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GENERALIZED HEXAGONS EMBEDDED IN METASYMPLECTIC SPACES

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Abstract. We consider thick generalized hexagons fully embedded in metasymplectic spaces, and we show that such an embedding either happens in a point residue (giving rise to a full embedding inside a dual polar space of rank 3), or happens inside a symplecton (giving rise to a full embedding in a polar space of rank 3), or is isometric (that is, point pairs of the hexagon have the same mutual position whether viewed in the hexagon or in the metasymplectic space–these mutual positions are *equality, collinearity, being special, opposition*). In the isometric case, we show that the hexagon is always a Moufang hexagon, its little projective group is induced by the collineation group of the metasymplectic space, and the metasymplectic space itself admits central collineations (hence, in symbols, it is of type $\mathsf{F}_{4,1}$). We allow non-thick metasymplectic spaces without non-thick lines and obtain a full classification of the isometric embeddings in this case.

1. Introduction

Generalized hexagons play a somewhat isolated but rather special role in the theory of spherical buildings because of its crystallographic Weyl group and split type (over each field there exists a simple algebraic group of *absolute*–that is, with a split torus–type G_2), but yet it does not arise as a residue in a higher rank spherical building. The latter can also be said about metasymplectic spaces (geometries related to weak spherical buildings of type F_4). It is therefore valuable and interesting to see such geometries embedded in bigger ambient geometries: this not only helps to understand better the embedded geometry, but also the ambient one. For example, in [6, 7], metasymplectic spaces fully embedded in the minuscule and the long root geometry of type E_6, and the long root geometry of type E_7, are all classified. In general, generalized hexagons are too sparse structures to say anything reasonable when they are

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embedded in a higher rank geometry. For example, there exists a full classification of fully embedded generalized quadrangles in projective space, but for hexagons we are far from such a classification. Only when the embedding satisfies extra conditions, called *flatness* in [14] and *regularity* in [13], one is able to classify (but a lot of fully embedded Moufang hexagons do not satisfy these conditions). Also, the generalized hexagons appearing in the conclusions of the classifications in [13, 14] are the duals of the long root hexagons, showing that projective spaces are not the natural home for the long root hexagons (that is, the Moufang hexagons, where the duality class is chosen such that they allow central elations, see below for precise definitions). Hence it makes sense to consider embeddings of hexagons into long root geometries. Since the natural homes of the split buildings of type G_2 are the buildings of type D_4, and since the long root geometry of type D_4 is a so-called *metasymplectic space*, it is a natural problem to consider (full) embeddings of hexagons into metasymplectic spaces. That problem has two main parts: first one would like to show that an embedding automatically leads to a nice hexagon, here a Moufang one, and secondly, one would like to know which Moufang hexagons exactly appear as an embedded one. In this paper we are concerned with the first part and leave the second part to a sequel.

There is another reason why the embedding problem of hexagons in metasymplectic spaces is exceptional compared to other embeddings. All metasymplectic spaces and all generalized hexagons are so-called *root filtration spaces*, see [3, 4], whereas for the other exceptional types, the notions of *root filtration space* and *long root geometry* coincide. The reason is that the Coxeter diagrams of types G_2 and F_4 have a symmetry that does not come from one of the underlying Dynkin diagram. Hence we have a richer family of geometries than just the long root ones, and this is true for both the embedded geometries as well as for the ambient geometries. The fact that some are not long root, but behave very much like them causes some challenges to overcome.

As already mentioned, the sparsity of generalized hexagons could be responsible for wild type of embeddings, or for our inability to prove structural results. For example, a tool that is often used in embedding questions is to prove restrictions on the way how subspaces embedd. However, subspaces of generalized hexagons are particularly unmanageable. As a comparison, subspaces of generalized quadrangles only come in three well known flavours: partial ovoids, partial line pencils and full subquadrangles. Nevertheless, we are able to show that fully embedded hexagons in metasymplectic spaces also come in three different flavours, two of which refer back to the older question of embedding hexagons in projective spaces. Loosely speaking the first part of our Main Result says:

Main Result 1.1. *A thick generalized hexagon fully embedded in a metasymplectic space is either embedded in a classical residue, or it is isometrically embedded, that is, points at maximal distance in the hexagon are also at maximal* *distance in the metasymplectic space, and points at distance* 2 *in the hexagon are also at "special" distance* 2 *in the metasymplectic space.*

The "classical residues" in the above refer to classical polar spaces and dual polar spaces. In the latter case, the hexagon is also isometrically embedded in a slightly broader sense. In the former case, a classification seems far away since it is not yet known whether the "natural" embedding of the split Cayley hexagon in a parabolic quadric of rank 3 is the only one in such quadric; in fact the uniqueness for the smallest example has been disproved by Coolsaet [5]. So we consider it out of scope of the present paper to further nail down the situation in these residues. About the isometric case, however, we can say more. The second part of our Main Result is, again loosely speaking:

Main Result 1.2. *If a thick generalized hexagon is isometrically and fully embedded in a metasymplectic space, then the hexagon is Moufang, it inherits its little projective group from the collineations group of the metasymplectic space, and both the hexagon and metasymplectic space are long root geometries, that is, in both the points are the centres of central elations.*

An explicit list of all possibilities (in the more general case of *lax* embedding instead of a full one), will be determined in a subsequent paper.

Concerning the proofs, we will be using a lot of properties of mainly metasymplectic spaces. We summarise these in a separate section (see Section 3). In particular we will need the notion of an *equator geometry* and an *extended equator geometry* in an arbitrary Lie incidence geometry of type $\mathsf{F}_{4,4}$. These notions were defined in [6] in the split case, so we will need to extend the definitions and properties (limiting ourselves to the things we need in our proof).

We now get down to precise definitions and statements of the Main Results.

2. Preliminaries and statement of the Main Results

We assume the reader is familiar with the basics of the theory of Tits buildings, cf. [16], and also with (classical) polar spaces. The present paper is concerned with the geometries arising from spherical buildings of types G_2 and F_4. These geometries are known in the literature as *generalized hexagons* and (thick) *metasymplectic spaces*, respectively. In order to fix notation and some common ground, we introduce abstract point-line geometries.

Point-line geometries. A *point-line geometry* Δ is a pair $(X, \mathscr{L})$ with $\mathscr{L} \subseteq 2^X$. The members of X are called the *points*, usually denoted with lower case Latin letters, those of $\mathscr{L}$ are the *lines*, usually denoted with upper case Latin letters. We will always deal with the situation, where two points are contained in at most one line, and lines have constant size at least 3. In general, a *thick line* is a line containing at least 3 points, and similarly for *thick points.* Points on a common line are called *collinear*; if two points x, y are collinear we write $x \perp y$. If the joining line is unique, we denote it by xy. The set of points collinear to a given point x is $x^\perp$ and for $Y \subseteq X$ we define $Y^\perp = \{x \in X \mid x \perp y, \forall y \in Y\}$.

Two subsets Y_1, Y_2 of X are said to be *collinear*, in symbols also $Y_1 \perp Y_2$, if each point of either is collinear to every point of the other. The *collinearity graph* of Δ is the graph with vertices the points of Δ, adjacent when collinear. The *distance* $\delta(x, y)$ between two points $x, y \in X$ is the distance between these points in the collinearity graph. The incidence graph is the bipartite graph on $X \cup \mathscr{L}$, where $x \in X$ is adjacent to $L \in \mathscr{L}$ if $x \in L$. A *subspace* Y of Δ is a subset of points with the property that, if some line $L \in \mathscr{L}$ has at least two points in common with Y, then $L \subseteq Y$. A *convex* subspace is a subspace Y with the additional property that, whenever $x, y \in Y$, then all points on any shortest path between x and y in the collinearity graph are contained in Y. We will frequently view a (convex) subspace as a point-line geometry, where the lines are inherited from Δ. Since X is a (convex) subspace of Δ, and since obviously the intersection of two (convex) subspaces is again a (convex) subspace, the intersection of all (convex) subspaces containing a common set $S \subseteq X$ is a (convex) subspace, and we say that S generates $\langle S \rangle$ (the intersection of all subspaces containing S), and the *hull of* S is $\mathsf{cl}(S)$ (the intersection of all convex subspaces containing S). Concerning the notation $\langle S \rangle$, we occasionally omit the braces for sets, and use commas instead of union symbols, e.g., for $x \in X$ and $S \subseteq X$, we write $\langle x, S \rangle$ for $\langle \{x\} \cup S \rangle$.

Embedded geometries. Let $\Gamma = (X, \mathscr{L})$ and $\Delta = (Y, \mathscr{M})$ be two point-line geometries. Then we say that Γ is fully embedded in Δ if $X \subseteq Y$ and $\mathscr{L} \subseteq \mathscr{M}$. We will sometimes omit the word 'fully'. Note that, if Γ is embedded in Δ, it is not necessarily true that X is a subspace of Y; in most cases it is not. If Δ is the point-line geometry arising from a projective space of dimension n over an associative division ring $\mathbb{K}$, denoted by $\mathsf{PG}(n, \mathbb{K})$, then we say that Γ *admits a projective embedding*. Also, if the embedding of Γ in Δ is such that the distance of any two points of Γ measured in the collinearity graph of Γ is the same as the distance between them measured in the collinearity graph of Δ, then we say that the embedding is *isometric*, or Γ is *isometrically embedded in* Δ.

Generalized hexagons. A point-line geometry $\Gamma = (X, \mathscr{L})$ is a generalized hexagon if the incidence graph has diameter 6 and girth 12. If all point and lines are thick, then we say that Γ is a thick generalized hexagon. If we call a *line pencil* the set P_x of lines containing a fixed point x, then the *dual* of Δ is the geometry $\Gamma^* = (\mathscr{L}, \{P_x \,|\, x \in X\})$. It is also a generalized hexagon. The distance between its points $L, M \in \mathscr{L}$ is denoted by $\delta^*(L, M)$. For lines $L, M \in \mathscr{L}$ at distance 3 in the dual, we denote by $R(L, M)$ the set of points of Γ collinear to some point of L and to some point of M (and the symbol R stands for "regulus'). A *collineation* of Γ is a permutation of X that induces a permutation of $\mathscr{L}$. A collineation is called *central* if it pointwise fixes $\Gamma_1(c)$ for some point $c \in X$, and $\Gamma_1^*(L)$ for each $L \in \mathscr{L}$ with $c \in L$. The point c is called the *centre* of the central collineation, which is also sometimes called a *central elation*. For the (original) definition of Moufang hexagon we refer the reader to

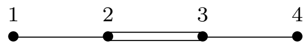

FIGURE 1. The Coxeter diagram of type F_4.

[17, §4.2] or [18, §5.2.1]. We content ourselves with mentioning an equivalent, though seemingly ostensibly stronger condition which, however, suffices for our purposes. Let U_c be the group of central elations with centre $c \in X$. If, up to duality, for every point $c \in X$, and some lines L, M at distance 3 in the dual and such that $|c^\perp \cap L| = |c^\perp \cap M| = 1$, the group U_c acts transitively on $R(L,M) \setminus \{c\}$, then we say that Γ is a *Moufang hexagon.* If the former condition is satisfied for every point c, then we say that the Moufang hexagon is of type $\mathsf{G}_{2,1}$; if it is satisfied for every point in the dual, then we say that Γ is a Moufang hexagon of type $\mathsf{G}_{2,2}$. The definition of Moufang hexagon we just gave is justified by Ronan's characterization of Moufang hexagons [11]. The *little projective group* of a Moufang hexagon is the group generated by all central elations or dual central elations (which are sometimes called *axial elations*).

Dual polar spaces of rank 3. We will also need the definition of a dual polar space of rank 3. Given a polar space Δ of rank 3, the associated *dual polar space* is the geometry Δ^* with point set the set of planes of Δ, and line set the set of plane pencils (a *plane pencil* is the set of planes through a given line). This is a geometry of which the collinearity graph has diameter 3, just as is the case for generalized hexagons.

Metasymplectic spaces. The geometries of importance in the present paper are the so-called "Lie incidence geometries" of types $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$. We now explain how these arise. Consider an abstract (weak) building Ω of type F_4, viewed as a simplicial complex, see [16]. Label the vertices of its (Coxeter) diagram as in Fig. 1. Let X be the set of vertices of type 1, and let $\mathscr{L}$ consist of the sets of vertices of type 1 on a common simplex together with a fixed vertex of type 2. Then $\Delta(X, \mathscr{L})$ is a metasymplectic space, called *of type* $\mathsf{F}_{4,1}$, where the 1 in the index refers to the type in Ω of the points of Γ. In this paper, we are only concerned with metasymplectic spaces having thick lines, that is, $|L| \geq 3$ for every $L \in \mathscr{L}$, hence we assume this throughout. This is equivalent to assuming that all residues of simplices of type $\{3,4\}$ correspond to (proper) projective planes. Using [12], one sees that either Ω is thick, or Δ arises from the line Grassmannian of a polar space of rank 4 (with thick lines).

In our definition above, we could also have considered the vertices of type 4 as points of a metasymplectic geometry, in view of the symmetry of the diagram. Nevertheless, the underlying Dynkin diagram is not symmetric, and it will turn out to be convenient for us to indeed distinguish between metasymplectic spaces of types $\mathsf{F}_{4,1}$ and $\mathsf{F}_{4,4}$. For this paper, it suffices to define the types as follows:

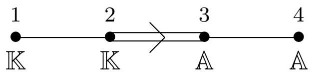

FIGURE 2. The Dynkin diagram of type $\mathsf{F_4}$ with Bourbaki labeling

(i) whenever Ω is not thick (that is, it is a line Grassmannian of a polar space of rank 4), we assume that the (thick) projective plane residues are the ones of simplices of type $\{3,4\}$, hence Γ is then of type $\mathsf{F}_{4,1}$;
(ii) whenever some projective plane residue arises from a non-commutative alternative division ring, then we assume that it conforms to a residue of a simplex of type $\{1,2\}$. Hence, here the corresponding metasymplectic spaces of type $\mathsf{F}_{4,1}$ have projective planes over commutative fields, whereas the corresponding metasympectic spaces of type $\mathsf{F}_{4,4}$ (taking vertices of type 4 as points) have projective planes over non-commutative division rings;
(iii) whenever all projective plane residues are defined over a field, then we choose the vertices of type 1 so that the polar space of rank 3 obtained as residue of a simplex of type 1 arises either from a non-degenerate alternating form in a vector space of dimension 6–or is a subspace of such–or from a non-degenerate Hermitian form in a vector space of dimension 6.

Hence the numbering is such that the coordinatizing structure $\mathbb{A}$ of the projective plane corresponding to types $3,4$ (residues of simplices of type $\{1,2\}$) is a quadratic alternative division algebra over the coordinatizing field $\mathbb{K}$ of the projective plane corresponding to types $1,2$; we have depicted this in Fig. 2. This conforms to the Bourbaki labelling of the vertices of the Dynkin diagram, except if $\mathbb{A}$ is a inseparable field extension of $\mathbb{K}$ in characteristic 2 (then one cannot distinguish the end vertices since in that case $\mathbb{K}$ is an inseparable field extension of $\mathbb{A}^2$, the field of squares of $\mathbb{A}$). We denote the corresponding building by $\mathsf{F_4}(\mathbb{K},\mathbb{A})$, where $\mathbb{A}$ should be given as an algebra over $\mathbb{K}$, and not just as an abstract division ring independent of $\mathbb{K}$. The corresponding metasymplectic spaces of type $\mathsf{F}_{4,1}$ will be denoted $\mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$ and those of type $\mathsf{F}_{4,4}$ will be denoted $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$. With this notation, it is sensible to denote the line Grassmannian of the hyperbolic quadric of Witt index 4 in $\mathsf{PG}(7,\mathbb{K})$, usually denoted by $\mathsf{D}_{4,2}(\mathbb{K})$, as $\mathsf{F}_{4,1}(\mathbb{K},1)$, putting $\mathbb{A}$ formally equal to the trivial algebra $\{\vec{o}\}$ of one element.

Main Results. We are now in a position to state our main results more precisely.

Theorem 2.1. *Let $\Gamma=(P,\mathscr{L})$ be a thick generalized hexagon, fully embedded in a metasymplectic space $\Delta=(X,\mathscr{M})$. Let Δ have type $\mathsf{F}_{4,i}$, $i\in\{1,4\}$, and let Ω be the underlying building of type $\mathsf{F_4}$. Then exactly one of the following holds.*

(i) *Γ is contained in a convex subspace of Δ isomorphic to a polar space of rank 3 and corresponding to the residue of a vertex of type $5-i$ in Ω.*
(ii) *Γ is contained in $p^\perp$ for some point $p \in X \setminus P$. This yields an isometric embedding of Γ in the dual polar space of rank 3 associated with the residue of p in Ω.*
(iii) *Γ is a Moufang hexagon of type $\mathsf{G}_{2,1}$ and is isometrically embedded in Δ. Also, Δ is of type $\mathsf{F}_{4,1}$.*

In the case that Ω is not thick, we can classify the isometric case, see Section 4.6 and Proposition 4.17 therein.

We now comment on the three possibilities above in the thick case.

Case (i) relates to an old open problem in the theory of generalized hexagons. The generalized hexagon $\mathsf{G}_{2,2}(\mathbb{K})$ related to a triality of type I_{id} (see [15]) of the hyperbolic quadric of Witt index 4 in $\mathsf{PG}(7,\mathbb{K})$, also known as the *split Cayley hexagon* (see [18]) embeds naturally into a parabolic polar space of rank 3, that is, a quadric in 6-dimensional projective space with equation $X_1X_2 + X_3X_4 + X_5X_6 = x_0^2$, see [15]. The conjecture is that this embedding is unique, whenever the lines of the hexagon contain at least four points; for the smallest case, where lines have size 3, there is a counterexample, see [5]. So a classification in Case (i) would mean a complete solution of the conjecture.

We do not know too much about Case (ii), except that it exists. Indeed, it is described in [8]. In view of that paper, one is tempted to conjecture that it does not occur in characteristic distinct from 2, but we have no clue. It would certainly need techniques very different from the ones in the present paper. Note also that the above reference describes an embedding of $\mathsf{G}_{2,2}(\mathbb{K})$ into a dual symplectic polar space of rank 3, that is, the corresponding embedding in Δ has the property that Γ sits inside $p^\perp \cap q^{\perp\!\!\perp}$ for some point q opposite p. We do not know whether there are examples in $p^\perp$ not having the latter property.

Case (iii) is most interesting. Every metasymplectic space of type $\mathsf{F}_{4,1}$ contains at least one fully embedded thick generalized hexagon via the full embeddings

$$\mathsf{G}_{2,1}(\mathbb{K}) \subseteq \mathsf{D}_{4,2}(\mathbb{K}) = \mathsf{F}_{4,1}(\mathbb{K},1) \subseteq \mathsf{F}_{4,1}(\mathbb{K},\mathbb{K}) \subseteq \mathsf{F}_{4,1}(\mathbb{K},\mathbb{A})$$

for every quadratic alternative division algebra $\mathbb{A}$ over $\mathbb{K}$, where $\mathsf{G}_{2,1}(\mathbb{K})$ denotes the dual of $\mathsf{G}_{2,2}(\mathbb{K})$ introduced before. In a sequel to the present paper, we intend to determine the precise list of isometrically but not necessarily fully embedded hexagons in metasymplectic spaces (this just means that the lines of the hexagon are subsets of lines of the metasymplectic space). The reason for abandoning the assumption of fullness is that it provides more interesting examples (but this can only be done in conjunction with the assumption of being isometric).

In the same vein, one could also relax the condition of thickness of the generalized hexagon. However, this case is completely solved. Indeed, since a non-thick hexagon with thick lines is contained in every Moufang hexagon of type $\mathsf{G}_{2,1}$, such a non-thick hexagon is embedded in every metasymplectic space of type $\mathsf{F}_{4,1}$. It is easy to see that this embedding is unique. However, more surprisingly, the same holds in the dual. Denote by $2\mathsf{PG}(2,\mathbb{A})$ the *double* of the projective plane $\mathsf{PG}(2,\mathbb{A})$ in the (dual) sense of [18], that is, the points of the geometry $2\mathsf{PG}(2,\mathbb{A})$ are the incident point-line pairs of $\mathsf{PG}(2,\mathbb{A})$ and the set of flags with common point or line is a typical line of $2\mathsf{PG}(2,\mathbb{A})$. Then it is shown in [9]:

Proposition 2.2. *The metasymplectic space* $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{A})$*, with* $\mathbb{A}$ *a quadratic alternative division algebra over* $\mathbb{K}$*, always contains a unique, up to isomorphism, fully embedded non-thick generalized hexagon isomorphic to* $2\mathsf{PG}(2,\mathbb{A})$*.*

Structure of the paper. The paper is organized as follows. In the next section, we gather the properties of metasympletic spaces that we need in our proof. In particular, we have to generalize the notion of extended equator geometry, introduced in [6] for $\mathsf{F}_{4,4}(\mathbb{K},\mathbb{K})$, to all metasymplectic spaces of type $\mathsf{F}_{4,4}$. This is new and will have independent interest. The actual proof of Theorem 2.1 is then presented in Section 4.

3. Metasymplectic spaces: an introduction and properties

3.1. Properties of metasymplectic spaces

Let $\Gamma = \mathsf{F}_4(\mathbb{K},\mathbb{A})$ be a building of type F_4 over $\mathbb{K}$ with associated quadratic alternative division algebra $\mathbb{A}$. For $i = 1, 4$, we let Γ_i be the metasymplectic parapolar space $\mathsf{F}_{4,i}(\mathbb{K},\mathbb{A})$. This means that we have a set of points, a set of lines, a set of planes and a set of *symplecta* and these are such that each line, each plane and each symplecton is a proper convex subset of the set of points. In particular, Γ_i is a partial linear space. The planes are projective planes when endowed with the lines of Γ they contain; the lines and planes contained in a symplecton render it a polar space of rank 3 which we denote by $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ if $i = 1$, and by $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ if $i = 4$. The opposition relation in Γ ([16], Chapter 7) acts on the types as the identity. The basic properties of Γ are the following, stated as facts. As noted on page 80 of [18], these can be proved using the diagram of type F_4; they also follow from [2].

Fact 3.1. *The symplecta, planes and lines of* Γ *through a given point* p*, endowed with the natural incidence relation, form a polar space* $\mathrm{Res}_{\Gamma_i}(p)$ *isomorphic to* $\mathsf{C}_{3,1}(\mathbb{A},\mathbb{K})$ *if* $i = 1$*, and* $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ *if* $i = 4$*, where the points of that polar space are the symplecta through* p*, the lines are the planes through* p*, and the planes are the lines through* p*.*

In particular, it follows that the isomorphism class of the geometry $\mathrm{Res}_{\Gamma_i}(p)$ does not depend on p. It is usually called the *point residual* of Γ_i. Another consequence is the following.

Corollary 3.2. *Every singular subspace of* Γ_i *is contained in some symplecton, and hence is either a point, a line or a projective plane.*

Fact 3.3. *Let x and y be two points of Γ_i. Then, precisely one of the following situations occurs.*

(0) $x = y$;

(1) *there is a unique line incident with both x and y. In this case, we call x and y collinear. We denote the unique line joining them by xy and write $x \perp y$;*

(2) *there is a unique symplecton incident with both x and y. In this case, there is no line incident with both x and y, and we call x and y symplectic, or say that $\{x, y\}$ is a symplectic pair, or say that x is symplectic to y. We denote the unique symplecton by $\xi(x, y)$ and write $x \perp\!\!\!\perp y$;*

(3) *there is a unique point z collinear with both x and y. In this case, we call x and y special, or say that $\{x, y\}$ is a special pair, or say that x is special to y, and denote this by $x \bowtie y$. The point z is denoted by $\mathfrak{c}(x, y)$. For every pair $\{x, z\}$ of collinear points, there is a point y such that $\mathfrak{c}(x, y) = z$;*

(4) *there is no point collinear with both x and y. In this case, x and y are opposite. For every point x there is at least one point y opposite x.*

Moreover, each of these possibilities occurs.

Fact 3.4. *Let x be a point and ξ a symplecton of Γ_i. Then precisely one of the following situations occurs.*

(0) $x \in \xi$;

(1) *the set of points of ξ collinear with x is a line L. Every point y of $\xi \setminus L$ which is collinear with each point of L is symplectic to x and $\xi(x, y)$ contains L. Every other point z of ξ (i.e., every point z of ξ collinear with a unique point z' of L) is special to x and $\mathfrak{c}(x, z) = z' \in L$. We say that x and ξ are close;*

(2) *there is a unique point u of ξ symplectic to x and $\xi \cap \xi(x, u) = \{u\}$. All points v of ξ collinear with u are special to x and $\mathfrak{c}(x, v) \notin \xi$. All points of ξ not collinear with u are opposite x. We say that x and ξ are far.*

Moreover, each of these possibilities occurs.

Fact 3.5. *The intersection of two different symplecta ξ and ζ is either empty, or a point, or a plane and each of these occurs.*

(1) *If $\xi \cap \zeta$ is a point x, then every point in $\xi \setminus x^\perp$ is far from ζ;*

(2) *If $\xi \cap \zeta$ is a plane π, then points $x \in \xi$ and $y \in \zeta$ are special to each other if and only if $x^\perp \cap \pi \neq y^\perp \cap \pi$.*

The next result reviews all possible mutual positions of a point and a line. It follows from including the line in a symplecton and then apply Fact 3.4.

Fact 3.6. *Let x be a point and L a line. Then exactly one of the following occurs.*

(1) $x \in L$;

(2) *$x \perp L$;*
(3) *$x \perp p \in L$ for exactly one point p, and $x \perp\!\!\!\perp q$ for all $q \in L \setminus \{p\}$;*
(4) *$x \bowtie p \in L$ for exactly one point p, and x is opposite q for all $q \in L \setminus \{q\}$;*
(5) *$x \perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with evidently $\mathfrak{c}(x, q) = p$;*
(6) *$x \perp\!\!\!\perp p \in L$ for exactly one point p, and $x \bowtie q$ for all $q \in L \setminus \{p\}$, with $\mathfrak{c}(x, q) = a \perp L$ for a unique point a (independent of q);*
(7) *$x \bowtie p$ for every $p \in L$. In this case there exists a unique line M such that $p \mapsto \mathfrak{c}(x, p)$ is a bijection from L to M.*

A corollary to this is the following.

Corollary 3.7. *Let π and π' be two planes intersecting in a point x. Let L and L' be lines in π and π', respectively, not containing x. Then $\perp\!\!\!\perp$ defines a bijection between L and L' if and only if there exists a point $p \in L$ special to all points of L' except for one, to which it is symplectic.*

Finally, Lemma 2(v) of [4] states:

Fact 3.8. *If $a \perp b \perp c \perp d$ is a path in Δ with $a \bowtie c$ and $b \bowtie d$, then a is opposite d.*

3.2. The equator and extended equator geometries

We now define the equator and extended equator geometries, see also [10], Proposition 6.26, and [6], Section 4.2.

Definition (Equator Geometry)**.** Let p, q be two opposite points of Γ_i. Let $\mathscr{S}_p$ denote the family of symplecta containing p. Then, by Fact 3.4, each member of $\mathscr{S}_p$ contains a unique point which is symplectic to q. The set of all such points is called the *equator geometry of the pair* $\{p, q\}$. It is usually denoted by $E(p, q)$. Using Fact 3.4(2), it is easy to see that $E(p, q) = p^{\perp\!\!\!\perp} \cap q^{\perp\!\!\!\perp}$ and hence this definition is symmetric in p, q.

The following was proved in Proposition 6.26 of [10] for $\Gamma_4 = \mathsf{F}_{4,4}(\mathbb{K}, \mathbb{K})$, but the proof remains valid for $\Gamma_4 = \mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$, with $\mathbb{A}$ any quadratic alternative division algebra. The reason is the following. In a polar space $\mathsf{C}_{3,1}(\mathbb{A}, \mathbb{K})$ (and we now use the symbol $\perp$ for collinearity in this polar space), taking two opposite lines L, M yields a set $L^\perp \cap M^\perp$ which coincides with $\{x, y\}^{\perp\perp}$ for each pair $\{x, y\}$ in $L^\perp \cap M^\perp$. We call such a set a *hyperbolic line* and denote it by $h(x, y)$.

Proposition 3.9. *Let p, q be two opposite points of Γ_4. Then, for any symplectic pair $\{u, v\}$ of points of $E(p, q)$, the hyperbolic line $h(u, v)$ is contained in $E(p, q)$. The geometry of points and hyperbolic lines of $E(p, q)$ is the point-line geometry of a polar space, which we also denote by $E(p, q)$, isomorphic to any point residual of Γ. A natural isomorphism from $E(p, q)$ to $\mathrm{Res}_{\Gamma_4}(p)$ is induced by the map $\varphi_{p,q}$ that sends a point $x \in E(p, q)$ to the symplecton $\xi(x, p)$.*

We need the following property of polar spaces of rank at least 3.

Lemma 3.10. *Any geometric hyperplane G' of any geometric hyperplane G of a polar space Π of rank at least 3 contains two non-collinear points.*

Proof. See [6], Lemma 4.2.3. □

We will also need the following similar statement, a proof of which is easy and left to the reader.

Lemma 3.11. *Any geometric hyperplane of a polar space Π of rank at least 3 contains two opposite lines.*

We observe one more property of $E(p,q)$.

Lemma 3.12. *Let p,q be opposite points of Γ_i, and $x,y \in E(p,q)$. Then either $x=y$, or $\{x,y\}$ is a symplectic pair, or x is opposite y.*

Proof. See [6], Lemma 4.2.4. □

We are now ready to define the extended equator geometry for opposite points p,q in Γ_4. The reason that we cannot do it for Γ_1 is that in the symplecta of Γ_4, the common perp $L^\perp \cap M^\perp$ of a pair of lines $\{L,M\}$ in generic position is determined by two of its points (since it represents a line in an ambient projective space), while this is not true for Γ_1 (except if $\mathbb{A}$ is an inseparable field extension of $\mathbb{K}$ in characteristic 2 or $\mathbb{K}=\mathbb{A}$ in characteristic 2).

Definition (Extended Equator Geometry)**.** Let p,q be two opposite points of Γ. Then define the point set

$$\widehat{E}(p,q) = \bigcup \{E(x,y) : x,y \in E(p,q),\ x \text{ opposite } y\}.$$

Note that, by Proposition 3.9 and Lemma 3.12, $E(p,q)$ contains pairs of opposite points. So, $\widehat{E}(p,q)$ is nonempty. The set $\widehat{E}(p,q)$, endowed with all the hyperbolic lines in it, is called the *extended equator geometry* for p,q. Further, p,q and $E(p,q)$ are contained in $\widehat{E}(p,q)$.

Standing hypothesis. From now on until the end of this subsection, we fix a pair of opposite points p,q of Γ_4 and write $E := E(p,q)$ and $\widehat{E} := \widehat{E}(p,q)$. The goal is to show that, endowed with all hyperbolic lines contained in it, $\widehat{E}$ is the unique (up to isomorphism) polar space in which each point residue is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$, and which we shall logically denote by $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$. The proof in Subsection 4.2 of [6] uses at various points special properties of hyperbolic lines that are not true in general, so we have to provide alternative arguments, and we choose to write down a complete proof for readability reasons. The proof will turn out to be slightly shorter and more efficient than the one in [6].

We first prove that $\widehat{E}$ does not contain collinear or special pairs of points and that it is closed under taking hyperbolic lines through symplectic pairs of its points. Then, we argue that the geometry of its points and hyperbolic lines is the point-line geometry of a polar space of type $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$.

Lemma 3.13. *Let $c \in \widehat{E} \setminus E$ be arbitrary and let $h(u,v)$ be an arbitrary hyperbolic line contained in E, $u,v \in E$, $u \neq v$. Then at least one of the two following properties hold*:

(i) *The point c and $h(u,v)$ are contained in a common equator geometry $E(x,y)$ for some opposite points $x,y \in E$;*

(ii) *all points of $h(u,v)$ are symplectic to c.*

Proof. Assume first that at least two points of $h(u,v)$ are symplectic to c. If c belongs to $\xi(u,v)$, then (ii) holds. Fact 3.4 implies that c is close to $\xi(u,v)$. Set $L = c^{\perp} \cap \xi(u,v)$, then $L \subseteq u^{\perp} \cap v^{\perp}$. Since $h(u,v) = \{u,v\}^{\perp\perp} \subseteq L^{\perp}$, we deduce (ii).

Hence, since hyperbolic lines have at least three points, we may assume that none of u and v are symplectic to c. Let $a,b \in E$ be opposite and such that $c \in E(a,b)$. Set $Q = a^{\perp\!\!\!\perp} \cap b^{\perp\!\!\!\perp} \cap E$. Then Q has the structure of a generalized quadrangle (endowed with the hyperbolic lines), because it is the set of points collinear with two opposite points in a rank 3 polar space.

Assume initially that $u \in Q$. If $v \in Q$, then take $(x,y) = (a,b)$ to obtain (i), so we may assume v and b are opposite. Since both c and Q belong to $E(a,b)$, each hyperbolic line in Q contains at least one point symplectic to c. Select two such hyperbolic lines L_1 and L_2 containing u and let $z_i \in L_i$ be a point symplectic to c, $i = 1,2$. Now we argue in the polar space E: the hyperbolic line $h(b,z_i)$ contains a unique point x_i symplectic to v, and since u is symplectic to both b and z_i, it is also symplectic to x_i. Moreover, x_1 and x_2 are opposite. hence we can set $(x,y) = (x_1,x_2)$ and (i) holds.

Now assume both u and v do not belong to Q. Note that $c^{\perp\!\!\!\perp} \cap Q$ is a geometric hyperplane of Q and so there exist opposite points z,z' in Q symplectic to c. If z and z' are both symplectic to both u and v, then we obtain (i) by setting $(z,z') = (x,y)$. Hence we may assume without loss of generality that u is not symplectic to z. If u is not symplectic to b either, then we can replace b by the unique point on $h(z,b)$ symplectic to u. Hence we may assume that $u \perp\!\!\!\perp b$. Let u' be the unique point on $h(b,u)$ symplectic to a. If u' were symplectic to c, then Fact 3.4(1) and the definition of hyperbolic line would imply that $c \perp\!\!\!\perp u$, a contradiction. Hence the previous paragraph implies that $h(u',u)$ and c belong to a common equator geometry $E(x',y')$ for certain opposite points $x',y' \in E$. But now since $u \in x'^{\perp\!\!\!\perp} \cap y'^{\perp\!\!\!\perp}$, the previous paragraph yields the assertion (i). $\square$

This implies the following.

Corollary 3.14. *Let $x \in \widehat{E}$. Then the set of points of E symplectic to or equal to x is a geometric hyperplane of E, viewed as a polar space, or coincides with it.*

Proof. Let L be a line of the polar space E (so L is a hyperbolic line contained in E). If $x \in E$, then the assertion is obvious. Now let $x \in \widehat{E} \setminus E$. Then, since equator geometries are polar spaces when endowed with the hyperbolic lines they contain, Lemma 3.13 implies that either one or all points of L are symplectic to x. This completes the proof of the corollary. □

We now reach our first main goal.

Lemma 3.15. *Let $x, y \in \widehat{E}$, $x \neq y$. Then either $\{x, y\}$ is a symplectic pair, or x is opposite y. If, moreover, $\{x, y\}$ is a symplectic pair, then $h(x, y)$ is completely contained in $\widehat{E}$.*

Proof. By Lemma 3.10 and Corollary 3.14, we can find two opposite points $a, b \in E(p, q)$ symplectic to both x and y. Hence $x, y \in E(a, b)$ and so the first assertion follows from Lemma 3.12. If, moreover, x and y are symplectic, then $h(x, y) \subseteq E(a, b) \subseteq \widehat{E}$, by Proposition 3.9 and the definition of $\widehat{E}$. □

We can now pin down the exact structure of the extended equator geometries.

Proposition 3.16. *The extended equator geometry $\widehat{E}(p, q)$, endowed with the hyperbolic lines contained in it, is a polar space isomorphic to $\mathsf{B}_{4,1}(\mathbb{K}, \mathbb{A})$.*

Proof. We check the Buekenhout-Shult axioms of a polar space as given in [1]. We repeat these axioms for the convenience of the reader.

(1) *Every (hyperbolic) line contains at least* 3 *points.* This holds by Lemma 3.15 and the fact that a hyperbolic line contains at least 3 points.

(2) *There is no point collinear with every other point.* By definition of $\widehat{E}(p, q)$, any point $x \in \widehat{E}(p, q)$ is contained in an equator geometry, which is, by Proposition 3.9, isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$, in which x has an opposite point.

(3) *One-or-all axiom, i.e., either exactly one or all points of a given line are collinear with a given point.* This requires some extra arguments compared to Proposition 4.2.11 of [6]. Let $h(a, b)$ be an arbitrary hyperbolic line, and z an arbitrary point (all in $\widehat{E}$). As before, there exist opposite points $x, y \in E$ such that $h(a, b) \subseteq E(x, y)$. In $x^{\perp\!\!\!\perp} \cap y^{\perp\!\!\!\perp} \cap E$ we find, using Lemma 3.13, two opposite points u, v symplectic to z (as a geometric hyperplane of a generalized quadrangle always contains opposite points). Then $h(a, b) \subseteq E(x, y)$ and $z \in \widehat{E}(x, y)$ and the assertion follows from Lemma 3.13 with (x, y) taking over the role of (p, q).

(4) *Finite rank, i.e., every nested family of singular subspaces is finite.* Again by Proposition 3.9, the residue in the point p is isomorphic to the polar space induced on the set of points of $\widehat{E}(p, q)$ symplectic to both p and q. Since in the whole of Γ, this is $E(p, q)$, it is also $E(p, q)$ in $\widehat{E}$. Hence the residue at p is a polar space isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{A})$ and as such has rank 3. We conclude that the rank of $\widehat{E}(p, q)$ is 4 and hence finite.

The argument above implies that $\widehat{E}(p,q)$ is a polar space isomorphic to $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$, as the residue in at least one point is isomorphic to $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$. The proposition is proved. $\square$

Just as in [6] one shows that $\widehat{E}(a,b)=\widehat{E}(p,q)$ for every pair $\{a,b\}$ of opposite points of $\widehat{E}(p,q)$. Also the results in Subsection 5.3 of [6] remain valid. We summarise these now. Let us call a maximal singular subspace of $\widehat{E}$, viewed as a polar space, a *hyperbolic solid*, as in [6].

Proposition 3.17. (1) *If a point is collinear to at least two points of $\widehat{E}$, then it is collinear to precisely all points of a hyperbolic solid.*

(2) *For every hyperbolic solid Σ in $\widehat{E}$, there exists a unique point $\beta(\Sigma)$ collinear to all points of Σ.*

(3) *For every hyperbolic plane π in $\widehat{E}$, the set*

$$\{\beta(\Sigma)\,|\,\pi\subseteq\Sigma \textit{ is a hyperbolic solid in } \widehat{E}\}$$

is a line of Γ_4.

(4) *Two hyperbolic solids Σ_1 and Σ_2 of $\widehat{E}$ share a unique point x if and only if $\beta(\Sigma_1)$ and $\beta(\Sigma_2)$ form a special pair of points of Γ_4, and in this case $\mathfrak{c}(\beta(\Sigma_1),\beta(\Sigma_2))=x$.*

(5) *Two hyperbolic solids Σ_1 and Σ_2 of $\widehat{E}$ are disjoint if and only if $\beta(\Sigma_1)$ and $\beta(\Sigma_2)$ are opposite points of Γ_4.*

(6) *The set $\widehat{T}(p,q)$ of points $\beta(\Sigma)$, with Σ ranging through all hyperbolic solids of $\widehat{E}$, with all induced lines, is isomorphic to the dual polar space $\mathsf{B}_{4,4}(\mathbb{K},\mathbb{A})$ corresponding to the polar space $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$.*

Finally, we provide some more information about the polar spaces $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$, which we will need in Section 4.5, in particular in Proposition 4.16. Recall that the 1-spaces of a vector space V are the points of a projective space $\mathsf{PG}(V)$ and V is called the *underlying vector space*. Each projective space of dimension at least 3 arises in this manner.

Proposition 3.18. *Let $\mathbb{A}$ be a quadratic alternative division algebra over $\mathbb{K}$, with $\mathbb{A}$ not an inseparable field extension if the characteristic of $\mathbb{K}$ is 2 (this also excludes $\mathbb{A}=\mathbb{K}$ in characteristic 2). Set $d:=\dim_{\mathbb{K}}\mathbb{A}$. Then the polar space $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ admits an embedding as a quadric in $\mathsf{PG}(7+d,\mathbb{K})$ and there exists a non-degenerate symmetric bilinear form β on the underlying vector space V such that for each point $p=\langle v\rangle$ of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$, $v\in V$, the hyperplane spanned by the lines of $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ through p corresponds to the (null set of the) linear form $\beta_v:V\to\mathbb{K}:w\mapsto\beta(v,w)$.*

Proof. It follows from 10.2 of [16] and the fact that $\mathsf{B}_{3,1}(\mathbb{K},\mathbb{A})$ is a point residue in $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ that $\mathsf{B}_{4,1}(\mathbb{K},\mathbb{A})$ is embedded in $\mathsf{PG}(7+d,\mathbb{K})$ and has the mentioned property with respect to the vector space $V:=\mathbb{K}^4\times\mathbb{A}\times\mathbb{K}^4$ and the symmetric bilinear form (the linearization of the quadratic form given in 10.2 of [16])

$$\begin{aligned}&\beta : V \times V \to \mathbb{K} :\\ &\quad ((x_{-4},\dots,x_{-1},X_0,x_1,\dots,x_4),(y_{-4},\dots,y_{-1},Y_0,y_1,\dots,y_4))\\ &\mapsto X_0\overline{Y}_0 + Y_0\overline{X}_0 + x_{-4}y_4 + x_{-3}y_3 + \cdots + x_4y_{-4},\end{aligned}$$

where $\mathbb{A} \to \mathbb{A} : X \mapsto \overline{X}$ is the standard involution in $\mathbb{A}$. It only remains to show that β is non-degenerate. It is routine to see that this is equivalent to showing that, whenever for some given $Y \in \mathbb{A}$ we have $X\overline{Y} + Y\overline{X} = 0$ for all $X \in \mathbb{A}$, then $Y = 0$. To see that this indeed holds, first set $X = 1$ to obtain $Y + \overline{Y} = 0$, hence $Y^{-1}XY = \overline{X}$ for all $X \in \mathbb{A}$. This implies that the standard involution is actually an automorphism, hence $\mathbb{A}$ is commutative (whence associative, too). But then the last equality reduces to $X = \overline{X}$ for all $X \in \mathbb{A}$, implying that $\mathbb{K} = \mathbb{A}$ and that the characteristic of $\mathbb{K}$ is equal to 2 (in view of the earlier derived property $Y + \overline{Y} = 0$). But that contradicts our assumptions on $\mathbb{A}$. □

4. Proof of the main results

4.1. A lemma on generalized hexagons

The following very general lemma will be used a number of times, although in the respective cases an appropriately weaker version would suffice. But for reasons of unification and convenience, we will nevertheless apply the lemma in such situation.

Lemma 4.1. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon. Let $*$ be a binary symmetric relation between pairs of points of Γ at mutual distance 3. Suppose that $*$ has the property that, whenever $p * q$ for $p, q \in P$, $\delta(p,q) = 3$, and $z \in \Gamma_1(p) \cap \Gamma_3(q)$, then $q * z$. Then $x * y$ for each pair of points of Γ at mutual distance 3.*

Proof. Let $p, q \in P$ be such that $\delta(p,q) = 3$ and $p * q$. First let $q' \in \Gamma_3(p)$ be such that $\delta(q,q') \le 2$. We claim that $p * q'$. Indeed, this follows directly from our assumptions if $q' \perp q$. So we may assume $\delta(q,q') = 2$. Denote by L and L' the lines containing q and q', respectively, and mutually intersecting, say in the point $c = \Gamma_1(q) \cap \Gamma_1(q')$. If $c \in \Gamma_3(p)$, then applying twice our assumption, we first obtain $c * p$ and then $q' * p$. So we may assume $\delta(p,c) = 2$. Select $p' \perp p$ arbitrarily in $\Gamma_3(c) \cap \Gamma_3(q)$, which is very well possible since lines contain at least three points. By applying our assumption to q, we obtain $p' * q$. Applied to p', we obtain $c * p'$. Denoting an arbitrary point of $L' \setminus \{c\}$ at distance 3 from p' by q'', we deduce similarly $p' * q''$. Now our assumption applied to q'' yields $q'' * p$, and subsequently applied to p, it finally yields $q' * p$.

Clearly, we deduce from the claim in the precious paragraph that all points of Γ at distance 3 from p are in relation $*$ to p. Now let $r, s \in P$ be arbitrary but at distance 3 from each other. Select $t \in \Gamma_3(p) \cap \Gamma_3(r)$. Letting t, p play the role of p, q, respectively, in the above, we find $t * r$. Now letting r, t play the role of p, q in the above argument, we deduce $r * s$. □

We now start our analysis of how a thick generalized hexagon $\Gamma = (X, \mathscr{L})$ can be embedded in a metasymplectic space $\Delta = (X, \mathscr{M})$. We refer to the mutual distance of points in Γ using the numerical distance, and to the one in Δ using the names *collinear, special, symplectic, opposite*. We begin with putting rather severe restrictions on the mutual positions of the points of Γ in Δ, and gradually lift these until we cover all possible cases.

4.2. Only collinear point pairs

Lemma 4.2. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Then there exists at least one pair of points of Γ that is not collinear in Δ.*

Proof. If all point pairs of Γ were collinear in Δ, then P is contained in a singular subspace of Δ, hence a plane. But then disjoint lines of Γ nevertheless meet nontrivally, a contradiction. $\square$

4.3. No special point pairs

Lemma 4.3. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is special in Δ. Then no pair of points of Γ is opposite in Δ and there exists a pair of points of Γ at distance 3 that is symplectic in Δ.*

Proof. Suppose for a contradiction that there is a pair $p, q \in P$ of opposite points. Let $L \in \mathscr{L}$ contain q. Since $L \in \mathscr{M}$, there exists, by Fact 3.6, a (unique) point on it special to p, contradicting our assumptions.

Now by Lemma 4.2, there exists at least one pair $p, q \in P$ of symplectic points. Suppose that $\delta(p, q) = 2$ (it is either 2 or 3 and if it is 3, there is nothing to prove anymore). Let $L \in \mathscr{L}$ be such that $q \in L$ and L contains points at distance 3 from p. If all members of $L \setminus \{q\}$ are collinear to p, then so is q, a contradiction. Hence by Fact 3.6, $L \setminus \{q\}$ contains a point symplectic to p and that point is at distance 3 from p. $\square$

We are now working towards Lemma 4.1. However, it will turn out we will use it for the dual of Γ.

Lemma 4.4. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is special in Δ. If $p, q \in P$ is a pair of points of Γ at distance 3 that is symplectic in Δ, then $\Gamma_1(p) \subseteq \xi(p, q)$.*

Proof. Suppose for a contradiction that some point $r \in \Gamma_1(p)$ is not contained in $\xi(p, q)$. Then r is close to $\xi(p, q)$ and since p is not collinear to q, Fact 3.4 implies that r and q are special, a contradiction. $\square$

Since we cannot guarantee that points of Γ at distance 3 from each other are symplectic in Δ, we cannot use Lemma 4.1 directly. So we try the dual.

Lemma 4.5. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is special in Δ. Then every pair of lines $L, M \in \mathscr{L}$ with $\delta^*(L, M) = 3$ is contained in a unique symp ξ of Δ. Also, every line of Γ intersecting L is contained in ξ.*

Proof. Let $L, M \in \mathscr{L}$ be such that $\delta^*(L, M) = 3$. If $L \perp M$ in Δ, then L and M are contained in a singular 3-space, a contradiction. Hence there exists at least one point pair $p \in L$, $q \in M$ that is symplectic. If $\delta(p, q) = 2$, then, as in the proof of Lemma 4.3, we can re-choose $q \in M$ such that $\delta(p, q) = 3$ and p and q are still symplectic. Then Lemma 4.4 ensures that $L, M \subseteq \xi(p, q)$. Suppose for a contradiction that some line $N \in \mathscr{L}$ intersecting L, say in the point r, lies outside $\xi(p, q)$. Pick $s \in N \setminus \{r\}$. Then s is close to $\xi(p, q)$ and so $K := s^\perp \cap \xi(p, q)$ is a line of Δ. Since Γ does not contain special pairs, all points of Γ lying in $\xi(p, q)$ are collinear to K. Let α be the plane of $\xi(p, q)$ through K containing p, and β the one containing q. Obviously $\alpha \neq \beta$. By Lemma 4.4 all points of $\Gamma_1(q)$ belong to $\xi(p, q)$; hence they all belong to β, as $q \notin \alpha$ (otherwise this would contradict $q \perp\!\!\!\perp p$). Select a point q' on a line $M' \neq M$ through q, with $M' \in \mathscr{L}$, $\delta^*(L, M') = 3$ and $q' \notin K$. Then there exists a point $p' \in L \setminus \{r\}$ at distance 3 from q'. Since $p' \perp\!\!\!\perp q'$, Lemma 4.4 again implies that every line $J \in \mathscr{L}$ through q' is contained in $\xi(p', q') = \xi(p, q)$. But such a line J is then also contained in β, leading to a triangle in Γ, a contradiction. $\square$

We have now everything in place to apply the dual of Lemma 4.1 with $*$ being the relation "being contained in the symp ξ", with ξ the symp containing two arbitrarily chosen lines L, M of Γ at mutual distance 3. We directly conclude:

Lemma 4.6. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is special in Δ. Then $P \subseteq \xi$ for a unique symp ξ of Δ.*

We can now move on to the next case, where we do have special point pairs, but no opposite ones.

4.4. There are special point pairs but no opposite ones

Lemma 4.7. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is opposite in Δ. Assume also that some point $p \in P$ is special to some point $q \in P$. Then there exists a point $q' \in P$ special to p and such that $\delta(p, q') = 3$.*

Proof. If $\delta(p, q) = 3$, then we set $q' = q$. So we may assume $\delta(p, q) = 2$. Let $L \in \mathscr{L}$ be a line trough q not containing a point of Γ at distance 1 from p. Then Fact 3.6 yields a point $q' \in L \setminus \{q\}$ special to p. $\square$

Lemma 4.8. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is opposite in Δ. Assume also that some point $p \in P$ is special to some point*

$q \in P$ at distance 3 from p. Then no member of $\Gamma_3(p) \cap \Gamma_1(q)$ is collinear or symplectic to p.

Proof. Clearly no point q' of $\Gamma_3(p) \cap \Gamma_1(q)$ is collinear to p as this would yield two distinct paths of length 2 joining p and the unique point of $\Gamma_2(p)$ on qq' (and then we would have a collinear, a symplectic and a special point to p on qq', contradicting Fact 3.6).

Assume for a contradiction that $z \in \Gamma_3(p) \cap \Gamma_1(q)$ is symplectic to p. Set $qz = M$. Then every point of $M \setminus \{z\}$ is special to p. Set $r = M \cap \Gamma_2(p)$ and $s = \Gamma_1(p) \cap \Gamma_1(r)$. It follows from Fact 3.6 that $s \perp M$. Let $L \in \mathscr{L}$ be any line through p not containing s. We claim that r is special to each point on L. Indeed, since $r \bowtie p$, the point r is special to each point of L, except possibly one, which is then collinear to r or symplectic to it. If some point of L were collinear to r, then there would be two paths of length 2 joining p with r, contradicting $p \bowtie r$. Hence we may assume that $r \perp\!\!\!\perp t$ for some $t \in L$. By looking at r and L, Fact 3.6 tells us that $t \perp s$ and by Corollary 3.7 $\perp\!\!\!\perp$ defines a bijection between L and M. Then again Fact 3.6 implies that s is the unique point in $t^\perp \cap t'^\perp$ (in Δ) for each $t' \in M \setminus \{r\}$. This contradicts the fact that there is a unique path of length 2 in Γ from t to a unique point of $M \setminus \{r\}$. The claim is proved.

This also shows that s and t are symplectic (indeed, they cannot be special as $s \bowtie t$ and $r \bowtie p$ would imply r opposite t by Fact 3.8). Now we choose $t \in L$ such that it belongs to $\Gamma_2(z)$. Suppose for a contradiction that $\xi(z,p) = \xi(s,t)$. Since t is not collinear to s, and p is not collinear to z, no point of pt is collinear to all points of sz. Hence, since both lines are contained in a common polar space, collinearity defines a bijection between sz and pt. It follows from Fact 3.4 that each point of $\langle r,s,z\rangle \setminus sz$ is special to each point of pt with centre on sz. Select $a \in rz \setminus \{r,z\}$. Then the unique point b of $\Gamma_1(a)$ at distance 1 from some point of L is such a centre and hence $b \in sz$. But then the hexagon line ab intersects rs, a contradiction. Hence $\xi(z,p) \neq \xi(s,t)$. Fact 3.5(2) applied to the symps $\xi(z,p)$ and $\xi(s,t)$ tells us that $t \bowtie z$ and $u \perp p$, with $\{u\} = \Gamma_1(t) \cap \Gamma_1(z)$. Also $sp \perp u \perp L$ by the same lemma. Since maximal subspaces of Δ are planes, we deduce that u is not collinear to r; so $t \bowtie x$ for all $x \in M$.

Now select $p' \in sp \setminus \{s,p\}$. Let u' be the unique point of $\Gamma_2(p')$ on uz and set $\{t'\} = \Gamma_1(p') \cap \Gamma_1(u')$. Clearly $p' \perp\!\!\!\perp z$ and $p \bowtie r$. Hence p', t', u' play the same role as p, t, u in the previous paragraphs. We conclude that u' is collinear to sp, so $z \in uu' \perp sp$, yielding $p \perp z$, a contradiction. □

Lemma 4.9. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that no pair of points of Γ is opposite. Assume also that some point $p \in P$ is special to some point $q \in P$ at distance 3 from p. Then all pairs of points of Γ at mutual distance 3 are special in Δ.*

Proof. By Lemma 4.8, the binary symmetric relation "is special to" satisfies the assumptions of Lemma 4.1. The assertion now follows from that lemma. $\square$

Our next aim is to show Lemma 4.12 below. We need two preliminary observations.

Lemma 4.10. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that all pairs of points of Γ at mutual distance 3 are special in Δ. Then no pair of points of Γ at mutual distance 2 is collinear.*

Proof. Let $p, q \in P$ be points at distance 3 and assume for a contradiction that some point $r \in \Gamma_2(p) \cap \Gamma_1(q)$ is collinear to p. Let $s \in \Gamma_1(p) \cap \Gamma_2(q) \cap \Gamma_3(r)$. Then $q \perp r \perp p \perp s$ is a path with $q \bowtie p$ and $r \bowtie s$, so that q is opposite s by Fact 3.8, a contradiction. $\square$

Lemma 4.11. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that all pairs of points of Γ at mutual distance 3 are special in Δ. Let $p, q \in P$ be two points at mutual distance 3. Let $r, s \in \Gamma_2(p) \cap \Gamma_1(q)$, $r \neq s$. If $p \perp\!\!\!\perp r$ and $p \perp\!\!\!\perp s$, then $r \perp\!\!\!\perp s$.*

Proof. Since q is close to $\xi(p, s)$, the point $\mathfrak{c}(p, q)$ lies on $q^\perp \cap \xi(p, s)$; in particular $s \perp \mathfrak{c}(p, q)$. Similarly, $r \perp \mathfrak{c}(p, q)$. Since $q \neq \mathfrak{c}(p, q)$, we now see that $s^\perp \cap r^\perp$ contains a line, hence $s \perp\!\!\!\perp r$. $\square$

Lemma 4.12. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that all pairs of points of Γ at mutual distance 3 are special in Δ. Then all pairs of points of Γ at mutual distance 2 are symplectic.*

Proof. Let $p, q \in P$ be two points at mutual distance 3. Let $r, s \in \Gamma_2(p) \cap \Gamma_1(q)$, $r \neq s$. Assume for a contradiction that $r \bowtie s$.

Set $a = \Gamma_1(p) \cap \Gamma_1(r)$ and $b = \Gamma_1(p) \cap \Gamma_1(s)$. We start by noting that, if $a \bowtie q$, then a and s are opposite by Fact 3.8, a contradiction. Hence, by Lemma 4.10, we find $q \perp\!\!\!\perp a$ and likewise $q \perp\!\!\!\perp b$. Set $c = \mathfrak{c}(p, q)$. Since $p \perp a$, the point p is close to $\xi(a, q)$, and so $a \perp c \in \xi(a, q)$. Similarly $b \perp c \in \xi(b, q)$. Hence $\xi(a, q) \cap \xi(b, q)$ is a plane π which contains c. Let $a_1 \in ar \setminus \{r\}$ and $b_1 \in bs \setminus \{s\}$ be arbitrary but such that $\delta(a_1, b_1) = 2$. Set $p_1 = \Gamma_1(a_1) \cap \Gamma_1(b_1)$. Then Lemma 4.11 applied to $a_1 \perp\!\!\!\perp q \perp\!\!\!\perp b_1$ reveals $a_1 \perp\!\!\!\perp b_1$. By Fact 3.5, $a_1^\perp \cap \pi = b_1^\perp \cap \pi$. Let x be the unique point of π collinear to ar. Then it follows from taking two different choices for a_1 that x is also collinear to bs. If a_1 exhausts $ar \setminus \{r\}$, then both $a_1^\perp \cap \pi$ and $b_1^\perp \cap \pi$ exhaust the line pencil in π with vertex x, except for the line $r^\perp \cap \pi$ and $s^\perp \cap \pi$, which must hence coincide. Fact 3.5 yields $r \perp\!\!\!\perp s$, the sought contradiction. $\square$

We can now close this case and show that the whole of Γ is collinear to a unique point outside Γ.

Lemma 4.13. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that all pairs of points of Γ at mutual distance 3 are special in Δ. Then there exists a unique point $x \in X$ such that $a \perp P$. We also have $x \notin P$.*

Proof. By the previous lemmas, collinearity of points is the same whether considered in Γ or in Δ. Hence we may use the symbol $\perp$ without any ambiguity. Select a path $p \perp r \perp s \perp q$ of points in P with $\delta(p, q) = 3$. By Lemma 4.12, there are symps $\xi(p, s)$ and $\xi(q, r)$, and since these have a line in common they intersect in a plane $\pi \supseteq rs$. Fact 3.5 implies that $x := \mathfrak{c}(p, q)$ lies in π. Since $r \perp x \perp p$, we see that x is collinear to all points of rp. Varying r and s we see that x is collinear to all points of Γ at distance 1 from p. Let $*$ be the binary symmetric relation "$\mathfrak{c}(a, b) = x$" between points of Γ at distance 3. Then the foregoing shows that $*$ satisfies the conditions of Lemma 4.1. Consequently x is the common centre of all pairs of (special) points of Γ at distance 3. In particular, $x \perp z$ for all $z \in P$. Obviously $x \notin P$ since for each point of Γ, there exist points of Γ symplectic and points special to it. $\square$

4.5. There exist opposite point pairs

We have now come to the last case.

Lemma 4.14. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon (fully) embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$. Assume that there exists a pairs of points of Γ that is opposite in Δ. Then the embedding is isometric.*

Proof. Let $p, q \in P$ be opposite. Clearly $\delta(p, q) = 3$. Let $L \in \mathscr{L}$ be any line through q. Then Fact 3.6 implies that the unique point of L at distance 2 from p is special to p and all other points of L are opposite. Hence Lemma 4.1 implies that all point pairs of Γ at mutual distance 3 are opposite in Δ. Our argument above then implies that all point pairs of Γ at mutual distance 2 are special in Δ. $\square$

Proposition 4.15. *Let $\Gamma = (P, \mathscr{L})$ be a generalized hexagon fully and isometrically embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$ isomorphic to $\mathsf{F}_{4,1}(\mathbb{K}, \mathbb{A})$ for some field $\mathbb{K}$ and some composition algebra $\mathbb{A}$ over $\mathbb{K}$. Then Γ is Moufang of type $\mathsf{G}_{2,1}$ and the little projective group of Γ is induced by the collineation group of Δ.*

Proof. Let $p \in P$ be arbitrary. Let θ be any central elation of Δ with centre p. Let $q \in P$ be opposite p. Let $L, M \in \mathscr{L}$ be two lines at distance 3 from each other with $p, q \in R(L, M)$. Since θ, as a central elation, stabilizes both L and M, it stabilizes $R(L, M)$. Hence θ stabilizes $\Gamma_3(p)$. Since θ also pointwise fixes every line of Δ, and hence of Γ, through p, it pointwise fixes $\Gamma_1(p)$. Since every point of $\Gamma_2(p)$ is uniquely determined in Δ as the centre of a special point pair consisting of a point of $\Gamma_3(p)$ and one of $\Gamma_1(p)$, we see that θ acts on Γ. Since the group of central elations with centre p acts simply transitively on $R(L,M) \setminus \{p\}$, we see that we have all central elations with centre p in Γ. Since p was arbitrary, we deduce that Γ is a Moufang hexagon of type $\mathsf{G}_{2,1}$ and the assertions follow. $\square$

Proposition 4.16. *Let $\Gamma = (P, \mathscr{L})$ be a thick generalized hexagon fully and isometrically embedded in the metasymplectic space $\Delta = (X, \mathscr{M})$ isomorphic to $\mathsf{F}_{4,4}(\mathbb{K}, \mathbb{A})$ for some field $\mathbb{K}$ and some composition algebra $\mathbb{A}$ over $\mathbb{K}$. Then $\mathbb{A}$ is inseparable in characteristic 2, Δ is hence isomorphic to $\mathsf{F}_{4,1}(\mathbb{A}^2, \mathbb{K})$, and Γ is hence a Moufang hexagon of type $\mathsf{G}_{2,1}$ with little projective group induced by the collineation group of Δ.*

Proof. Select an ordinary hexagon $p \perp r_1 \perp s_1 \perp q \perp s_2 \perp r_2 \perp p$ in Γ (hence with $\delta(p,q) = \delta(r_1, s_2) = \delta(r_2, s_1) = 3$), and an additional path $p \perp r_3 \perp s_3 \perp q$, with $r_3 \notin \{r_1, r_2\}$. We now use the assertions and notation of Proposition 3.17. The point r_1 is collinear to p and also to every member of $\widehat{E}(p,q)$ obtained by intersecting a symplecton through pr_1 with a symplecton through the line qs_1. Hence $r_1 \in \widehat{T}(p,q)$. Similarly for r_i, s_i, $i = 1,2,3$. Hence the line $r_i s_i$, $i \in \{1,2,3\}$, also belongs to $\widehat{T}(p,q)$. If a_i runs through $r_i s_i$, then $W_{a_i} := a_i^\perp \cap \widehat{E}(p,q)$ runs through the set of singular (hyperbolic) 3-spaces of $\widehat{E}(p,q)$ containing the singular (hyperbolic) plane $\pi_i := (r_i s_i)^\perp \cap \widehat{E}(p,q)$. If $a_1 \bowtie a_2$, then $\mathfrak{c}(a_1, a_2) = W_{a_1} \cap W_{a_2}$. However, this way we see that, if $a_1 \bowtie a_3$, both $\mathfrak{c}(a, 1, a_2)$ and $\mathfrak{c}(a_1, a_3)$ belong to W_{a_1}, which does not contain special pairs. We conclude $a_1 \bowtie a_2 = a_1 \bowtie a_3$. Hence $R(r_1 s_1, r_2 s_2) = R(r_1 s_1, r_3 s_3)$, implying $\pi_1^{\perp\!\!\perp} \cap \pi_2^{\perp\!\!\perp} = \pi_1^{\perp\!\!\perp} \cap \pi_3^{\perp\!\!\perp}$. If $\mathbb{A}$ is not an inseparable field extension in characteristic 2, then we use Proposition 3.18 to see $\widehat{E}(p,q)$ embedded as a quadric in a projective space $\mathsf{PG}(V)$ such that the underlying vector space V is endowed with a non-degenerate bilinear form β with the property that points $\langle v \rangle$ and $\langle w \rangle$ of $\widehat{E}(p,q)$ are colinear if and only if $\beta(v,w) = 0$. Hence, by non-degeneracy of β, we conclude $\pi_3 \subseteq \langle \pi_1, \pi_2 \rangle$ (generation in $\mathsf{PG}(V)$). However, $\langle \pi_1, \pi_2 \rangle$ intersects the quadric in a Klein quadric, which does not contain three mutually disjoint planes. $\square$

4.6. The non-thick isometric case

We now specialize to non-thick metasymplectic spaces. Since we assume thick lines, these are, as already mentioned, the line Grassmannians of polar spaces of rank 4. The symplecta are polar spaces of rank 3 and the point-perps are degenerate dual polar spaces of rank 3, more exactly, the direct product of a thick line and a generalized quadrangle with thick lines. Embeddings of hexagons in such structures remain completely mysterious, and are presumably wild.

Now suppose the embedding of the hexagon $\Gamma = (P, \mathscr{L})$ is isometric in the line Grassmannian of the polar space $\Delta^* = (X^*, \mathscr{M}^*)$. Let $x \perp y \perp z \perp u$ be a path in Γ with $\delta(x,u) = 3$. The line xy of Γ corresponds to a planar line pencil that we denote by $\Pi(p, \alpha)$, where p is the vertex of the pencil and α the plane. Since the embedding is isometric with distance 2 in Γ corresponding to special in the metasymplectic space, the line $yz \in \mathscr{L}$ corresponds to a line pencil $\Pi(q, \beta)$, with $p \neq q$ and $\alpha \neq \beta$, but of course $q \in \alpha$ and $p \in \beta$. Then the line zu corresponds to a line pencil $\Pi(r, \gamma)$, with $r \in \beta$ and $q \in \gamma$. It follows that $qr = p^\perp \cap \gamma$. Now u corresponds to a line L of γ through r distinct from qr (the latter corresponds to z). A line of Γ through u distinct from zu corresponds to a line pencil with vertex s in $\gamma \setminus qr$. This means that p and s are not collinear, but p and both q and r are. This implies that, if we identify each line of Γ with the vertex of the corresponding line pencil, then we obtain a flat and polarized, hence regular (with the terminology of [13]) but not necessarily full embedding of the dual Γ^* of Γ in any ambient projective space of Δ^* (note that Δ^* indeed admits a projective embedding as it has rank 4). Since the line pencils are full, the classification of regularly embedded hexagons in [13] implies that Γ^* is isomorphic to the split Cayley hexagon $\mathsf{G}_{2,2}(\mathbb{K})$ for some field $\mathbb{K}$, and hence Γ is isomorphic to $\mathsf{G}_{2,1}(\mathbb{K})$. Also, the polar space Δ^* is always isomorphic to $\mathsf{B}_{3,1}(\mathbb{K}, \mathbb{K})$, our notation for the parabolic polar space of rank 3 (a parabolic quadric in $\mathsf{PG}(6, \mathbb{K})$). Hence we have shown:

Proposition 4.17. *Let Γ be a thick generalized hexagon fully and isometrically embedded in the non-thick metasymplectic space Δ. Then Γ is isomorphic to $\mathsf{G}_{2,1}(\mathbb{K})$ for some field $\mathbb{K}$, and Δ is the line Grassmannian of $\mathsf{B}_{3,1}(\mathbb{K})$.*

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