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[**Review Aritcle**]

SPECTRAL PROPERTIES OF THE NEUMANN-POINCARÉ OPERATOR AND CLOAKING BY ANOMALOUS LOCALIZED RESONANCE: A REVIEW

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ABSTRACT. This is a review paper on recent development on the spectral theory of the Neumann-Poincaré operator. The topics to be covered are convergence rate of eigenvalues of the Neumann-Poincaré operator and surface localization of the single layer potentials of its eigenfunctions. Study on these topics is motivated by their relations with the cloaking by anomalous localized resonance. We review on this topic as well.

1. INTRODUCTION AND PRELIMINARIES

This paper is a review on recent development on the spectral theory of the Neumann-Poincaré (abbreviated to NP) operator. In this paper we restrict ourselves to the following topics and refer to recent review papers [1, 2] for other topics:

(i) Decay rate of eigenvalues of the NP operator,
(ii) Surface localization of the surface plasmons, which are the single layer potentials of the eigenfunctions of the NP operator,
(iii) Applications to the cloaking by anomalous localized resonance.

Let us begin by introducing the NP operator. It is an integral operator defined on the boundary of a bounded domain. Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) whose boundary $\partial\Omega$

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may have several but finite connected components and is assumed to be $C^{1,\alpha}$ for some $\alpha > 0$. The NP operator $\mathcal{K}_{\partial\Omega}$ acting on a function φ on $\partial\Omega$ is defined by

$$\mathcal{K}_{\partial\Omega}[\varphi](x) = \int_{\partial\Omega} \partial_{n_y}\Gamma(x-y)\varphi(y)\, d\sigma(y)\ , \quad x \in \partial\Omega,$$

where $\Gamma(x)$ is the fundamental solution to the Laplacian, *i.e.*,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|\ , & d = 2\ , \\ -\dfrac{1}{4\pi}|x|^{-1}\ , & d = 3\ . \end{cases}$$

Here and afterwards, ∂_n denotes the outward normal derivative on $\partial\Omega$ and ∂_{n_y} does that with respect to y variables. The L^2-adjoint $\mathcal{K}^*_{\partial\Omega}$ of $\mathcal{K}_{\partial\Omega}$ is also called the NP operator. This does not cause any confusion since they have the same spectral structures. It is helpful to write down $\mathcal{K}^*_{\partial\Omega}$ explicitly:

$$\mathcal{K}^*_{\partial\Omega}[\varphi](x) = \frac{1}{\omega_d}\int_{\partial\Omega} \frac{\langle x-y, n_x\rangle}{|x-y|^d}\varphi(y)\, d\sigma(y)\ , \quad x \in \partial\Omega,$$

where $\omega_2 = 2\pi$ and $\omega_3 = 4\pi$.

Let $\mathcal{S}_{\partial\Omega}$ be the single layer potential, namely,

$$\mathcal{S}_{\partial\Omega}[\varphi](x) := \int_{\partial\Omega} \Gamma(x-y)\varphi(y)\, d\sigma(y)\ , \quad x \in \mathbb{R}^d.$$

Since $\mathcal{S}_{\partial\Omega}$ maps $H^{-1/2}(\partial\Omega)$ into $H^{1/2}(\partial\Omega)$ ($H^s(\partial\Omega)$ is the usual Sobolev space on $\partial\Omega$), $\langle\ ,\ \rangle_*$, defined by

$$\langle \varphi, \psi\rangle_* := -\langle \varphi, \mathcal{S}_{\partial\Omega}[\psi]\rangle,$$

where the righthand side is the $H^{-1/2} - H^{1/2}$ duality pairing, becomes a bilinear form on $H^{-1/2}(\partial\Omega)$. In fact, $\langle\ ,\ \rangle_*$ is an inner product on $H_0^{-1/2}(\partial\Omega)$ (functions with the mean zero) in two dimensions, and on $H^{-1/2}(\partial\Omega)$ in three dimensions. Moreover, the NP operator $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint on $H^{-1/2}(\partial\Omega)$ with respect to this inner product [3]. We refer to the survey paper [1] and references therein for this fact.

If $\partial\Omega$ is $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$, then $\mathcal{K}^*_{\partial\Omega}$ is a compact operator on $H^{-1/2}(\partial\Omega)$ (see [4] for a proof of this fact). Since $\mathcal{K}^*_{\partial\Omega}$ is self-adjoint, the spectrum of $\mathcal{K}^*_{\partial\Omega}$ on $H^{-1/2}(\partial\Omega)$ consists of eigenvalues of finite multiplicities (except 0 which can be of infinite multiplicity if it is an eigenvalue) accumulating to 0 and 0 which is either an eigenvalue or a continuous spectrum. We mention that the NP eigenvalues (the eigenvalues of the NP operator) lie in the interval $(-\frac{1}{2}, \frac{1}{2}]$.

Let $\lambda_1, \lambda_2, \ldots$ ($1/2 > |\lambda_1| \ge |\lambda_2| \ge \ldots$) be eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ counting multiplicities, and $\psi_1, \psi_2, \ldots$ are the corresponding (normalized) eigenfunctions. Let ψ_0 be eigenfunction corresponding to the eigenvalue $1/2$. The following addition formula is proved in [5]: if $\partial\Omega$ is

$\mathcal{C}^{1,\alpha}$, then for $x \in \overline{\Omega}$ and $z \in \mathbb{R}^d \setminus \overline{\Omega}$

$$\Gamma(x-z) = -\sum_{j=1}^{\infty} \mathcal{S}_{\partial\Omega}[\psi_j](z)\mathcal{S}_{\partial\Omega}[\psi_j](x) + \mathcal{S}_{\partial\Omega}[\psi_0](z). \tag{1.1}$$

If Ω is a ball, then this is the expansion of $\Gamma(x-z)$ in terms of the spherical harmonics; in terms of ellipsoidal harmonics if Ω is an ellipsoid (see [2]). The formula (1.1) shows in particular that if $z \in \mathbb{R}^d \setminus \partial\Omega$, then $\mathcal{S}_{\partial\Omega}[\psi_j](z) \to 0$ as $j \to \infty$. Thus $\mathcal{S}_{\partial\Omega}[\psi_j]$ is localized at the surface $\partial\Omega$. We are interested in how fast λ_j and $\mathcal{S}_{\partial\Omega}[\psi_j](z)$ tend to 0 as $j \to \infty$, which corresponds to the first two topics mentioned at the beginning of Introduction. As the third topic indicates, these questions are related to the cloaking by anomalous localized resonance (abbreviated to CALR).

To discuss CALR, we consider the following problem:

$$\begin{cases} \nabla \cdot \epsilon_\delta \nabla u = f & \text{in } \mathbb{R}^d, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.2}$$

Here ϵ_δ represents the distribution of the dielectric constants in the presence of the inclusion Ω. We assume that it is 1 after normalization on the background $\mathbb{R}^d \setminus \overline{\Omega}$ and $-c + i\delta$ on the inclusion Ω for some constant $c > 0$. The dielectric constant $-c$ indicates that Ω is a meta material and δ is the lossy parameter tending to 0. We choose c so that the number

$$\frac{1-c+i\delta}{1+c-i\delta}$$

approaches to the accumulation point of the eigenvalues, namely, 0. So, we take $c = 1$. Then, the distribution of dielectric constants is given by

$$\epsilon_\delta = (-1+i\delta)\chi_\Omega + \chi_{\mathbb{R}^d\setminus\overline{\Omega}},$$

where χ_A denotes the indicator function of the set A. The source function f is compactly supported in $\mathbb{R}^d \setminus \overline{\Omega}$ and satisfies the condition $\int_{\mathbb{R}^d} f dx = 0$ to guarantee existence of the solution to (1.2). Significant examples of such source functions are polarizable dipoles, that is, $f(x) = a \cdot \nabla\delta_z(x)$, where a is a constant vector, δ_z is the Dirac delta function at z located outside Ω.

Let u_δ be the solution to the problem (1.2) and let

$$E_\delta := \Im \int_{\mathbb{R}^d} \epsilon_\delta |\nabla u_\delta|^2 \, dx = \delta \int_\Omega |\nabla u_\delta|^2 \, dx \tag{1.3}$$

($\Im$ for the imaginary part). The problem of CALR is to find domains Ω and the sources f such that

$$E_\delta \to \infty \quad \text{as } \delta \to 0, \tag{1.4}$$

and u_δ is bounded outside some radius r. As explained in [6], the quantity E_δ represents the time averaged electromagnetic power produced by the source f dissipated into heat. So, (1.4) implies an infinite amount of energy dissipated per unit time in the limit $\delta \to 0$ which is unphysical. If we scale the source f by a factor of $1/\sqrt{E_\delta}$, then $u_\delta/\sqrt{E_\delta}$ approaches zero outside the radius r and the normalized source is invisible from the outside: CALR occurs.

We refer to the review paper [7] for physics related to CALR and for a comprehensive list of relevant references.

We now discuss the NP spectral nature of CALR, particularly, its connection to the problems (i) and (ii) stated at the beginning of this section. The problem (1.2) can be rephrased as

$$\begin{cases} \Delta u = f & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ \Delta u = 0 & \text{in } \Omega, \\ u|_- = u|_+ & \text{on } \partial\Omega, \\ (-1+i\delta)\partial_n u|_- = \partial_n u_\delta|_+ & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } |x| \to \infty. \end{cases} \tag{1.5}$$

Here and afterwards, subscripts $+$ and $-$ indicate the traces on $\partial\Omega$ from outside and inside of Ω, respectively.

Let F be the Newtonian potential of f, *i.e.*,

$$F(x) = \int_{\mathbb{R}^d} \Gamma(x-y) f(y) dy, \quad x \in \mathbb{R}^d,$$

and we seek the solution u_δ to (1.2) in the form of

$$u_\delta(x) = F(x) + \mathcal{S}_{\partial\Omega}[\varphi_\delta](x), \quad x \in \mathbb{R}^d \tag{1.6}$$

for some $\varphi_\delta \in H_0^{-1/2}(\partial\Omega)$. The third condition (continuity of the potential) in (1.5) is automatically fulfilled since $\mathcal{S}_{\partial\Omega}[\varphi_\delta]$ is continuous across $\partial\Omega$. The fourth condition (continuity of the flux) takes the following form:

$$(-1+i\delta)\partial_n \mathcal{S}_{\partial\Omega}[\varphi_\delta]|_- - \partial_n \mathcal{S}_{\partial\Omega}[\varphi_\delta]|_+ = (2-i\delta)\partial_n F.$$

Thanks to the well-known jump relation (see, for example, [8])

$$\partial_n \mathcal{S}_{\partial\Omega}[\varphi]\big|_\pm(x) = \Big(\pm\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\Big)[\varphi](x), \quad x \in \partial\Omega, \tag{1.7}$$

we see that φ_δ satisfies the integral equation

$$(\mu_\delta I - \mathcal{K}^*_{\partial\Omega})[\varphi_\delta] = \partial_n F \quad \text{on } \partial\Omega, \tag{1.8}$$

where

$$\mu_\delta := \frac{i\delta}{2(-2+i\delta)}. \tag{1.9}$$

Let ψ_j be the normalized eigenfunction corresponding to the eigenvalue λ_j as before. Then one can easily see from (1.8) that φ_δ admits the spectral decomposition

$$\varphi_\delta = \sum_{j=1}^\infty \frac{\langle \partial_n F, \psi_j\rangle_*}{\mu_\delta - \lambda_j}\psi_j.$$

Thanks to the condition $\int_{\mathbb{R}^d} f dx = 0$, we have $\int_\Omega |\nabla F|^2\, dx < \infty$. It thus follows from (1.6) that

$$\int_\Omega |\nabla u_\delta|^2\, dx \approx \int_\Omega |\nabla \mathcal{S}_{\partial\Omega}[\varphi_\delta]|^2\, dx.$$

Here and afterwards, $A \approx B$ means that there are positive constants c, C such that $cA \le B \le CA$. On the other hand, we have

$$\int_\Omega |\nabla \mathcal{S}_{\partial\Omega}[\varphi_\delta]|^2 dx = \int_{\partial\Omega} \left(-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)[\varphi_\delta]\mathcal{S}_{\partial\Omega}[\varphi_\delta] d\sigma \approx \sum_{j=1}^\infty \frac{|\langle \partial_n F, \psi_j \rangle_*|^2}{|\mu_\delta - \lambda_j|^2},$$

where the first identity follows from (1.7). Because of (1.9), $|\mu_\delta - \lambda_j|^2 \approx \delta^2 + \lambda_j^2$. It then follows from (1.3) that

$$E_\delta \approx \delta \sum_{j=1}^\infty \frac{|\langle \partial_n F, \psi_j \rangle_*|^2}{\delta^2 + \lambda_j^2}.$$

In particular, if $f = a \cdot \nabla \delta_z$ for some $z \in \mathbb{R}^d \setminus \overline{\Omega}$, then it is proved in [5] that

$$|\langle \partial_n F, \psi_n \rangle_*| \approx |a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_n](z)|.$$

In fact, if $f = a \cdot \nabla \delta_z$, then $F(x) = a \cdot \nabla_x \Gamma(x - z)$, and hence

$$\langle \partial_n F, \psi_j \rangle_* = -a \cdot \nabla \int_{\partial\Omega} \partial_{n_x} \Gamma(x - z) \mathcal{S}_{\partial\Omega}[\psi_j](x)\, d\sigma(x).$$

It can be seen from (1.1) and (1.7) that

$$\partial_{n_x}\Gamma(x - z) = -\sum_{j=1}^\infty \mathcal{S}_{\partial\Omega}[\psi_j](z)\partial_n \mathcal{S}_{\partial\Omega}[\psi_j](x) = \sum_{j=1}^\infty \left(\frac{1}{2} - \lambda_j\right)\mathcal{S}_{\partial\Omega}[\psi_j](z)\psi_j(x).$$

It then follows that

$$\langle \partial_n F, \psi_j \rangle_* = \left(\frac{1}{2} - \lambda_j\right) a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z).$$

As a consequence, we have

$$E_\delta \approx \delta \sum_{j=1}^\infty \frac{|a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)|^2}{\delta^2 + \lambda_j^2}. \tag{1.10}$$

Here and throughout this paper, we write $A \lesssim B$ to imply that there is a constant C independent of the parameter (in this case it is δ). The meaning of $A \gtrsim B$ is analogous, and $A \approx B$ means that both $A \lesssim B$ and $A \gtrsim B$ hold.

The estimate (1.10) clearly shows how CALR is related to the properties mentioned in (i) and (ii), namely, decay rates of λ_j and $a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)$ as $j \to \infty$: To achieve the property $E_\delta \to \infty$ as $\delta \to 0$, the faster decay of λ_j and the slower decay of $a \cdot \nabla \mathcal{S}_{\partial\Omega}[\psi_j](z)$ are desired.

It is the purpose of this paper to review recent results on decay estimates of NP eigenvalues in two and three dimensions. We then discuss known results on CALR. We also review recent results on the spectral structure of the NP operator in elasticity in relation to CALR.

2. NP SPECTRUM AND CALR OF THE LAPLACE OPERATOR

2.1. **Decay rate of NP eigenvalues.** In this section, we review progresses on the problem (i) mentioned at the beginning of Introduction, namely, the decay rate of the NP eigenvalues on domains with smooth boundaries.

If a bounded domain in $\mathbb{R}^d$ has the $C^{1,\alpha}$ boundary, then the corresponding NP operator is compact and has eigenvalues accumulating to 0. If we denote the eigenvalues by λ_j which are enumerated in descending order counting multiplicities, namely,

$$|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \cdots (\to 0), \tag{2.1}$$

then the question is how fast they tend to 0. It turns out that the decay rates in two and three dimensions are completely different, which is due to different regularities of the integral kernels of the NP operator. If the boundary of the domain is $C^{k,\alpha}$ ($k \geq 1,\ 0 < \alpha \leq 1$), the NP kernel gains regularity indefinitely as k or α increases in two dimensions. However, it does not in three dimensions if $k \geq 2$. In fact, the decay rate of NP eigenvalues on a two-dimensional domain with the $C^{k,\alpha}$ boundary is $o(j^{-k+1-\alpha+0})$ (see Theorem 2.3 for a precise statement) while that in three dimensions is $o(j^{-\alpha/2+0})$ if $k = 1$ (see Theorem 2.2) and $j^{-1/2}$ if $k \geq 2$ (see Theorem 2.1). Here $o(j^{-\alpha/2+0})$ means $o(j^{-\alpha/2+\delta})$ for any $\delta > 0$. NP eigenvalues on a two-dimensional domain with the analytic boundary decays exponentially fast [9]. NP operators on domains with corners have continuous spectrum as shown in [10, 11, 12, 13, 14, 15]. It is worth mentioning that for the curvilinear polygonal domains in two dimensions, there is only absolutely continuous spectrum [13, 16]. We are not aware of a domain with nonempty singularly continuous spectrum.

Figures 1 and 2 schematically depict the decay rates in two- and three-dimensions.

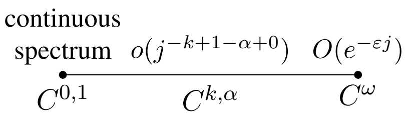

FIGURE 1. Smoothness of boundary and decay rate in 2D

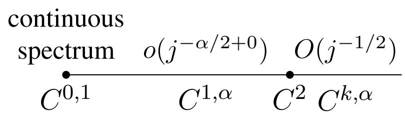

FIGURE 2. Smoothness of boundary and decay rate in 3D

We begin the review on the decay rates with the Weyl-type asymptotics of the NP eigenvalues in three dimensions which is proved by Miyanishi.

Theorem 2.1 ([17]). *Let $\Omega \subset \mathbb{R}^3$ be a $C^{2,\alpha}$ bounded domain for some $\alpha > 0$. Then*

$$|\lambda_j| \sim \Big(\frac{3W(\partial\Omega) - 2\pi\chi(\partial\Omega)}{128\pi} \Big)^{1/2} j^{-1/2} \quad \text{as } j \to \infty, \tag{2.2}$$

where $W(\partial\Omega)$ and $\chi(\partial\Omega)$ are the Willmore energy and the Euler characteristic of $\partial\Omega$, respectively.

Here, $|\lambda_j| \sim Cj^{-1/2}$ means that $|\lambda_j|j^{1/2} \to C$ as $j \to \infty$. The Willmore energy on $\partial\Omega$ is defined to be

$$W(\partial\Omega) := \int_{\partial\Omega} H^2(x)\, d\sigma$$

where $H(x)$ is the mean curvature of the surface $\partial\Omega$. The Weyl-type asymptotics of the positive and negative NP eigenvalues on the three-dimensional domains with the C^∞ boundary are obtained in [18]. As a consequence, it is proved that if the Gaussian curvature is negative at a point on the boundary, then there are infinitely many negative NP-eigenvalues.

A natural question arises: what is the decay rate on $C^{1,\alpha}$ domains in three dimensions? Filling up the gap between $C^{0,1}$ and $C^{2,0+}$, the following result is obtained in [4].

Theorem 2.2 ([4])**.** *If* $\Omega \subset \mathbb{R}^3$ *is a bounded* $C^{1,\alpha}$ *domain for some* $0 < \alpha < 1$*, then it holds that*

$$|\lambda_j| = o(j^{-\alpha/2+\delta}) \quad (j \to \infty) \tag{2.3}$$

for any $\delta > 0$*.*

The following result for the two-dimensional case is obtained in the same paper.

Theorem 2.3 ([4])**.** *If* $\Omega \subset \mathbb{R}^2$ *is a bounded* $C^{k,\alpha}$ *domain for some integer* $k \geq 1$ *and* $0 < \alpha < 1$*, then it holds that*

$$|\lambda_j| = o(j^{-k+1-\alpha+\delta}) \quad (j \to \infty) \tag{2.4}$$

for any $\delta > 0$*.*

The decay rate estimate (2.4) is an improvement over the result of [19] where it is proved that $|\lambda_j| = O(j^{-k-\alpha+3/2})$ for a $C^{k,\alpha}$ bounded domain $\Omega \subset \mathbb{R}^2$ (this estimate when $\alpha = 0$ is obtained in [20]).

As mentioned before, if $\partial\Omega$ is merely $C^{0,1}$, then the NP operator admits essential spectrum and may not have infinitely many eigenvalues. Thus in this case the expected decay rate is j^0. On the other hand, if $\Omega \subset \mathbb{R}^3$ has the $C^{2,\alpha}$ boundary, then the decay rate is $j^{-1/2}$ and this rate is optimal according to (2.2). The estimate (2.3) interpolates naturally between $C^{0,1}$ and $C^{2,\alpha}$ cases. Thus we expect that (2.3) is optimal in some sense even if we don't have a proof. We also expect (2.4) for two dimensions to be optimal since it interpolates naturally between $C^{0,1}$ and C^ω cases where eigenvalues decays to 0 exponentially fast.

The difference of the decay rates between two and three dimensions as appearing in (2.2), (2.3) and (2.4) (depicted in Figures 1 and 2) is due to the difference of the regularity properties of the integral kernels of the NP operators. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain whose boundary is $C^{k,\alpha}$-smooth. If $k = 1$ and $\alpha > 0$, then the integral kernel, denoted by $K(x, y)$, of the NP operator satisfies

$$|K(x, y)| \lesssim |x - y|^{-2+\alpha}, \quad x, y \in \partial\Omega,$$

and gains regularity as α increases and we have faster decay as shown in (2.3). However, if $k \geq 2$, then it does not gain regularity even if k or α increases and, as a result, the decay rate is fixed at $j^{-1/2}$ as shown in (2.2). In two dimensions, $K(x, y)$ gains regularity indefinitely as k or α increase as shown in [4]. For example, if $\partial\Omega$ is C^∞, then $K(x, y)$ is C^∞ on $\partial\Omega \times \partial\Omega$; if

$\partial\Omega$ is C^ω, so is $K(x, y)$. Delgado and Ruzhansky prove a theorem in [21] which characterizes the Schatten class of an integral operator on the L^2-space on a compact manifold in terms of the Sobolev regularity of its integral kernel. Theorems 2.2 and 2.3 are proved using this theorem.

We now review the result of [9] for two-dimensional domains with real analytic boundaries. Let Ω be a bounded domain in $\mathbb{R}^2$ such that $\partial\Omega$ is real analytic. Let S^1 be the unit circle and $Q : S^1 \to \partial\Omega \subset \mathbb{C}$ be a regular parametrization of $\partial\Omega$. Then Q admits an extension as an analytic mapping from an annulus $A_\epsilon := \{\tau \in \mathbb{C} : e^{-\epsilon} < |\tau| < e^\epsilon\}$ for some $\epsilon > 0$ onto a tubular neighborhood of $\partial\Omega$ in $\mathbb{C}$. Let

$$q(t) := Q(e^{it}), \quad t \in \mathbb{R} \times i(-\epsilon, \epsilon).$$

Then q is an analytic function from $\mathbb{R} \times i(-\epsilon, \epsilon)$ onto a tubular neighborhood of $\partial\Omega$. It satisfies $q(t + 2\pi) = q(t)$ for any $t \in \mathbb{R}$. Let ϵ_q be the supremum of ϵ with the following property:

If $q(t) = q(s)$ for $t \in [-\pi, \pi] \times i(-\epsilon, \epsilon)$ and $s \in [-\pi, \pi)$, then $t = s$.

The number ϵ_q is called the maximal Grauert radius for q. Let

$$\epsilon_{\partial\Omega} := \sup_q \epsilon_q,$$

where the supremum is taken over all regular real analytic parametrization q of $\partial\Omega$. The number $\epsilon_{\partial\Omega}$ is called the modified maximal Grauert radius of $\partial\Omega$.

The following theorem holds.

Theorem 2.4 ([9]). *Let Ω be a bounded planar domain with the analytic boundary $\partial\Omega$ and $\epsilon_{\partial\Omega}$ be the modified maximal Grauert radius of $\partial\Omega$. Let $\{\lambda_j\}_{j=0}^\infty$ be the eigenvalues of the NP operator enumerated as in* (2.1). *Then for any $\epsilon < \epsilon_{\partial\Omega}$, there exists a constant C with the property:*

$$|\lambda_{2j-1}| = |\lambda_{2j}| \le Ce^{-\epsilon j}$$

for all j.

The proof relies on the Weyl-Courant min-max principle and a Paley-Wiener type lemma.

2.2. **Asymptotics of NP eigenvalues with axially symmetric eigenfunctions.** In this subsection, we review results in [22] on NP eigenvalues with axially symmetric eigenfunctions on axially symmetric domains in three dimensions. A typical example of such a domain is a torus. More concretely, let $\Sigma \subset \mathbb{R}^2$ be a bounded domain with the Lipschitz boundary such that $\overline{\Sigma} \subset \mathbb{R} \times (0, \infty)$. We then define Ω by

$$\Omega := \{(x, y\cos\eta, y\sin\eta) \mid (x, y) \in \Sigma, \ \ \eta \in [-\pi, \pi)\}. \tag{2.5}$$

For example, if Σ is a disk, then Ω is a solid torus. We say that the function $F : \partial\Omega \to \mathbb{C}$ is axially symmetric if $F(x, y\cos\eta, y\sin\eta)$ is independent of η. In [22], the decay rate and asymptotics of the NP eigenvalues whose eigenfunctions are axially symmetric are investigated.

This study is strongly motivated by the work [23] where it is observed by computational experiments that the single layer potential of axially symmetric eigenfunctions on a torus seem less localized near the boundary of the domain compared to other eigenfunctions. Based on the formula (1.10), this observation leads us to investigation of the decay rate of the corresponding eigenvalues as the first step for the challenging exploration of occurrence (or non-occurrence) of CALR on three-dimensional axially symmetric domains.

We define for $p = (x, y) \in \partial\Sigma$

$$v_p := -n_{p,2} \tag{2.6}$$

where $n_{p,2}$ is the second component of the outward normal vector n_p at p, namely, $n_p = (n_{p,1}, n_{p,2})$. We also introduce the distance

$$\delta(p, p') := \frac{|p - p'|}{2(yy')^{1/2}} \quad (p = (x, y),\ p' = (x', y') \in \partial\Sigma).$$

For a given function f on $\partial\Sigma$, let $E[f]$ be the extension to $\partial\Omega$ defined by

$$E[f](x, y\cos\eta, y\sin\eta) = y^{-1/2} f(x, y).$$

It is proved in [22] that the following relation holds:

$$\mathcal{K}^*_{\partial\Omega}[E[f]](x, y\cos\eta, y\sin\eta) = y^{-1/2}\mathcal{K}^*_0[f](p) \quad (p = (x, y)), \tag{2.7}$$

where $\mathcal{K}^*_0$ is the integral operator of the form

$$\mathcal{K}^*_0[f](p) := \int_{\partial\Sigma} K^*_0(p, p') f(p')\, d\sigma(p')$$

with

$$K^*_0(p, p') = K^*_{\partial\Sigma}(p, p') A_0(\delta(p, p')) - \frac{v_p}{4\pi y} B_0(\delta(p, p')).$$

Here, $K^*_{\partial\Sigma}(p, p')$ is the integral kernel of the NP operator $\mathcal{K}^*_{\partial\Sigma}$ on $\partial\Sigma$ and

$$A_0(\delta) := \delta^2 \int_0^{\pi/2} \frac{1}{(\delta^2 + \sin^2\varphi)^{3/2}}\, d\varphi, \quad B_0(\delta) = \int_0^{\pi/2} \frac{\sin^2\varphi}{(\delta^2 + \sin^2\varphi)^{3/2}}\, d\varphi.$$

The relation (2.7) shows that the subspace of axially symmetric functions is invariant under $\mathcal{K}^*_{\partial\Omega}$. Furthermore, it shows that investigating the spectral property of $\mathcal{K}^*_{\partial\Omega}$ amounts to that of $\mathcal{K}^*_0$ on $H^{-1/2}(\partial\Sigma)$ equipped with the inner product

$$\langle f, g\rangle_0 := \langle E[f], E[g]\rangle_*$$

for $f, g \in H^{-1/2}(\partial\Sigma)$. With this inner product, $\mathcal{K}^*_0$ is self-adjoint.

It is further proved, by investigating the asymptotic behaviour of the integrals A_0, B_0 as $\delta \to 0$, that

$$\mathcal{K}^*_0 = \mathcal{K}^*_{\partial\Sigma} - M\mathcal{S}_{\partial\Sigma} + \mathcal{R}^* \tag{2.8}$$

where M is the multiplication operator by $\frac{v_p}{2y}$ and $\mathcal{S}_{\partial\Sigma}$ is the single layer potential on $\partial\Sigma$. The difference $\mathcal{R}^*$ is a smoothing operator and the singularity of the integral kernel of $\mathcal{K}^*_0$ comes from that of either $\mathcal{K}^*_{\partial\Sigma}$ or $\mathcal{S}_{\partial\Sigma}$.

If $\partial\Sigma$ is $C^{1,\alpha}$, then $|K^*_{\partial\Sigma}(p, p')| \lesssim |p - p'|^{-1+\alpha}$, which is more singular than the singularity $\log|p - p'|$ of $\mathcal{S}_{\partial\Sigma}$. If $\partial\Sigma$ is $C^{k,\alpha}$ ($k \geq 2$), then $K^*_{\partial\Sigma}(p, p')$ is bounded, and hence the singularity of $K^*_0(p, p')$ is $\log|p - p'|$. Because of this difference of singularities of $K^*_0(p, p')$, we obtain different decay estimates depending on the regularity of $\partial\Sigma$, which is natural if we compare them with decay rates in three dimensions as described in Theorem 2.1 and 2.2.

We denote by ρ_j eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ with axially symmetric eigenfunctions, namely, the eigenvalues of the operator $\mathcal{K}^*_0$. If $\partial\Sigma$ is $C^{1,\alpha}$, then $\mathcal{K}^*_0$ is compact and has eigenvalues of finite multiplicities converging to 0. We enumerate all such eigenvalues $\{\rho_j\}_{j=1}^\infty$ in the descending order

$$|\rho_1| \ge |\rho_2| \ge |\rho_3| \ge \cdots \to 0.$$

Theorem 2.5 ([22]). *Let Ω be the axially symmetric domain defined by* (2.5).

(1) *If $\partial\Sigma$ is $C^{1,\alpha}$ for some $\alpha > 0$, then it holds that*

$$|\rho_j| = o(j^{-\alpha+\delta}) \quad (j \to \infty)$$

for all $\delta > 0$.

(2) *If $\partial\Sigma$ is $C^{k,\alpha}$ for some $k \ge 2$ and $\alpha > 0$, then it holds that*

$$|\rho_j| = O(j^{-1} \log j) \quad (j \to \infty).$$

If $\partial\Omega$ is C^∞, then we obtain asymptotics of the positive and negative eigenvalues with axially symmetric eigenfunctions. We enumerate all positive eigenvalues $\rho_j^+ > 0$ and negative eigenvalues $-\rho_j^- < 0$ of $\mathcal{K}^*_0$ in descending orders:

$$\rho_1^\pm \ge \rho_2^\pm \ge \rho_3^\pm \ge \cdots .$$

Theorem 2.6 ([22]). *Let Ω be the axially symmetric domain defined by* (2.5). *If $\partial\Sigma$ is C^∞, then*

$$|\rho_j| \sim C_0 j^{-1}, \quad \rho_j^\pm \sim C_0^\pm j^{-1}$$

where C_0 and $C_0^\pm$ are defined by

$$C_0^\pm = \mp\frac{1}{4\pi} \int_{\{p=(x,y)\in\partial\Sigma \mid \mp v_p > 0\}} \frac{v_p}{y}\, d\sigma(p)$$

and

$$C_0 = C_0^+ + C_0^-.$$

The coefficients $C_0^\pm$ have significant geometric meanings. Suppose that Σ is convex. Because of (2.6), if $v_p > 0$, then n_p is downward, and hence $(x, y\cos\eta, y\sin\eta)$ is a concave point on $\partial\Omega$. So, C_0^- is the integration over the concave part of Σ; C_0^+ over the convex part. The connection between negative NP eigenvalues and concavity of the domain has been proved in [24, 18] that if there exists a point on the boundary where the Gaussian curvature is negative, then the NP operators has infinitely many negative eigenvalues. Since $C_0^\pm > 0$, Theorem 2.6 that there are infinitely many positive and negative NP eigenvalues with axially symmetric eigenfunctions.

Theorem 2.6 requires $\partial\Sigma$ to be C^∞. It is because it's proof uses a result of [25] which relies on calculus of pseudo-differential operators.

If $\partial\Sigma$ is Lipschitz, then we can observe an interesting fact from the formula (2.8). Since $M\mathcal{S}_{\partial\Sigma} - \mathcal{R}^*$ compact, $\mathcal{K}^*_0$ is a compact perturbation of $\mathcal{K}^*_{\partial\Sigma}$. We emphasize that $\mathcal{K}^*_0$ and

$\mathcal{K}^*_{\partial\Sigma}$ are self-adjoint on $H^{-1/2}(\partial\Sigma)$ with different inner products. However, since these inner products induce equivalent norms, one can prove that $\mathcal{K}^*_0$ and $\mathcal{K}^*_{\partial\Sigma}$ have the same essential spectra on $H^{-1/2}(\partial\Sigma)$ using the Weyl criterion. As a consequence, we have the following theorem.

Theorem 2.7 ([22])**.** *If $\partial\Sigma$ is Lipschitz, then the inclusion relation*

$$\sigma_{\text{ess}}(\mathcal{K}^*_{\partial\Sigma}, H^{-1/2}(\partial\Sigma)) \subset \sigma_{\text{ess}}(\mathcal{K}^*_{\partial\Omega}, H^{-1/2}(\partial\Omega))$$

holds.

Theorem 2.7 gives examples of three-dimensional domains with non-trivial essential spectrum. For instance, we apply Theorem 2.7 together with the result of [15] to obtain the following corollary:

Corollary 2.8 ([22])**.** *If $\partial\Sigma$ is a curvilinear polygon with the minimum angle $\alpha \in (0, 2\pi)$, then*

$$\left[-\left| \frac{1}{2} - \frac{\alpha}{2\pi} \right|, \left| \frac{1}{2} - \frac{\alpha}{2\pi} \right| \right] \subset \sigma_{\text{ess}}(\mathcal{K}^*_{\partial\Omega}, H^{-1/2}(\partial\Omega)).$$

2.3. **CALR.** It is in [26] that the phenomena of anomalous resonance is related to the invisibility cloaking and CALR is proved to occur when the source function f in (1.2) is given by $f = a \cdot \nabla\delta_z$ for some $z \in \mathbb{R}^d \setminus \overline{\Omega}$ and Ω is an annulus. It is proved that there is a virtual radius r_* such that if $|z| < r_*$, then (1.4) holds and CALR takes place; if $|z| > r_*$, then E_δ is bounded regardless of δ. In fact, $r_* = \sqrt{r_e^3/r_i}$, where r_e and r_i are outer and inner radii of the annulus, respectively. This result is extended to other sources in [6] by relating CALR with the spectral properties of the NP operator. CALR occurs on confocal ellipses as proved in [27]. Since CALR is a phenomenon occurring at the limit point of NP eigenvalues, the structure does not have to be doubly connected. The only requirement for geometry of the domain is that 0 is not an eigenvalue of the NP operator. In fact, it is proved in [5] that CALR occurs on ellipses, which we review in this section.

In three dimensions no domain is known where CALR occurs. It is known that CALR does not take place on balls [28] and strictly convex domains in three dimensions [23]. We review this result as well.

CALR on ellipses. The elliptic coordinates (ρ, ω) for $x = (x_1, x_2) = (x_1(\rho, \omega), x_2(\rho, \omega))$ is given by

$$x_1(\rho, \omega) = R\cos\omega\cosh\rho, \quad x_2(\rho, \omega) = R\sin\omega\sinh\rho, \quad \rho > 0,\ 0 \le \omega < 2\pi.$$

When this holds, we write $\rho = \rho_x$ and $\omega = \omega_x$. The ellipse $\partial\Omega$ is given by

$$\partial\Omega = \{x \in \mathbb{R}^2 \ :\ \rho_x = \rho_0\} \tag{2.9}$$

for some $\rho_0 > 0$. The number ρ_0 is called the elliptic radius of Ω. If we vary ρ_0, then varied ellipses are confocal.

It is known (see, for example, [29] for a proof) that eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ are

$$\lambda_n = \pm\frac{1}{2e^{2n\rho_0}}, \quad n = 1, 2, \cdots,$$

and corresponding eigenfunctions are

$$\phi^c_n(\omega) := \Xi(\rho_0, \omega)^{-1}\cos n\omega, \quad \phi^s_n(\omega) := \Xi(\rho_0, \omega)^{-1}\sin n\omega, \quad n = 1, 2, \cdots,$$

where

$$\Xi = \Xi(\rho_0, \omega) := R\sqrt{\sinh^2\rho_0 + \sin^2\omega}.$$

Using the explicit forms of eigenvalues and eigenfunctions, it is proved in [5] using (1.10) that if $f(x) = a \cdot \nabla\delta_z(x)$ for some $z \notin \overline{\Omega}$, then

$$E_\delta \approx \begin{cases} \delta^{-2+\rho_z/\rho_0}|\log\delta| & \text{if } \rho_0 < \rho_z < 3\rho_0, \\ \delta|\log\delta|^2 & \text{if } \rho_z = 3\rho_0, \\ \delta & \text{if } \rho_z > 3\rho_0, \end{cases}$$

as $\delta \to 0$. It is also proved that the solution u_δ is bounded outside a bounded set. Thus, if the location z of the source satisfies $\rho_0 < \rho_z \le 2\rho_0$, then $E_\delta \to \infty$, and CALR takes place. If $2\rho_0 < \rho_z$, then $E_\delta \to 0$, and CALR does not take place.

Non-occurrence of CALR on strictly convex domains in 3D. Here we review the result of [23] which proves non-occurrence of CALR on strictly convex three-dimensional domains. Let Ω be a strictly convex domain in $\mathbb{R}^3$ such that $\partial\Omega$ is C^∞-smooth. Then all eigenvalues, except possibly finitely many, of $\mathcal{K}^*_{\partial\Omega}$ are positive (see [18]). We may modify $\mathcal{K}^*_{\partial\Omega}$ on a finite-dimensional subspace so that the modified operator, denoted by $\mathcal{K}^*$, is positive-definite pseudo-differential operator of order -1. Thus, for each real number s there are constants c_s and C_s such that

$$c_s\|\varphi\|_{H^{s-1/2}(\partial\Omega)} \le \|\mathcal{K}^*_{\partial\Omega}[\varphi]\|_{H^{s+1/2}(\partial\Omega)} \le C_s\|\varphi\|_{H^{s-1/2}(\partial\Omega)} \tag{2.10}$$

for all $\varphi \in H^s(\partial\Omega)$. Let $\lambda_1, \lambda_2, \ldots$ ($|\lambda_1| \ge |\lambda_2| \ge \ldots$) be eigenvalues of $\mathcal{K}^*_{\partial\Omega}$ counting multiplicities, and $\psi_1, \psi_2, \ldots$ are the corresponding (normalized) eigenfunctions. We infer from (2.10) that there is j_0 such that

$$c_s\|\psi_j\|_{H^{s-1/2}(\partial\Omega)} \le \|\mathcal{K}^*_{\partial\Omega}[\psi_j]\|_{H^{s+1/2}(\partial\Omega)} \le C_s\|\psi_j\|_{H^{s-1/2}(\partial\Omega)} \tag{2.11}$$

and $\lambda_j > 0$ for all $j \ge j_0$.

Let K be a compact set in $\mathbb{R}^3 \setminus \partial\Omega$. Since $\text{dist}(K, \partial\Omega) > 0$, for any positive integer k and for any real number s, there is a constant $M_{k,s}$ such that

$$\|\mathcal{S}_{\partial\Omega}[\psi_j]\|_{C^k(K)} \le M_{k,s}\|\psi_j\|_{H^{-s-1/2}(\partial\Omega)}.$$

It then follows from (2.11) that

$$\|\mathcal{S}_{\partial\Omega}[\psi_j]\|_{C^k(K)} \le C_{k,s}\|(\mathcal{K}^*_{\partial\Omega})^s[\psi_j]\|_{H^{-1/2}(\partial\Omega)} = C_{k,s}\lambda_j^s$$

for some constant $C_{k,s}$ depending on k and s, but independent of j. In particular, we have

$$\|\nabla\mathcal{S}_{\partial\Omega}[\psi_j]\|_{L^\infty(K)} \lesssim \lambda_j^s. \tag{2.12}$$

The estimate (2.12) shows that $\mathcal{S}_{\partial\Omega}[\psi_j]$ is localized too fast for CALR to occur. In fact, since $\lambda_j > 0$ for all $j \ge j_0$, we infer from (2.2) that there is a positive constant $C_{\partial\Omega}$ such that $\lambda_j \sim C_{\partial\Omega} j^{-1/2}$ as $j \to \infty$. It then follows from (2.12) that

$$\|\nabla \mathcal{S}_{\partial\Omega}[\psi_j]\|_{L^\infty(K)} \lesssim j^{-s}$$

for any $s > 0$. Then, (1.10) yields

$$E_\delta \lesssim \delta \sum_{j=1}^{\infty} \frac{j^{-2s}}{\delta^2 + j^{-1}} \to 0 \ \ (\delta \to 0),$$

which shows that CALR does not take place no matter where z is located.

Discussions. The two-dimensional domains, annuli and ellipses, where CALR is known to take place are those whose NP eigenvalues and corresponding eigenfunctions are known. It would be interesting to investigate CALR on two-dimensional domains with real analytic boundaries. One can attempt to prove that if the location z of the source function $f(x) = a \cdot \nabla \delta_z(x)$ is sufficiently close to Ω, then CALR takes place; if z is far away from Ω, then CALR does not take place. In this case, it is known that eigenvalues decay exponentially fast [9]. But, it is not known how the single layer potentials of eigenfunctions are localized at $\partial\Omega$.

As mentioned before, no three-dimensional domains where CALR take place. It would be interesting to find one, if there is any.

3. NP SPECTRUM AND CALR FOR THE LAMÉ SYSTEM

The NP operator for the Lamé system of the linear elasticity is defined similarly to that for the Laplace operator. However, there is a significant difference between two NP operators: Unlike the one for the Laplace operator, the NP operator for the Lamé system (abbreviated to the eNP operator) is not compact even if the boundary of the domain is smooth. Recently, it is proved that if the domain has a smooth boundary, then the corresponding eNP operator is polynomially compact and spectrum consists of eigenvalues accumulating to some numbers determined by Lamé constants. CALR occurring at different accumulation points are investigated when the domain is an ellipse. It is further proved in a recent preprint that vector fields on the boundary of the domain can be canonically decomposed so that each linear factor of the polynomial of the eNP operator is compact on each subspace, and those subspaces characterize eigenspaces. We review these results in this section.

3.1. **Spectral structure of the eNP operator.** Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded Lipschitz domain. Let (λ, μ) be a Lamé constants for Ω satisfying the strong convexity condition:

$$\mu > 0, \quad d\lambda + 2\mu > 0. \tag{3.1}$$

The corresponding Lamé operator $\mathcal{L}_{\lambda,\mu}$ is defined by

$$\mathcal{L}_{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \nabla(\nabla \cdot u),$$

and the corresponding conormal derivative ∂_ν on $\partial\Omega$ is defined by

$$\partial_\nu u := \lambda(\nabla \cdot u) n + 2\mu (\widehat{\nabla} u) n,$$

where $\widehat{\nabla}$ denotes the symmetric gradient, namely, $\widehat{\nabla}u := \frac{1}{2}(\nabla u + \nabla u^T)$ (T for the transpose). Let $\Gamma = (\Gamma_{ij})_{i,j=1}^d$ be the Kelvin matrix of fundamental solutions to the Lamé operator $\mathcal{L}_{\lambda,\mu}$, namely,

$$\Gamma_{ij}(x) := \begin{cases} \dfrac{\alpha_1}{2\pi}\delta_{ij}\ln|x| - \dfrac{\alpha_2}{2\pi}\dfrac{x_i x_j}{|x|^2}, & \text{if } d = 2, \\ -\dfrac{\alpha_1}{4\pi}\dfrac{\delta_{ij}}{|x|} - \dfrac{\alpha_2}{4\pi}\dfrac{x_i x_j}{|x|^3}, & \text{if } d = 3, \end{cases} \tag{3.2}$$

where

$$\alpha_1 := \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\lambda + 2\mu}\right) \quad \text{and} \quad \alpha_2 := \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda + 2\mu}\right),$$

and δ_{ij} is the Kronecker's delta (see [30]).

The eNP operator, which we denote by $\mathcal{K}_e$ (the subscript e is added to distinguish it from the NP operator for the Laplace operator and $\partial\Omega$ is deleted from the subscript for ease of notation), is defined by

$$\mathcal{K}_e[f](x) := \text{p.v.} \int_{\partial\Omega} (\partial_{\nu_y}\Gamma(x-y))^T f(y)\, d\sigma(y), \quad x \in \partial\Omega.$$

Like the NP operator $\mathcal{K}_{\partial\Omega}$ for the Laplacian, the eNP operator $\mathcal{K}_e$ can be realized as a self-adjoint operator on $H^{1/2}(\partial\Omega)^d$ by introducing a new inner product (see [31]).[1] Unlike $\mathcal{K}_{\partial\Omega}$ which is compact on $H^{1/2}(\partial\Omega)$ if $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$, the eNP operator is not compact on $H^{1/2}(\partial\Omega)^d$ even if $\partial\Omega$ is smooth (see [32]). We will see that the eNP operator is polynomially compact on $H^{1/2}(\partial\Omega)^d$ and each linear factor of the polynomial is compact on a subspace of $H^{1/2}(\partial\Omega)^d$ whose sum is $H^{1/2}(\partial\Omega)^d$. To prove this, a decomposition theorem for surface vector fields is proved in [33]. We review the decomposition theorem in a separate subsection since it is of independent interest.

Decomposition theorem for surface vector fields. Set $\mathcal{H} := H^{1/2}(\partial\Omega)$. For each $f \in \mathcal{H}^d$, let $v_-^f \in H^1(\Omega)^d$ be the unique solution to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Similarly, we define v_+^f for the unique solution to

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^d \setminus \overline{\Omega}, \\ u = f & \text{on } \partial\Omega, \\ u = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{3.3}$$

such that $\nabla v_+^f \in L^2(\mathbb{R}^d \setminus \overline{\Omega})^d$ if $d = 2$ and $v_+^f \in H^1(\mathbb{R}^d \setminus \overline{\Omega})^d$ if $d = 3$. Note that we need to impose the condition $\int_{\partial\Omega} f = 0$ to ensure the existence of the solution to (3.3) if $d = 2$. We

[1] Here we work with $\mathcal{K}_e$ instead of $\mathcal{K}_e^*$ while for the Laplace operator we work with $\mathcal{K}_{\partial\Omega}^*$. We do so because the decomposition theorem is more transparently described if we deal with $H^{1/2}(\partial\Omega)^d$ than with $H^{-1/2}(\partial\Omega)^d$.

define subspaces of $\mathcal{H}^d$ as follows:

$$\mathcal{H}_{\text{div}} := \{f \in \mathcal{H}^d \mid \nabla \cdot v_-^f = 0 \text{ in } \Omega,\ \ \nabla \cdot v_+^f = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega}\},$$
$$\mathcal{H}_{\text{div,rot}}^- := \{f \in \mathcal{H}^d \mid \nabla \cdot v_-^f = 0,\ \nabla \times v_-^f = 0 \text{ in } \Omega\},$$
$$\mathcal{H}_{\text{div,rot}}^+ := \{f \in \mathcal{H}^d \mid \nabla \cdot v_+^f = 0,\ \nabla \times v_+^f = 0 \text{ in } \mathbb{R}^d \setminus \overline{\Omega}\}.$$

If $d = 2$, $\nabla\times$ is replaced by rot[2].

In two dimensions, the decomposition theorem is an immediate consequence of the jump relation of the Cauchy integral. For a complex function f, the Cauchy integral $\mathcal{C}$ is defined by

$$\mathcal{C}[f](z) := \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(w)}{w-z}\,dw, \quad z \in \mathbb{C} \setminus \partial\Omega.$$

Let $\mathcal{C}_b$ be the Cauchy transform on $\partial\Omega$, namely,

$$\mathcal{C}_b[f](z) := \frac{1}{2\pi i}\text{p.v.}\int_{\partial\Omega} \frac{f(w)}{w-z}\,dw, \quad z \in \partial\Omega.$$

The following jump relation is well-known:

$$\mathcal{C}[f]|_\pm = \left(\mathcal{C}_b \mp \frac{1}{2}I\right)[f].$$

See [34, (17.2)] for the case when $\partial\Omega$ smooth, and [35, 36, 37] for the Lipschitz case. Thus we have a decomposition

$$\overline{f} = \overline{\mathcal{C}[f]|_-} - \overline{\mathcal{C}[f]|_+},$$

where $\overline{\mathcal{C}[f]|_-}$ extends to Ω as an anti-holomorphic function and $\overline{\mathcal{C}[f]|_+}$ to $\mathbb{C}\setminus\overline{\Omega}$. By identifying the vector field $f = (f_1, f_2)$ with the complex function $f_1 - if_2$, this decomposition yields the following theorem.

Theorem 3.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with the Lipschitz boundary.*

$$\mathcal{H}^2 = \mathcal{H}_{\text{div,rot}}^- \oplus \mathcal{H}_{\text{div,rot}}^+. \tag{3.4}$$

The sum in (3.4) is actually the orthogonal sum with respect to the inner product on $\mathcal{H}^2$ defined by the (inverse of) single layer potential, namely,

$$\|f\|_*^2 = -\langle f, \mathcal{S}_e^{-1}[f]\rangle$$

where $\mathcal{S}_e$ is the single layer potential defined by the fundamental solution (3.2).

In three dimensions, we have the following theorem.

Theorem 3.2 ([33])**.** *Let Ω be a bounded domain in $\mathbb{R}^3$ with the Lipschitz boundary. The subspaces $\mathcal{H}_{\text{div,rot}}^-$, $\mathcal{H}_{\text{div,rot}}^+$ and $\mathcal{H}_{\text{div}}$ are infinite-dimensional, and it holds that*

$$\mathcal{H}^3 = \mathcal{H}_{\text{div,rot}}^- \oplus \mathcal{H}_{\text{div,rot}}^+ \oplus (\mathcal{H}_{\text{div,rot}}^- + \mathcal{H}_{\text{div,rot}}^+)^\perp,$$

[2]The results reviewed in this section hold to be true even when $d \geq 3$. We describe the results for $d = 2, 3$ for convenience.

where the decomposition is orthogonal, and

$$(\mathcal{H}^-_{\rm div,rot} + \mathcal{H}^+_{\rm div,rot})^\perp \subset \mathcal{H}_{\rm div}.$$

In particular, we have

$$\mathcal{H}^3 = \mathcal{H}^-_{\rm div,rot} \oplus \mathcal{H}^+_{\rm div,rot} + \mathcal{H}_{\rm div}. \tag{3.5}$$

We do not know whether the decomposition (3.5) is orthogonal for a general domain Ω, or equivalently,

$$\mathcal{H}^\pm_{\rm div,rot} \cap \mathcal{H}_{\rm div} = \{0\}.$$

Regarding this question the following theorem is proved.

Theorem 3.3 ([33]). *Let Ω be a bounded domain in $\mathbb{R}^3$. If $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 1/2$, then the following inequality holds:*

$$\dim \mathcal{H}_{\rm div}/(\mathcal{H}^-_{\rm div,rot} + \mathcal{H}^+_{\rm div,rot})^\perp = \dim \mathcal{H}_{\rm div} \cap (\mathcal{H}^-_{\rm div,rot} + \mathcal{H}^+_{\rm div,rot}) \le b_1(\partial\Omega),$$

where $b_1(\partial\Omega)$ denotes the first Betti number of $\partial\Omega$.

In particular, if $\partial\Omega$ is simply connected, then $b_1(\partial\Omega) = 0$ and the following corollary is obtained.

Corollary 3.4 ([33]). *Let Ω be a bounded domain in $\mathbb{R}^3$. If $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 1/2$ and simply connected, then the orthogonal decomposition*

$$\mathcal{H}^3 = \mathcal{H}^-_{\rm div,rot} \oplus \mathcal{H}^+_{\rm div,rot} \oplus \mathcal{H}_{\rm div}$$

holds.

It is helpful to review the proof of Theorem 3.2 briefly. The Lamé operator can be written as

$$\mathcal{L}_{\lambda,\mu} u = -\mu \nabla \times (\nabla \times u) + \frac{\mu}{2k_0} \nabla(\nabla \cdot u),$$

where

$$k_0 := \frac{\mu}{2(\lambda + 2\mu)}. \tag{3.6}$$

The corresponding conormal derivative is given by

$$\partial^D_\nu u(x) := -\mu n_x \times (\nabla \times u)(x) + \frac{\mu}{2k_0} (\nabla \cdot u)(x) n_x$$

for all $x \in \partial\Omega$. We introduce the div-free eNP operator $\mathcal{K}^D_e$ defined by

$$\mathcal{K}^D_e[f](x) := \text{p.v.} \int_{\partial\Omega} (\partial^D_{\nu_y} \Gamma(x-y))^T f(y)\, d\sigma(y).$$

We call $\mathcal{K}^D_e$ div-free eNP operator because the corresponding double layer potential produces divergence-free solutions of the Lamé system. It is worth mentioning that the divergence-free solutions of the Lamé system are solutions for any pair of Lamé parameters. It is then proved in [33] that

$$\mathcal{H}^\pm_{\rm div,rot} = \text{Ker} \left(\mathcal{K}^D_e \pm \frac{1}{2} I \right)$$

and

$$\mathrm{Ran}\left(\mathcal{K}_e^D + \frac{1}{2}I\right)\left(\mathcal{K}_e^D - \frac{1}{2}I\right) \subset \mathcal{H}_{\mathrm{div}},$$

from which Theorem 3.2 follows.

Spectral structure of the eNP operator. As mentioned before, $\mathcal{K}_e$ is not compact even if $\partial\Omega$ is smooth. However, it is proved that eNP operator is polynomially compact. In fact, it is proved in [31] that if $\Omega \subset \mathbb{R}^2$ and $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$, then $\mathcal{K}_e^2 - k_0^2 I$ (k_0 is the number defined in (3.6)) is compact and the spectrum of $\mathcal{K}_e$ on $\mathcal{H}^2$ consists of eigenvalues converging to k_0 and $-k_0$; in [38] that if $\Omega \subset \mathbb{R}^3$ and $\partial\Omega$ is C^∞, then $\mathcal{K}_e(\mathcal{K}_e^2 - k_0^2 I)$ is compact and the spectrum of $\mathcal{K}_e$ on $\mathcal{H}^3$ consists of eigenvalues converging to k_0, $-k_0$ and 0. Refinements of these results are obtained in [33]. One refinement is that the linear factors of $\mathcal{K}_e^2 - k_0^2 I$ or $\mathcal{K}_e(\mathcal{K}_e^2 - k_0^2 I)$ are compact on subspaces appearing in the decompositions (3.4) or (3.5), respectively. The other refinement is characterization of eigenspaces in terms of those subspaces.

The two-dimensional result is as follows, which yields the result of [31] as corollary.

Theorem 3.5 ([33]). *If Ω is a bounded domain in $\mathbb{R}^2$ whose boundary is $C^{1,\alpha}$ for some $\alpha > 0$, then $\mathcal{K}_e + k_0 I$ and $\mathcal{K}_e - k_0 I$ are compact on $\mathcal{H}^-_{\mathrm{div,rot}}$ and $\mathcal{H}^+_{\mathrm{div,rot}}$, respectively.*

We then have characterizations of eigenspaces. Here and afterwards, $\| \ \|_*$ is the norm induced by the inner product $\langle \ , \ \rangle_*$.

Theorem 3.6 ([33]). *Let Ω be a bounded domain in $\mathbb{R}^2$ whose boundary is $C^{1,\alpha}$ for some $\alpha > 0$. Let $\{f_j\}_{n=1}^\infty$ be an orthonormal system in $\mathcal{H}^2$ consisting of eigenfunctions of $\mathcal{K}_e$ and let $\lambda_j \in \mathbb{R}$ be the corresponding eigenvalues, i.e., $\mathcal{K}_e[f_j] = \lambda_j f_j$. Let $P^\pm$ be the orthogonal projections onto $\mathcal{H}^\pm_{\mathrm{div,rot}}$, respectively. The followings hold.*

(i) *If $\lambda_j \to k_0$, then $\|(P^+ - I)[f_j]\|_* \to 0$.*
(ii) *If $\lambda_j \to -k_0$, then $\|(P^- - I)[f_j]\|_* \to 0$.*

Moreover, estimates on the rotation and the divergence of the solutions to the interior and exterior boundary value problems with the eigenfunctions f_j as the boundary values are obtained [33].

In three dimensions, the following theorem is obtained.

Theorem 3.7 ([33]). *Let Ω be a bounded domain in $\mathbb{R}^3$ whose boundary is $C^{1,\alpha}$ for some $\alpha > 0$.*

(i) *If $\alpha > 0$, then the operators $\mathcal{K}_e + k_0 I$ and $\mathcal{K}_e - k_0 I$ are compact on $\mathcal{H}^-_{\mathrm{div,rot}}$ and $\mathcal{H}^+_{\mathrm{div,rot}}$, respectively.*
(ii) *If $\alpha > 1/2$, then the operator $\mathcal{K}_e$ is compact on $\mathcal{H}_{\mathrm{div}}$.*

In (ii) above, the condition $\alpha > 1/2$ is assumed to guarantee the multiplication operator by the normal vector n to be bounded from $\mathcal{H}$ into $\mathcal{H}^3$. As an immediate consequence of this theorem, the result of [38] is extended to domains with $C^{1,\alpha}$ boundaries:

Corollary 3.8 ([33]). *If Ω is a bounded domain in $\mathbb{R}^3$ whose boundary is $C^{1,\alpha}$ for some $\alpha > 1/2$, then the operator $\mathcal{K}_e(\mathcal{K}_e^2 - k_0^2 I)$ is compact on $\mathcal{H}^3$.*

We use infinite dimensionality of the subspaces $\mathcal{H}^-_{\mathrm{div,rot}}$, $\mathcal{H}^+_{\mathrm{div,rot}}$, and $\mathcal{H}_{\mathrm{div}}$ to prove that the operators $\mathcal{K}_e^2 - k_0^2 I$, $\mathcal{K}_e(\mathcal{K}_e - k_0 I)$, and $\mathcal{K}_e(\mathcal{K}_e + k_0 I)$ are not compact on $\mathcal{H}^3$. As a consequence we obtain the following theorem, which was proved in [38] when $\partial\Omega$ is C^∞.

Theorem 3.9 ([33]). *If Ω is a bounded domain in $\mathbb{R}^3$ whose boundary is $C^{1,\alpha}$ for some $\alpha > 1/2$, then the eigenvalues of the eNP operator $\mathcal{K}_e$ on $\mathcal{H}^3$ consist of three infinite real sequences converging to 0, k_0, and $-k_0$.*

Then, the following theorem is obtained.

Theorem 3.10 ([33]). *Let Ω be a bounded domain in $\mathbb{R}^3$ whose boundary is $C^{1,\alpha}$ for some $\alpha > 1/2$. Let $\{f_j\}_{n=1}^\infty$ be an orthonormal system in $\mathcal{H}^3$ consisting of eigenfunctions of $\mathcal{K}_e$. Let $\lambda_j \in \mathbb{R}$ be the corresponding eigenvalues, i.e., $\mathcal{K}_e[f_j] = \lambda_j f_j$. Then f_j can be decomposed into the sum $f_j = f_j^+ + f_j^- + f_j^o$, where*

$$f_j^\pm \in \mathcal{H}^\pm_{\mathrm{div,rot}}, \quad f_j^o \in (\mathcal{H}^+_{\mathrm{div,rot}} + \mathcal{H}^-_{\mathrm{div,rot}})^\perp \subset \mathcal{H}_{\mathrm{div}},$$

and the following statements hold as $n \to \infty$:

(i) *If $\lambda_j \to k_0$, then $\|f_j - f_j^+\|_{\mathcal{H}^3} \to 0$.*
(ii) *If $\lambda_j \to -k_0$, then $\|f_j - f_j^-\|_{\mathcal{H}^3} \to 0$.*
(iii) *If $\lambda_j \to 0$, then $\|f_j - f_j^o\|_{\mathcal{H}^3} \to 0$.*

Like the two-dimensional case, estimates on the curl and the divergence of the solutions to the interior and exterior problems are also obtained.

3.2. **Convergence rates and asymptotics of eigenvalues.** In this subsection, we review recent results on estimates of the convergence and asymptotic behaviour of eigenvalues.

Let $a = +1, -1$ in two dimensions and $a = +1, 0, -1$ in three dimensions, and let λ_j^a be eigenvalues of the eNP operator converging to ak_0 as $j \to 0$. They are enumerated counting multiplicities in such a way that

$$|\lambda_1^a - ak_0| \geq |\lambda_2^a - ak_0| \geq |\lambda_3^a - ak_0| \geq \cdots .$$

The following results for the two-dimensional case are proved in [39].

Theorem 3.11 ([39]). *Let Ω be a bounded domain in $\mathbb{R}^2$ whose boundary is $C^{k,\alpha}$ with $k+\alpha > 2$ and $0 \leq \alpha < 1$, It holds for $a = +1, -1$ that*

$$\lambda_j^a = ak_0 + o(j^\delta) \quad \text{as } j \to \infty \tag{3.7}$$

for any $\delta > -(k+\alpha) + 3/2$.

To present the result for the domains with real analytic boundaries, we need the notion of the parametrization by a Riemann mapping. Let U be the unit disk and let $\Phi : U \to \Omega$ be a Riemann mapping. Then, $q(s) = \Phi(e^{is})$ is called the parametrization of $\partial\Omega$ by Φ.

Theorem 3.12 ([39]). *Let Ω be a bounded domain in $\mathbb{R}^2$ whose boundary is real analytic. Let q be a parametrization of $\partial\Omega$ by a Riemann mapping and let ϵ_q be its modified maximal Grauert radius. It holds that*

$$\lambda_j^a = ak_0 + o(e^{-\epsilon j}) \quad \text{as } j \to \infty \tag{3.8}$$

for any $\epsilon < \epsilon_q/8$.

It is worth mentioning that (3.8) is not optimal as was shown in also shown in [39] by an example.

Quite recently, the leading order asymptotics of λ_j^a are derived in [40] when $\Omega \subset \mathbb{R}^3$ and $\partial\Omega$ is C^∞. Precise asymptotics of positive and negative parts of $\lambda_j^a - ak_0$ are obtained. Here we write the asymptotics of $|\lambda_j^a - ak_0|$ just for simplicity.

Theorem 3.13 ([40]). *Let Ω be a bounded smooth domain in $\mathbb{R}^3$. It holds that*

$$|\lambda_j^a - ak_0| \sim (A_a W(\partial\Omega) + B_a \chi(\partial\Omega))^{\frac{1}{2}} j^{-\frac{1}{2}}, \quad \text{as } j \to \infty,$$

where A_a and B_a are constants determined only by the Lamé constants λ, μ, and $W(\partial\Omega)$ and $\chi(\partial\Omega)$ are Willmore energy and the Euler characteristic of the surface $\partial\Omega$, respectively.

3.3. **CALR on ellipses.** We now review the result of [31] on CALR for the Lamé system on ellipses. Ellipses are the only domains where CALR takes place. No domain is known where CALR does not take place.

Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{1,\alpha}$ boundary. Let (λ, μ) be the Lamé constants of $\mathbb{R}^2 \setminus \Omega$ satisfying the strong convexity condition (3.1). Let $(\widetilde{\lambda}, \widetilde{\mu})$ be Lamé constants of Ω. We assume that $(\widetilde{\lambda}, \widetilde{\mu})$ is of the form

$$(\widetilde{\lambda}, \widetilde{\mu}) := (c + i\delta)(\lambda, \mu),$$

where $c < 0$ and $\delta > 0$. Since eigenvalues of the eNP operator accumulate at either k_0 or $-k_0$, we choose c so that $k(c) = k_0$ or $-k_0$, where

$$k(c) := \frac{c+1}{2(c-1)}.$$

Let $\mathbb{C}$ and $\widetilde{\mathbb{C}}$ be the isotropic elasticity tensor corresponds to (λ, μ) and $(\widetilde{\lambda}, \widetilde{\mu})$, respectively, and let

$$\mathbb{C}_\Omega = \widetilde{\mathbb{C}}\chi_\Omega + \mathbb{C}\chi_{\mathbb{R}^2 \setminus \overline{\Omega}},$$

where χ denotes the indicator function as before. We consider the following transmission problem:

$$\begin{cases} \nabla \cdot \mathbb{C}_\Omega \widehat{\nabla} u = f & \text{in } \mathbb{R}^2, \\ u(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases}$$

where the source function f is taken to be a polarizable dipole, namely,

$$f = A\nabla\delta_z.$$

Here, A is a 2×2 constant matrix not of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}.$$

In this case, the relevant energy is given by

$$E_\delta = \delta \sum_{j=1}^{\infty} \frac{|A\nabla \mathcal{S}_e[\psi_j](z)|^2}{\delta^2 + |k_\delta(c) - \lambda_j|^2},$$

where λ_j is eigenvalues of the eNP operator and ψ_j is the corresponding normalized eigenfunctions.

The following theorem is proved in [31].

Theorem 3.14 ([31]). *Let Ω be the ellipse defined by* (2.9).

(i) *If $k(c) = k_0$, then*

$$E_\delta \approx \begin{cases} |\log \delta|\, \delta^{-2+\rho_z/\rho_0} & \text{if } \rho_0 < \rho_z \le 3\rho_0, \\ \delta & \text{if } \rho_z > 3\rho_0, \end{cases}$$

as $\delta \to 0$.

(ii) *If $k(c) = -k_0$, then*

$$E_\delta \approx \begin{cases} |\log \delta|^3 \delta^{-3/2+\rho_z/2\rho_0} & \text{if } \rho_0 < \rho_z \le 5\rho_0, \\ \delta & \text{if } \rho_z > 5\rho_0, \end{cases}$$

as $\delta \to 0$.

As a consequence of this theorem, it is proved that if $k(c) = k_0$, then CALR occurs if $\rho_0 < \rho_z \le 2\rho_0$. It is worth mentioning that this cloaking region coincides with that for Laplace equation explained in subsection 2.3. If $k(c) = -k_0$, then CALR occurs if $\rho_0 < \rho_z \le 3\rho_0$. It is interesting to observe that the cloaking region is different from that for the case $k(c) = k_0$.

3.4. **Discussions.** The decay estimate (2.2) for the Laplace operator in two dimensions strongly suggests that there is a room for the estimate (3.7) to be improved. In three dimensions the decay rate of eigenvalues of the eNP operator for domains less regular than C^∞ is not known. These are interesting problems to investigate. No three-dimensional domain is known where the CALR occurs or does not occur. This is also an intriguing problem.

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