod{5}. \tag{3.39}$$

Employing (2.1) and (2.8) in (3.39), we obtain

$$\sum_{n=0}^{\infty} \overline{\mathscr{E}\mathscr{O}}(2n) q^n \equiv \frac{q^3\eta^3 f_{25}^3}{f_5} \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{k(k+1)} \pmod{5}. \tag{3.40}$$

Extracting the terms involving the powers of $q^{5n+\ell+3}$ from both sides of (3.40) and then using (2.9), we prove (3.38). □

4. Congruences for $\mathscr{E}\mathscr{O}_e(n)$

Theorem 4.1. *Let $p \geq 5$ be a prime and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e\left(4 \cdot p^{2\alpha} n + \frac{p^{2\alpha}-1}{6}\right) q^n \equiv f_1 \pmod{2}, \tag{4.1}$$

$$\mathscr{E}\mathscr{O}_e\left(4 \cdot p^{2\alpha+1}(pn+j) + \frac{p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod{2}. \tag{4.2}$$

Proof. From (1.6), we note that

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(n) q^n = \frac{f_4^2}{f_2^2}. \tag{4.3}$$

Employing (2.10) in (4.3), we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(n) q^n \equiv f_4 \pmod{2}. \tag{4.4}$$

Extracting the terms involving q^{4n} from (4.4) and replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(4n) q^n \equiv f_1 \pmod{2}. \tag{4.5}$$

The remaining part of the proof is similar to proofs of the identities (3.30) and (3.31). □

Theorem 4.2. *Let $p \geq 5$ be a prime with $\left(\frac{-3}{p}\right) = -1$ and $1 \leq j \leq (p-1)$. Then for all integers $n \geq 0$ and $\alpha \geq 0$, we have*

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e\left(4 \cdot p^{2\alpha} n + \frac{13 \cdot p^{2\alpha}-1}{6}\right) q^n \equiv 2 f_4^3 f_1 \pmod{4}, \tag{4.6}$$

$$\mathscr{E}\mathscr{O}_e\left(4 \cdot p^{2\alpha+1}(pn+j) + \frac{13 \cdot p^{2\alpha+2}-1}{6}\right) \equiv 0 \pmod{4}. \tag{4.7}$$

Proof. From (1.6), we note that

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(n) q^n = \frac{f_4^2}{f_2^2}. \tag{4.8}$$

Extracting the terms involving q^{2n} from (4.8) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(2n) q^n = \frac{f_2^2}{f_1^2}. \tag{4.9}$$

Multiplying the numerator and denominator by f_1^2, we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(2n)q^n = \frac{f_2^2 f_1^2}{f_1^4}. \tag{4.10}$$

Using (2.11) in (4.10), we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(2n)q^n \equiv f_1^2 \pmod 4. \tag{4.11}$$

Using (2.3) in (4.11), we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(2n)q^n \equiv \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8} \pmod 4. \tag{4.12}$$

Extracting the terms involving q^{2n+1} from (4.12) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(4n+2)q^n \equiv 2\frac{f_1 f_8^2}{f_4} \pmod 4. \tag{4.13}$$

Using (2.11) in (4.13), we obtain

$$\sum_{n=0}^{\infty} \mathscr{E}\mathscr{O}_e(4n+2)q^n \equiv 2f_1 f_4^3 \pmod 4. \tag{4.14}$$

The remaining part of the proof is similar to proofs of the identities (3.21) and (3.22). □

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