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ORDER, TYPE AND ZEROS OF ANALYTIC AND MEROMORPHIC FUNCTIONS OF $[p,q]-\phi$ ORDER IN THE UNIT DISC

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Abstract. In this paper, we investigate the $[p,q]-\phi$ order and $[p,q]-\phi$ type of f_1+f_2, f_1f_2 and $\frac{f_1}{f_2}$, where f_1 and f_2 are analytic or meromorphic functions with the same $[p,q]-\phi$ order and different $[p,q]-\phi$ type in the unit disc. Also, we study the $[p,q]-\phi$ order and $[p,q]-\phi$ type of different f and its derivative. At the end, we investigate the relationship between two different $[p,q]-\phi$ convergence exponents of f. We extend some earlier precedent well known results.

1. Introduction and Preliminaries

We use $\mathbb{C}$ to denote the complex plane and $\Delta=\{z:|z|<1\}$ to denote the unit disc of the complex plane $\mathbb{C}$. By a meromorphic function f, we mean a meromorphic function in the complex plane or a meromorphic function in the unit disc. Let us consider that readers are familiar with the fundamental results and the standard notations of the Nevanlinna theory in the complex plane or in the unit disc which are available in [5, 11, 12, 19, 21, 22]. In addition, let us recall some notations such as $m(r,f)$ and $N(r,f)$. Let $n(r,f)$ be the number of poles of a function f (counting multiplicities) in $|z|\le r$. Then we define the integrated counting function $N(r,f)$ by

$$N(r,f)=\int_0^r \frac{n(t,f)-n(0,f)}{t}dt+n(0,f)\log r,$$

and we define the proximity function $m(r,f)$ by

$$m(r,f)=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f\left(re^{i\phi}\right)\right|d\phi,$$

where $\log^+ x=\max\{0,\log x\}$. We should think of $m(r,f)$ as a measure that how close f is to infinity on $|z|=r$. Nevertheless, within that context, we recall that $T(r,f)$ stands for the Nevanlinna characteristic function of the meromorphic function f defined on each positive real value r by

$$T(r,f)=m(r,f)+N(r,f).$$

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And $M(r,f)$ stands for the so called maximum modulus function defined for each non-negative real value r by

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

In order to describe the growth of order of entire or meromorphic functions more precisely, we use some notations for $r \in [0,\infty)$, $\exp_1 r = e^r$ and $\exp_{p+1} r = \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{p+1} r = \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Also $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$ and $\log_{-1} r = \exp_1 r$. Moreover, we denote the linear measure for a set $E \subset [0,\infty)$, by $mE = \int_E dt$ and logarithmic measure for a set $E \subset (1,\infty)$, by $m_l E = \int_E \frac{dt}{t}$.

2. Preliminaries

Throughout this paper we assume that p, q always denote positive integers with $1 \le q \le p$. Now, we recall the following well known definitions.

Definition 2.1. [18] The order $\rho(f)$ and the lower order $\lambda(f)$ of a meromorphic function f in Δ is defined by

$$\rho(f) = \limsup_{r\to1^-} \frac{\log T(r,f)}{\log(\frac{1}{1-r})}, \qquad \lambda(f) = \liminf_{r\to1^-} \frac{\log T(r,f)}{\log(\frac{1}{1-r})}.$$

For an analytic function f in Δ, we define the order $\rho_M(f)$ and the lower order $\lambda_M(f)$ in Δ is defined by

$$\rho_M(f) = \limsup_{r\to1^-} \frac{\log(\log M(r,f))}{\log(\frac{1}{1-r})}, \qquad \lambda_M(f) = \liminf_{r\to1^-} \frac{\log(\log M(r,f))}{\log(\frac{1}{1-r})}.$$

Definition 2.2. [3], [8] Let f be an meromorphic function in Δ. Then, the iterated p-order $\rho_p(f)$ and iterated p-lower order $\lambda_p(f)$ of f is defined by

$$\rho_p(f) = \limsup_{r\to1^-} \frac{\log_p T(r,f)}{\log(\frac{1}{1-r})}, \qquad \lambda_p(f) = \liminf_{r\to1^-} \frac{\log_p T(r,f)}{\log(\frac{1}{1-r})}.$$

For an analytic function f in Δ, we define the iterated p-order $\rho_{M,p}(f)$ and the iterated p-lower order $\lambda_{M,p}(f)$ in Δ is defined by

$$\rho_{M,p}(f) = \limsup_{r\to1^-} \frac{\log_{p+1} M(r,f)}{\log(\frac{1}{1-r})}, \qquad \lambda_{M,p}(f) = \liminf_{r\to1^-} \frac{\log_{p+1} M(r,f)}{\log(\frac{1}{1-r})}.$$

Remark 2.3. For $p=1$, we have $\rho_1(f) = \rho(f)$. If f is an analytic function in Δ, then we have $\rho(f) \le \rho_M(f) \le \rho(f)+1$([19]) and $\rho_{M,p}(f) = \rho_p(f)$ for $p \ge 2$([11]).

Definition 2.4. [2] Let f be a meromorphic function in Δ, then the $[p,q]$- order and $[p,q]$- lower order of f are respectively defined by

$$\rho_{[p,q]}(f) = \limsup_{r\to1^-} \frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})}, \qquad \lambda_{[p,q]}(f) = \liminf_{r\to1^-} \frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})}.$$

If f is an analytic function in Δ, then the $[p,q]$- order and $[p,q]$- lower order of f are respectively defined by

$$\rho_{M,[p,q]}(f) = \limsup_{r\to1^-} \frac{\log_{p+1} M(r,f)}{\log_q(\frac{1}{1-r})}, \qquad \lambda_{M,[p,q]}(f) = \liminf_{r\to1^-} \frac{\log_{p+1} M(r,f)}{\log_q(\frac{1}{1-r})}.$$

Definition 2.5. [2] Let f be a meromorphic function in Δ with $0 < \rho_{[p,q]}(f) = \alpha_1 < \infty$. Then the $[p,q]$- type of f is defined by

$$\tau_{[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p-1} T(r,f)}{[\log_{q-1}(\frac{1}{1-r})]^{\alpha_1}}.$$

For an analytic function f in Δ, and the $[p,q]$- type about maximum modulus of f of $[p,q]$-order with $0 < \rho_{M,[p,q]}(f) = \alpha_2 < \infty$ is defined by

$$\tau_{M,[p,q]}(f) = \limsup_{r\to 1^-} \frac{\log_{p} M(r,f)}{[\log_{q-1}(\frac{1}{1-r})]^{\alpha_2}}.$$

Definition 2.6. Let f be a meromorphic function in Δ with $0 < \lambda_{[p,q]}(f) = \beta_1 < \infty$. Then the lower $[p,q]$- type of f is defined by

$$\underline{\tau}_{[p,q]}(f) = \liminf_{r\to 1^-} \frac{\log_{p-1} T(r,f)}{[\log_{q-1}(\frac{1}{1-r})]^{\beta_1}}.$$

For an analytic function f in Δ, and the lower $[p,q]$- type about maximum modulus of f of $[p,q]$-order with $0 < \lambda_{M,[p,q]}(f) = \beta_2 < \infty$ is defined by

$$\underline{\tau}_{M,[p,q]}(f) = \liminf_{r\to 1^-} \frac{\log_{p} M(r,f)}{[\log_{q-1}(\frac{1}{1-r})]^{\beta_2}}.$$

Remark 2.7. From Definitions 2.5 and 2.6 it is easy to see that $\tau_{[p,q]} \leq \tau_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f)$.

Recently, in 2009, Chyzhykov et al. [4] introduced the idea of ϕ- order of f, where f is an analytic or meromorphic function in the unit disc Δ as follows.

Definition 2.8. [4] Let $\phi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function, the ϕ-order of a meromorphic function f in the unit disc Δ is defined by

$$\rho(f,\phi) = \limsup_{r\to 1^-} \frac{\log T(r,f)}{\log \phi(r)}.$$

In particular, $\rho(f, \frac{1}{1-r})$ is the order of f and $\rho(f, \log\frac{1}{1-r})$ is the logarithmic order of f.
Very recently, in 2014, Shen et al. [16] introduced the idea of the $[p,q]-\phi$ order and $[p,q]-\phi$ lower order of a meromorphic function f in the complex plane. On the basis of Definitions 2.4, 2.8 and the idea of the $[p,q]-\phi$ order and $[p,q]-\phi$ lower order introduced by Shen et al. [16], it is natural for us to define the $[p,q]-\phi$ order and $[p,q]-\phi$ lower order of a meromorphic function f in the unit disc Δ.

Definition 2.9. Let $\phi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function, the $[p,q]-\phi$ order and $[p,q]-\phi$ lower order of a meromorphic function f in a unit disc Δ is defined by respectively

$$\rho_{[p,q]}(f,\phi) = \limsup_{r\to 1^-} \frac{\log_p T(r,f)}{\log_q \phi(r)} \qquad \lambda_{[p,q]}(f,\phi) = \liminf_{r\to 1^-} \frac{\log_p T(r,f)}{\log_q \phi(r)}.$$

For an analytic function f in Δ, we also defined $[p,q]-\phi$ order and $[p,q]-\phi$ lower order of f respectively by

$$\rho_{M,[p,q]}(f,\phi) = \limsup_{r\to 1^-} \frac{\log_{p+1} M(r,f)}{\log_q \phi(r)} \qquad \lambda_{M,[p,q]}(f,\phi) = \liminf_{r\to 1^-} \frac{\log_{p+1} M(r,f)}{\log_q \phi(r)}.$$

REMARK 2.10. If $\phi(r) = \frac{1}{1-r}$, then $\rho_{[p,q]}(f,\phi) = \rho_{[p,q]}(f)$ and $\lambda_{[p,q]}(f,\phi) = \lambda_{[p,q]}(f)$, where f is a meromorphic function in Δ. Also, $\rho_{M,[p,q]}(f,\phi) = \rho_{M,[p,q]}(f)$ and $\lambda_{M,[p,q]}(f,\phi) = \lambda_{M,[p,q]}(f)$, where f is an analytic function in Δ.

On the basis of Definitions 2.5 and 2.8, it is natural for us to define the $[p,q]-\phi$ type and the $[p,q]-\phi$ lower type of a meromorphic function f in the unit disc Δ.

DEFINITION 2.11. Let $\phi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function, the $[p,q]-\phi$ type of a meromorphic function f in Δ, with $0 < \rho_{[p,q]}(f,\phi) = \alpha_3 < \infty$ is defined by

$$\tau_{[p,q]}(f,\phi) = \limsup_{r\to 1^-} \frac{\log_{p-1} T(r,f)}{[\log_{q-1}\phi(r)]^{\alpha_3}}.$$

If f is an analytic function in Δ, and the $[p,q]$- type with respect to maximum modulus of f of $[p,q]-\phi$ order with $0 < \rho_{M,[p,q]}(f,\phi) = \alpha_4 < \infty$ is defined by

$$\tau_{M,[p,q]}(f,\phi) = \limsup_{r\to 1^-} \frac{\log_{p} M(r,f)}{[\log_{q-1}\phi(r)]^{\alpha_4}}.$$

DEFINITION 2.12. Let $\phi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function, the $[p,q]-\phi$ lower type of a meromorphic function f in Δ, with $0 < \lambda_{[p,q]}(f,\phi) = \beta_3 < \infty$ is defined by

$$\underline{\tau}_{[p,q]}(f,\phi) = \liminf_{r\to 1^-} \frac{\log_{p-1} T(r,f)}{[\log_{q-1}\phi(r)]^{\beta_3}}.$$

If f is an analytic function in Δ and the $[p,q]-\phi$ lower type with respect to maximum modulus of f of $[p,q]-\phi$ order with $0 < \lambda_{M,[p,q]}(f,\phi) = \beta_4 < \infty$ is defined by

$$\underline{\tau}_{M,[p,q]}(f,\phi) = \liminf_{r\to 1^-} \frac{\log_{p} M(r,f)}{[\log_{q-1}\phi(r)]^{\beta_4}}.$$

DEFINITION 2.13. Let $\phi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function and $a \in \mathbb{C}\cup\{\infty\}$, we use $n(r,\frac{1}{f-a})$ to denote the unintegrated counting function for a sequence of a- point of a meromorphic function f in Δ, then the $[p,q]-\phi$ exponents of convergence of a-points of $f(z)$ about $n(r,\frac{1}{f-a})$ is defined by

$$\lambda^{n}_{[p,q]}(f,\phi,a) = \limsup_{r\to 1^-} \frac{\log_p n(r,\frac{1}{f-a})}{\log_q \phi(r)}.$$

DEFINITION 2.14. Let $\phi : [0,1) \to (0,\infty)$ be a non-decreasing unbounded function and $a \in \mathbb{C}\cup\{\infty\}$, let $N(r,\frac{1}{f-a})$ be the integrated counting function for a sequence of a- point of a meromorphic function f in Δ, then the $[p,q]-\phi$ exponents of convergence of a-points of $f(z)$ about $N(r,\frac{1}{f-a})$ is defined by

$$\lambda^{N}_{[p,q]}(f,\phi,a) = \limsup_{r\to 1^-} \frac{\log_p N(r,\frac{1}{f-a})}{\log_q \phi(r)}.$$

In the value distribution theory of entire or meromorphic functions two important growth indicators order and type are relevant in this topic. Many authors have investigated the growth of entire or meromorphic functions in the complex plane or unit disc (see [5], [9], [10], [12], [17], [18], [19], [20], [21], [22]). In the following, we list some classical results related to growth indicators, in the complex plane.

THEOREM 2.15. [5] *If f_1 and f_2 are meromorphic functions of finite order with $\rho(f_1) = \alpha_5$ and $\rho(f_2) = \alpha_6$, then $\rho(f_1 + f_2) \leq \max\{\alpha_5, \alpha_6\}$, $\rho(f_1 f_2) \leq \max\{\alpha_5, \alpha_6\}$, $\rho(\frac{f_1}{f_2}) \leq \max\{\alpha_5, \alpha_6\}$. Moreover, if $\alpha_5 < \alpha_6$, then $\rho(f_1 + f_2) = \rho(f_1 f_2) = \rho(\frac{f_1}{f_2}) = \alpha_6$.*

THEOREM 2.16. [22] *If f_1, f_2 are meromorphic functions of finite order then $\lambda(f_1 + f_2) \leq \min\{\max\{\rho(f_1), \lambda(f_2)\}, \max\{\rho(f_2), \lambda(f_1)\}\}$, $\lambda(f_1 f_2) \leq \min\{\max\{\rho(f_1), \lambda(f_2)\}, \max\{\rho(f_2), \lambda(f_1)\}\}$. Moreover, if $\rho(f_1) < \lambda(f_2)$, then $\lambda(f_1 + f_2) = \lambda(f_1 f_2) = \lambda(f_2)$; or if $\rho(f_2) < \lambda(f_1)$, then $\lambda(f_1 + f_2) = \lambda(f_1 f_2) = \lambda(f_1)$.*

THEOREM 2.17. [12] *If f_1 and f_2 are two entire functions of finite order satisfying $\rho(f_1) = \rho(f_2) = \alpha_7$, then the following hold:*
(i) If $\tau_M(f_1) = 0$ and $0 < \tau_M(f_2) < \infty$, then $\rho(f_1 f_2) = \alpha_7, \tau_M(f_1 f_2) = \tau_M(f_2)$.
(ii) If $\tau_M(f_1) < \infty$ and $0 < \tau_M(f_2) = \infty$, then $\rho(f_1 f_2) = \alpha_7, \tau_M(f_1 f_2) = \infty$.

THEOREM 2.18. [20] *If f_1 and f_2 are two entire functions satisfying $\rho_p(f_1) = \rho_p(f_2) = \alpha_8, 0 \leq \tau_{M,p}(f_1) < \tau_{M,p}(f_2) \leq \infty$, then the following hold:*
(i) If $p \geq 1$, then $\rho_p(f_1 + f_2) = \alpha_8, \tau_{M,p}(f_1 + f_2) = \tau_{M,p}(f_2)$;
(ii) If $p > 1$, then $\rho_p(f_1 f_2) = \alpha_8, \tau_{M,p}(f_1 f_2) = \tau_{M,p}(f_2)$.

THEOREM 2.19. [20] *Let $p \geq 1$ and $f(z)$ be an entire function or a meromorphic function in the complex plane satisfying $0 < \rho_p(f) < \infty$. If $p \geq 1$, then $\rho_p(f) = \rho_p(f')$ and $\tau_{M,p}(f') = \tau_{M,p}(f)$; if $p > 1$, then $\rho_p(f) = \rho_p(f')$ and $\tau_p(f') = \tau_p(f)$.*

From Theorems 2.15-2.19, we can easily obtain the following similar propositions in the unit disc Δ.

Propositions (i) If f_1 and f_2 are meromorphic functions in Δ, satisfying $\rho_{[p,q]}(f_1) = \alpha_9$ and $\rho_{[p,q]}(f_2) = \alpha_{10}$, then $\rho_{[p,q]}(f_1 \pm f_2) \leq max\{\alpha_9, \alpha_{10}\}$, $\rho_{[p,q]}(f_1 f_2) \leq max\{\alpha_9, \alpha_{10}\}$ and $\rho_{[p,q]}(\frac{f_1}{f_2}) \leq max\{\alpha_9, \alpha_{10}\}$. Also, if $\alpha_9 \neq \alpha_{10}$, then $\rho_{[p,q]}(f_1 \pm f_2) = \rho_{[p,q]}(f_1 f_2) = \rho_{[p,q]}(\frac{f_1}{f_2}) = max\{\alpha_9, \alpha_{10}\}$.
(ii) If f_1 and f_2 are meromorphic functions in Δ, then $\lambda_{[p,q]}(f_1 + f_2) \leq max\{\rho_{[p,q]}(f_1), \lambda_{[p,q]}(f_2)\}$ or $\lambda_{[p,q]}(f_1 + f_2) \leq max\{\rho_{[p,q]}(f_2), \lambda_{[p,q]}(f_1)\}$ and $\lambda_{[p,q]}(f_1 f_2) \leq max\{\rho_{[p,q]}(f_1), \lambda_{[p,q]}(f_2)\}$ or $\lambda_{[p,q]}(f_1 f_2) \leq max\{\rho_{[p,q]}(f_2), \lambda_{[p,q]}(f_1)\}$.
(iii) If f_1 and f_2 are meromorphic functions in Δ satisfying $\rho_{[p,q]}(f_1) < \lambda_{[p,q]}(f_2) \leq \infty$, then $\lambda_{[p,q]}(f_1 + f_2) = \lambda_{[p,q]}(f_1 f_2) = \lambda_{[p,q]}(\frac{f_1}{f_2}) = \lambda_{[p,q]}(f_2)$.
(iv) If f_1 and f_2 are analytic functions in Δ satisfying $\rho_{M,[p,q]}(f_1) = \alpha_{11}$ and $\rho_{M,[p,q]}(f_2) = \alpha_{12}$, then $\rho_{M,[p,q]}(f_1 \pm f_2) \leq max\{\alpha_{11}, \alpha_{12}\}$ and $\rho_{M,[p,q]}(f_1 f_2) \leq max\{\alpha_{11}, \alpha_{12}\}$. If $\alpha_{11} \neq \alpha_{12}$, then $\rho_{M,[p,q]}(f_1 \pm f_2) = \rho_{M,[p,q]}(f_1 f_2) = max\{\alpha_{11}, \alpha_{12}\}$.
(v) If f_1 and f_2 are analytic functions in Δ, then $max\{\lambda_{M,[p,q]}(f_1 \pm f_2), \lambda_{M,[p,q]}(f_1 f_2)\} \leq max\{\rho_{M,[p,q]}(f_1), \lambda_{M,[p,q]}(f_2)\}$ or
$max\{\lambda_{M,[p,q]}(f_1 \pm f_2), \lambda_{M,[p,q]}(f_1 f_2)\} \leq max\{\rho_{M,[p,q]}(f_2), \lambda_{M,[p,q]}(f_1)\}$.
(vi) If f_1 and f_2 are analytic functions of $[p,q]$-order in Δ, for any $r \in [0,1)$, by the inequality $T(r,f) \leq log^+ M(r,f) \leq \frac{4}{1-r} T(\frac{1+r}{2}, f)$ ([5], [19]), using this we can easily obtain that if $p = q \geq 2$ and $\rho_{[p,q]}(f) > 1$, or $p > q \geq 1$, then $\rho_{[p,q]}(f) = \rho_{M,[p,q]}(f)$ and $\tau_{[p,q]}(f) = \tau_{M,[p,q]}(f)$. Similarly, we can show that if $p = q \geq 2$ and $\lambda_{[p,q]}(f) > 1$, or $p > q \geq 1$,then $\lambda_{[p,q]}(f) = \lambda_{M,[p,q]}(f)$ and $\underline{\tau}_{[p,q]}(f) = \underline{\tau}_{M,[p,q]}(f)$.

3. Main Results

The main purpose of this paper is to make use of concept of $[p, q] - \phi$ order and $[p, q]-\phi$ type to generalise the above results in unit disc Δ. In this direction we obtain the following results.

THEOREM 3.1. *Let f_1 and f_2 be meromorphic functions in Δ satisfying* $0 < \rho_{[p,q]}(f_1, \phi) = \rho_{[p,q]}(f_2, \phi) = \alpha_{13} < \infty$ *and* $0 \leq \beta_1 = \tau_{[p,q]}(f_1, \phi) < \tau_{[p,q]}(f_2, \phi) = \beta_2 \leq \infty$. *Then* $\rho_{[p,q]}(f_1 + f_2, \phi) = \rho_{[p,q]}(f_1 f_2, \phi) = \rho_{[p,q]}(\frac{f_1}{f_2}, \phi) = \alpha_{13}$ *and the following hold:*
(i) If $p > 1$ *and* $p \geq q \geq 1$, *then* $\tau_{[p,q]}(f_1 + f_2, \phi) = \tau_{[p,q]}(f_1 f_2, \phi) = \tau_{[p,q]}(\frac{f_1}{f_2}, \phi) = \tau_{[p,q]}(f_2, \phi)$.
(ii) If $p = q = 1$, *then* $\beta_2 - \beta_1 \leq max\{\tau(f_1 + f_2, \phi), \tau(f_1 f_2, \phi), \tau(\frac{f_1}{f_2}, \phi)\} \leq \beta_2 + \beta_1$.

THEOREM 3.2. *Let f_1 and f_2 be meromorphic functions in Δ satisfying* $0 < \rho_{[p,q]}(f_1, \phi) = \lambda_{[p,q]}(f_2, \phi) < \infty$ *and* $0 \leq \tau_{[p,q]}(f_1, \phi) < \underline{\tau}_{[p,q]}(f_2, \phi) \leq \infty$. *Then* $\lambda_{[p,q]}(f_1 + f_2, \phi) = \lambda_{[p,q]}(f_1 f_2, \phi)$
$= \lambda_{[p,q]}(\frac{f_1}{f_2}, \phi) = \lambda_{[p,q]}(f_1, \phi)$. *And If* $p > 1$ *and* $p \geq q \geq 1$, *we have* $\underline{\tau}_{[p,q]}(f_1 + f_2, \phi) = \underline{\tau}_{[p,q]}(f_1 f_2, \phi) = \underline{\tau}_{[p,q]}(\frac{f_1}{f_2}, \phi) = \underline{\tau}_{[p,q]}(f_2, \phi)$.

THEOREM 3.3. *Let f_1 and f_2 be analytic functions in Δ satisfying* $0 < \rho_{M,[p,q]}(f_1, \phi) = \rho_{M,[p,q]}(f_2, \phi) = \alpha_{14} < \infty$ *and* $0 \leq \tau_{M,[p,q]}(f_1, \phi) < \tau_{M,[p,q]}(f_2, \phi) \leq \infty$. *Then* $\rho_{M,[p,q]}(f_1 + f_2, \phi) = \alpha_{14}$ *and* $\tau_{M,[p,q]}(f_1 + f_2, \phi) = \tau_{M,[p,q]}(f_2, \phi)$.

THEOREM 3.4. *Let f be an analytic function of $[p, q]-\phi$order in Δ, then* $\rho_{M,[p,q]}(f, \phi) = \rho_{M,[p,q]}(f', \phi)$ *and* $\lambda_{M,[p,q]}(f, \phi) = \lambda_{M,[p,q]}(f', \phi)$. *If* $0 < \rho_{M,[p,q]}(f, \phi) < \infty$ *or* $0 < \lambda_{M,[p,q]}(f, \phi) < \infty$,*then* $\tau_{M,[p,q]}(f, \phi) = \tau_{M,[p,q]}(f', \phi)$, $\underline{\tau}_{M,[p,q]}(f, \phi) = \underline{\tau}_{M,[p,q]}(f', \phi)$.

THEOREM 3.5. *Let f be a meromorphic function of $[p, q] - \phi$ order in Δ, then*
(i) If $p \geq q \geq 1$, *then* $\rho_{[p,q]}(f, \phi) = \rho_{[p,q]}(f', \phi)$, $\lambda_{[p,q]}(f, \phi) = \lambda_{[p,q]}(f', \phi)$. *Also if* $0 < \rho_{[p,q]}(f, \phi) < \infty$ *or* $0 < \lambda_{[p,q]}(f, \phi) < \infty$,
(a) if $p \geq q \geq 2$ *and* $p > 1$, *then* $\tau_{[p,q]}(f, \phi) = \tau_{[p,q]}(f', \phi)$, $\underline{\tau}_{[p,q]}(f, \phi) = \underline{\tau}_{[p,q]}(f', \phi)$.
(b) if $p > q = 1$, *then* $\tau_{[p,q]}(f, \phi) = \tau_{[p,q]}(f', \phi)$.
(ii) If $p = q = 1$, *then* $\rho(f, \phi) = \rho(f', \phi)$, $\lambda(f, \phi) = \lambda(f', \phi)$ *and* $\tau_{[1,1]}(f', \phi) \leq 2\tau_{[1,1]}(f, \phi)$, $\underline{\tau}_{[1,1]}(f', \phi) \leq 2\underline{\tau}_{[1,1]}(f, \phi)$.

THEOREM 3.6. *Let f be a meromorphic function of $[p, q]-\phi$order in Δ,$a \in \mathbb{C}\cup\{\infty\}$. Then the following hold:*
(i) If $p > q \geq 1$, *then* $\lambda^N_{[p,q]}(f, \phi, a) = \lambda^n_{[p,q]}(f, \phi, a)$.
(ii) If $p = q = 1$, *then* $\lambda^N(f, \phi, a) \leq \lambda^n(f, \phi, a) + 1$.
(iii)If $p = q \geq 2$, *then* $\lambda^N_{[p,q]}(f, \phi, a) \leq \lambda^n_{[p,q]}(f, \phi, a) \leq max\{\lambda^N_{[p,q]}(f, \phi, a), 1\}$.
Furthermore, we have $\lambda^N_{[p,q]}(f, \phi, a) = \lambda^n_{[p,q]}(f, \phi, a)$ *if* $\lambda^N_{[p,q]}(f, \phi, a) \geq 1$ *and if* $\lambda^N_{[p,q]}(f, \phi, a) < 1$ *then* $\lambda^N_{[p,q]}(f, \phi, a) \leq \lambda^n_{[p,q]}(f, \phi, a) \leq 1$.

REMARK 3.7. The conclusions of Theorem 3.6 also hold between $\overline{\lambda}^n_{[p,q]}(f, \phi, a)$ and $\overline{\lambda}^N_{[p,q]}(f, \phi, a)$.

4. Preliminary Lemmas

To prove the above theorems, we need some lemmas as follows.

Lemma 4.1. [5] *Let f_1, f_2,...,f_m be meromorphic functions in Δ, where $m \geq 2$ is a positive integer. Then*

(i) $T(r, f_1 f_2 \dots f_m) \leq \sum_{i=1}^{m} T(r, f_i)$,

(ii) $T(r, f_1 + f_2 + \cdots + f_m) \leq \sum_{i=1}^{m} T(r, f_i) + \log m$.

Lemma 4.2. [7,19] *Let f be a meromorphic function in Δ, and let $k \geq 1$ be an integer. Then*

$$m(r, \tfrac{f^k}{f}) = S(r, f),$$

where $S(r, f) = O\{\log^+ T(r, f) + \log(\frac{1}{1-r})\}$, *possibly outside a set $E_1 \subset [0, 1)$ with* $\int_{E_1} \frac{dt}{1-t} < \infty$.

Lemma 4.3. [1] *Let $g : (0, 1) \to \mathbb{R}$ and $h : (0, 1) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ holds outside of an exceptional set $E_2 \subset [0, 1)$ for which $\int_{E_2} \frac{dt}{1-t} < \infty$. Then there exists a constant $d \in (0, 1)$ such that if $s(r) = 1 - d(1 - r)$, then $g(r) \leq h(s(r))$ for all $r \in [0, 1)$.*

Lemma 4.4. [6,17] *Suppose that f is meromorphic in Δ with $f(0) = 0$. Then*

$$m(r, f) \leq [1 + \phi(\tfrac{r}{R})]T(r, f') + N(R, f'),$$

where $0 < r < R < 1, \phi(t) = \frac{1}{\pi} \log \frac{1+t}{1-t}, 0 < t < 1$.

5. Proof of Main Results

Proof of Theorem 3.1. Given that $0 \leq \beta_1 = \tau_{[p,q]}(f_1, \phi) < \tau_{[p,q]}(f_2, \phi) = \beta_2 < \infty$, by Definition 2.11, we can write for any given $\epsilon > 0$ and $r \to 1^-$, that

$$T(r, f_1) \leq exp_{p-1} \{(\beta_1 + \epsilon)[log_{q-1}\phi(r)]^{\alpha_{13}}\}, \tag{5.1}$$

$$T(r, f_1) \leq exp_{p-1} \{(\beta_1 + \epsilon)[log_{q-1}\phi(r)]^{\alpha_{13}}\}. \tag{5.2}$$

Using Eqs.(5.1) and (5.2) and using Lemma 4.1 we get

$$T(r, f_1 + f_2) \leq T(r, f_1) + T(r, f_2) + \log 2$$

$$\leq \exp_{p-1} \left\{(\beta_1 + \epsilon)[\log_{q-1} \phi(r)]^{\alpha_{13}}\right\} + \exp_{p-1} \left\{(\beta_2 + \epsilon)[\log_{q-1} \phi(r)]^{\alpha_{13}}\right\} + \log 2$$

$$\leq 2 \exp_{p-1} \left\{(\beta_2 + \epsilon)[\log_{q-1} \phi(r)]^{\alpha_{13}}\right\} + \log 2. \tag{5.3}$$

Therefore

$$\rho_{[p,q]}(f_1 + f_2, \phi) \leq \alpha_{13}. \tag{5.4}$$

Similarly, for any given $\epsilon > 0$, there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \to 1^-$ satisfying

$$T(r_n, f_1) \leq \exp_{p-1} \left\{(\beta_1 + \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{13}}\right\} \tag{5.5}$$

and

$$T(r_n, f_2) \geq \exp_{p-1} \left\{(\beta_2 - \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{13}}\right\}. \tag{5.6}$$

Again, from Eqs. (5.5) and (5.6) and Lemma 4.1, we get

$$T(r_n, f_1 + f_2) \geq T(r_n, f_2) - T(r_n, f_1) - \log 2$$

(5.7)
$$\geq \exp_{p-1}\left\{(\beta_2-\epsilon)[\log_{q-1}\phi(r_n)]^{\alpha_{13}}\right\} - \exp_{p-1}\left\{(\beta_1+\epsilon)[\log_{q-1}\phi(r_n)]^{\alpha_{13}}\right\} - \log 2.$$

Now by Eq. (5.6), we get

$$\rho_{[p,q]}(f_1+f_2,\phi) \geq \alpha_{13}. \tag{5.8}$$

From Eqs. (5.4) and (5.8) we have,

$$\rho_{[p,q]}(f_1+f_2,\phi) = \alpha_{13}.$$

Moreover, for $p=q=1$, from Eq. (5.3), we can write

$$\tau_{[p,q]}(f_1+f_2,\phi) \leq \beta_1+\beta_2.$$

If $p>1$ and $p \geq q \geq 1$, then

$$\tau_{[p,q]}(f_1+f_2,\phi) \leq \beta_2.$$

For $p=q=1$, from Eq. (5.6), we can get

$$\tau_{[p,q]}(f_1+f_2,\phi) \geq \beta_2-\beta_1$$

and for $p>1$, $p \geq q \geq 1$, we have

$$\tau_{[p,q]}(f_1+f_2,\phi) \geq \beta_2.$$

Therefore, for $p=q=1$,

$$\beta_2-\beta_1 \leq \tau_{[p,q]}(f_1+f_2,\phi) \leq \beta_2+\beta_1.$$

For $p>1$ and $p \geq q \geq 1$, then $\tau_{[p,q]}(f_1+f_2,\phi) = \beta_2$.
Since,

$$T(r, f_1\cdot f_2) \leq T(r,f_1)+T(r,f_2)$$
$$T(r, f_1\cdot f_2) \geq T(r,f_2)-T(r,f_1)-o(1)$$

and

$$T(r,\tfrac{1}{f_2}) = T(r,f_2)+o(1),$$

by the above proof, for $p>1$ and $p \geq q \geq 1$, we obtain that

$$\rho_{[p,q]}(f_1+f_2,\phi) = \rho_{[p,q]}(\tfrac{f_1}{f_2},\phi) = \alpha_{13}$$
$$\tau_{[p,q]}(f_1+f_2,\phi) = \tau_{[p,q]}(\tfrac{f_1}{f_2},\phi) = \tau_{[p,q]}(f_2,\phi)$$

If $p=q=1$, then we obtain that

$$\beta_2-\beta_1 \leq \max\{\tau_{[p,q]}(f_1+f_2,\phi), \tau_{[p,q]}(\tfrac{f_1}{f_2},\phi)\} \leq \beta_2+\beta_1.$$

This completes the proof of the theorem. □

Note: Theorem 3.1 also true for $\tau_{[p,q]}(f_2,\phi)=\beta_2=\infty$.

Proof of Theorem 3.2. Suppose that $0 \leq \beta_3 = \tau_{[p,q]}(f_1, \phi) < \underline{\tau}_{[p,q]}(f_2, \phi) = \beta_4 < \infty$. Assume that $\rho_{[p,q]}(f_1, \phi) = \lambda_{[p,q]}(f_2, \phi) = \lambda_1$ and by the Definitions 2.11 and 2.12, it is easy to see that for any given $\epsilon > 0$, there exists a sequence $\{r_n\}_{n\in\mathbb{N}} \to 1^-$ satisfying

$$T(r_n, f_1) < \exp_{p-1}\left\{(\beta_3 + \epsilon)[\log_{q-1}\phi(r_n)]^{\lambda_1}\right\} \tag{5.9}$$

and

$$T(r_n, f_2) < \exp_{p-1}\left\{(\beta_4 + \epsilon)[\log_{q-1}\phi(r_n)]^{\lambda_1}\right\}. \tag{5.10}$$

From Eqs. (5.9) and (5.10) and Lemma 4.1, we get

$$T(r_n, f_1 + f_2) \leq T(r_n, f_1) + T(r_n, f_2) + \log 2$$

$$\leq \exp_{p-1}\left\{(\beta_3 + \epsilon)[\log_{q-1}\phi(r_n)]^{\lambda_1}\right\} + \exp_{p-1}\left\{(\beta_4 + \epsilon)[\log_{q-1}\phi(r_n)]^{\lambda_1}\right\} + \log 2$$

$$\leq 2\exp_{p-1}\left\{(\beta_4 + \epsilon)[\log_{q-1}\phi(r_n)]^{\lambda_1}\right\} + \log 2. \tag{5.11}$$

Therefore,

$$\lambda_{[p,q]}(f_1 + f_2, \phi) \leq \lambda_1.$$

On the other hand, for any given $\epsilon > 0$ and $r \to 1^-$, we have

$$T(r, f_1) \leq \exp_{p-1}\left\{(\beta_3 + \epsilon)[\log_{q-1}\phi(r)]^{\lambda_1}\right\} \tag{5.12}$$

and

$$T(r, f_2) \geq \exp_{p-1}\left\{(\beta_4 - \epsilon)[\log_{q-1}\phi(r)]^{\lambda_1}\right\}. \tag{5.13}$$

Again, from Eqs.(5.12) and (5.13) and Lemma 4.1, we get

$$T(r, f_1 + f_2) \geq T(r, f_2) - T(r, f_1) - \log 2$$

$$\geq \exp_{p-1}\left\{(\beta_4 - \epsilon)[\log_{q-1}\phi(r)]^{\lambda_1}\right\} - \exp_{p-1}\left\{(\beta_3 + \epsilon)[\log_{q-1}\phi(r)]^{\lambda_1}\right\} - \log 2. \tag{5.14}$$

Hence by Eq. (5.14), we have $\lambda_{[p,q]}(f_1 + f_2, \phi) \geq \lambda_1$. Therefore, we obtain that $\lambda_{[p,q]}(f_1 + f_2, \phi) = \lambda_1$.

Moreover, for $p > 1$ and $p \geq q \geq 1$, by Eq. (5.11) we have, $\underline{\tau}_{[p,q]}(f_1 + f_2, \phi) \leq \beta_4$ and by Eq. (5.14), we get $\underline{\tau}_{[p,q]}(f_1 + f_2, \phi) \geq \beta_4$ for $p > 1, p \geq q \geq 1$.

Thus for $p > 1,\ p \geq q \geq 1$, we have $\underline{\tau}_{[p,q]}(f_1 + f_2, \phi) = \underline{\tau}_{[p,q]}(f_2, \phi)$.

Again since,

$$T(r, f_1 f_2) \leq T(r, f_1) + T(r, f_2)$$

$$T(r, f_1 f_2) \geq T(r, f_2) - T(r, f_1) - o(1)$$

and

$$T(r, \tfrac{1}{f_2}) = T(r, f_2) + o(1).$$

Therefore, by the above proof for $p > 1$ and $p \geq q \geq 1$, we have

$$\lambda_{[p,q]}(f_1 f_2, \phi) = \lambda_{[p,q]}(\tfrac{f_1}{f_2}, \phi) = \lambda_{[p,q]}(f_2, \phi)$$

and

$$\underline{\tau}_{[p,q]}(f_1 f_2, \phi) = \underline{\tau}_{[p,q]}(\tfrac{f_1}{f_2}, \phi) = \underline{\tau}_{[p,q]}(f_2, \phi).$$

This proves the theorem. □

Note: The above Theorem 3.2 true for $\tau_{[p,q]}(f_1, \phi) < \underline{\tau}_{[p,q]}(f_2, \phi) = \infty$.

Proof of Theorem 3.3. Let $0 \leq \beta_5 = \tau_{M,[p,q]}(f_1, \phi) < \tau_{M,[p,q]}(f_2, \phi) = \beta_6 < \infty$, by the Definition 2.11, for any given $\epsilon(0 < 2\epsilon < \beta_6 - \beta_5$, there exists a sequence $\{r_n\}_{n \in \mathbb{N}} \to 1^-$ satisfying

$$M(r_n, f_1) \leq \exp_p \left\{(\beta_5 + \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{14}}\right\} \tag{5.15}$$

and

$$M(r_n, f_2) > \exp_p \left\{(\beta_6 - \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{14}}\right\}. \tag{5.16}$$

We can choose a sequence $\{z_n\}_{n=1}^{\infty}$ satisfying $|z_n| = r_n(n = 1, 2, \cdots)$ and $|f_2(z_n)| = M(r_n, f_2)$, then using Eqs. (5.15) and (5.16), we have

$$M(r_n, f_1 + f_2) \geq |f_1(z_n) + f_2(z_n)| \geq |f_2(z_n) - f_1(z_n)| \geq M(r_n, f_2) - M(r_n, f_1).$$

So, $M(r_n, f_1 + f_2) \geq \exp_p \left\{(\beta_6 - \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{14}}\right\} - \exp_p \left\{(\beta_6 - \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{14}}\right\}$ $\geq \frac{1}{2} \exp_p \left\{(\beta_6 - \epsilon)[\log_{q-1} \phi(r_n)]^{\alpha_{14}}\right\}$, for $r_n \to \infty$. Hence, $\rho_{M,[p,q]}(f_1 + f_2, \phi) \geq \alpha_{14}$ and $\tau_{M,[p,q]}(f_1 + f_2, \phi) \geq \beta_6$.
On the other hand, we have

$$\begin{gathered} M(r, f_1 + f_2) \leq M(r, f_1) + M(r, f_2) \leq \\ \exp_p \left\{(\beta_5 + \epsilon)[\log_{q-1} \phi(r)]^{\alpha_{14}}\right\} + \exp_p \left\{(\beta_6 + \epsilon)[\log_{q-1} \phi(r)]^{\alpha_{14}}\right\} \\ \leq 2 \exp_p \left\{(\beta_6 + \epsilon)[\log_{q-1} \phi(r)]^{\alpha_{14}}\right\}. \end{gathered}$$

Therefore, we have $\rho_{M,[p,q]}(f_1 + f_2, \phi) \leq \alpha_{14}$ and $\tau_{M,[p,q]}(f_1 + f_2, \phi) \leq \beta_6$. Thus we can get $\rho_{M,[p,q]}(f_1 + f_2, \phi) = \alpha_{14}$ and $\tau_{M,[p,q]}(f_1 + f_2, \phi) = \beta_6$. This completes the proof of the theorem. □

Note: Theorem 3.3 also true for $\tau_{M,[p,q]}(f_1, \phi) < \tau_{M,[p,q]}(f_2, \phi) = \infty$.

Proof of Theorem 3.4. Since f is an analytic function in the unit disc Δ, and $f(z) = f(0) + \int_0^z f'(\xi) d\xi$, where $(|z| = r < 1)$, thus we have

$$\begin{aligned} M(r, f) &\leq |f(0)| + |\int_0^z f'(\xi) d\xi| \\ &\leq |f(0)| + rM(r, f') \\ &\leq |f(0)| + M(r, f') \end{aligned}$$

i.e.,

$$M(r, f') \geq M(r, f) - |f(0)|. \tag{5.17}$$

Using Definition 2.11 and Eq. (5.17), we obtain

$$\rho_{M,[p,q]}(f', \phi) \geq \rho_{M,[p,q]}(f, \phi)$$

and

$$\lambda_{M,[p,q]}(f', \phi) \geq \lambda_{M,[p,q]}(f, \phi).$$

On the other hand, we take a point z_0 in the circle $|z| = r \in (0, 1)$, satisfying $|f'(z_0)| = M(r, f')$. Again, by the Cauchy integral formula for derivatives, we know that

$$f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z_0)^2},$$

where $C_r = \{\xi : |\xi - z_0| = s(r) - r\}$ and $s(r) = 1 - d(1-r), d \in (0,1)$. So,

$$\begin{aligned} M(r, f') &= |f'(z_0)| \\ &= |\frac{1}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z_0)^2} d\xi| \\ &= \frac{1}{2\pi}|\int_0^{2\pi} \frac{f(z_0 + s(r)e^{i\theta})}{(s(r))^2 e^{2i\theta}} s(r) i e^{i\theta} d\theta| \\ &\le \frac{1}{2\pi}\int_0^{2\pi} |\frac{f(z_0 + s(r)e^{i\theta})}{s(r)}| d\theta \\ &= \frac{1}{2\pi}\frac{1}{s(r)}\int_0^{2\pi} |f(z_0 + s(r)e^{i\theta})| d\theta \\ &\le \frac{1}{s(r)} M(s(r), f). \end{aligned}$$

Therefore, we have

$$M(r, f') \le \frac{M(s(r), f)}{1 - d(1-r)}. \tag{5.18}$$

Now by Eq. (5.18), we get

$$\rho_{M,[p,q]}(f', \phi) \le \rho_{M,[p,q]}(f, \phi) \text{ and } \lambda_{M,[p,q]}(f', \phi) \le \lambda_{M,[p,q]}(f, \phi).$$

Hence, we have

$$\rho_{M,[p,q]}(f, \phi) = \rho_{M,[p,q]}(f', \phi) \text{ and } \lambda_{M,[p,q]}(f, \phi) = \lambda_{M,[p,q]}(f', \phi), \tag{5.19}$$

if $0 < \rho_{M,[p,q]}(f, \phi) < \infty$. Again, by Eqs. (5.17) and (5.19), we can get $\tau_{M,[p,q]}(f', \phi) \ge \tau_{M,[p,q]}(f, \phi)$.

If $p \ge q = 1$, then using Eqs. (5.18) and (5.19), we can obtain that

$$\frac{\log_p M(r, f')}{(\phi(r))^{\rho_{M,[p,1]}(f',\phi)}} \le \max\{\frac{\log_p \frac{1}{1-d(1-r)}}{(\phi(r))^{\rho_{M,[p,1]}(f,\phi)}}, \frac{\log_p M(s(r), f)}{(\phi(r))^{\rho_{M,[p,1]}(f,\phi)}}\}.$$

Let $d \to 1^-$, then $s(r) \to r$, therefore $\tau_{M,[p,q]}(f', \phi) \le \tau_{M,[p,q]}(f, \phi)$, then $\tau_{M,[p,q]}(f, \phi) = \tau_{M,[p,q]}(f', \phi)$. If $p \ge q \ge 2$, then

$$\frac{\log_p M(r, f')}{[\log_{q-1}(\phi(r))]^{\rho_{M,[p,q]}(f',\phi)}} \le \max\{\frac{\log_p \frac{1}{1-d(1-r)}}{[\log_{q-1}(\phi(r))]^{\rho_{M,[p,q]}(f,\phi)}}, \frac{\log_p M(s(r), f)}{[\log_{q-1}(\phi(r))]^{\rho_{M,[p,q]}(f,\phi)}}\}$$

thus we have $\tau_{M,[p,q]}(f', \phi) \le \tau_{M,[p,q]}(f, \phi)$. Therefore, we get $\tau_{M,[p,q]}(f, \phi) = \tau_{M,[p,q]}(f', \phi)$.

If $0 < \lambda_{M,[p,q]}(f, \phi) < \infty$ then in similar way we can obtain $\underline{\tau}_{M,[p,q]}(f, \phi) = \underline{\tau}_{M,[p,q]}(f', \phi)$. This proves the theorem. □

Proof of Theorem 3.5. By Lemma 4.2, we get

$$\begin{aligned} T(r,f') &= m(r,f') + N(r,f') \\ &\le m(r,f) + m(r,\frac{f'}{f}) + 2N(r,f) \\ &\le 2T(r,f) + m(r,\frac{f'}{f}) \\ &= 2T(r,f) + S(r,f) \\ &= 2T(r,f) + O(log^{+}T(r,f) + log(\frac{1}{1-r})) \qquad (r \notin E_1 \subset [0,1)) \end{aligned}$$

Therefore, we get

$$T(r,f') \le (2+\epsilon)T(r,f) + O(\log(\frac{1}{1-r})). \tag{5.20}$$

From (5.20) and using Lemma 4.3, we have
If $p \ge q \ge 1$ then we have $\rho_{[p,q]}(f',\phi) \le \rho_{[p,q]}(f,\phi)$ and $\lambda_{[p,q]}(f',\phi) \le \lambda_{[p,q]}(f,\phi)$.
If $p > 1$, then we have $\tau_{[p,q]}(f',\phi) \le \tau_{[p,q]}(f,\phi)$ and $\underline{\tau}_{[p,q]}(f',\phi) \le \underline{\tau}_{[p,q]}(f,\phi)$.
If $p = q = 1$, then we have $\tau_{[1,1]}(f',\phi) \le 2\tau_{[1,1]}(f,\phi)$ and $\underline{\tau}_{[1,1]}(f',\phi) \le \underline{\tau}_{[p,q]}(f,\phi)$.

Let $R = s(r) = 1 - d(1-r), d \in (0,1)$ in Lemma 4.4, we have

$$\begin{aligned} T(r,f) &= m(r,f) + N(r,f) \\ &\le [1 + \phi(\frac{r}{R})]T(r,f') + N(R,f') + N(r,f) \\ &= [1 + \frac{1}{\pi}log\frac{1+\frac{r}{1-d(1-r)}}{1-\frac{r}{1-d(1-r)}}]T(s(r),f') + N(s(r),f') + N(r,f) \\ &= [1 + \frac{1}{\pi}log\frac{1-d+dr+r}{1-d+dr-r}]T(s(r),f') + N(s(r),f') + N(r,f) \\ &\le [1 + \frac{1}{\pi}log\frac{1-d+dr+r}{1-d+dr-r}]T(s(r),f') + T(s(r),f') + N(r,f) \\ &< [2 + \frac{1}{\pi}log\frac{3}{(1-d)(1-r)}]T(s(r),f') + N(r,f). \end{aligned}$$

Therefore, we obtain that

$$T(r,f) < [2 + \frac{1}{\pi}log\frac{3}{(1-d)(1-r)}]T(s(r),f') + N(r,f). \tag{5.21}$$

By similar proof of Theorem 3.4 and Eq. (5.21), we can obtain the following statements.
If $p \ge q \ge 1$, then we have $\rho_{[p,q]}(f',\phi) \ge \rho_{[p,q]}(f,\phi)$ and $\lambda_{[p,q]}(f',\phi) \ge \lambda_{[p,q]}(f,\phi)$.
If $p \ge q \ge 2$, then we have $\tau_{[p,q]}(f',\phi) \ge \tau_{[p,q]}(f,\phi)$ and $\underline{\tau}_{[p,q]}(f',\phi) \ge \underline{\tau}_{[p,q]}(f,\phi)$.
If $p > q = 1$, then we have $\tau_{[p,q]}(f,\phi) \le \tau_{[p,q]}(f',\phi)$.
Therefore the following statements hold:
If $p \ge q \ge 1$, then $\rho_{[p,q]}(f,\phi) = \rho_{[p,q]}(f',\phi)$, $\lambda_{[p,q]}(f,\phi) = \lambda_{[p,q]}(f',\phi)$.
If $0 < \rho_{[p,q]}(f,\phi) < \infty$ or $0 < \lambda_{[p,q]}(f,\phi) < \infty$,
(a) if $p \ge q \ge 2$ and $p > 1$, then $\tau_{[p,q]}(f,\phi) = \tau_{[p,q]}(f',\phi)$, $\underline{\tau}_{[p,q]}(f,\phi) = \underline{\tau}_{[p,q]}(f',\phi)$.
(b) if $p > q = 1$, then $\tau_{[p,q]}(f,\phi) = \tau_{[p,q]}(f',\phi)$.

For $p=q=1$, then $\rho(f,\phi)=\rho(f',\phi)$, $\lambda(f,\phi)=\lambda(f',\phi)$ and $\tau_{[1,1]}(f',\phi)\leq 2\tau_{[1,1]}(f,\phi)$, $\underline{\tau}_{[1,1]}(f',\phi)\leq \underline{\tau}_{[p,q]}(f,\phi)$. □

Proof of Theorem 3.6. Without any loss of generality, assume that $f(a)\neq 0$, by $N\left(r,\frac{1}{f-a}\right)=\int_0^r \frac{n\left(t,\frac{1}{f-a}\right)-n\left(0,\frac{1}{f-a}\right)}{t}dt \quad (0<r<1)$,
we have
(5.22)

$$n\left(r,\frac{1}{f-a}\right)\leq \frac{1}{\log\left(1+\frac{1-r}{2r}\right)}\int_r^{r+\frac{1-r}{2}}\frac{n\left(t,\frac{1}{f-a}\right)}{t}dt\leq \frac{1}{\log\left(1+\frac{1-r}{2r}\right)}N\left(\frac{1+r}{2},\frac{1}{f-a}\right).$$

where $0<r<1, \log\left(1+\frac{1-r}{2r}\right)\sim\frac{1-r}{2r}$ as $r\to 1^-$. Using Eq. (5.22), we have
(5.23)

$$\limsup_{r\to 1^-}\frac{\log_p n\left(r,\frac{1}{f-a}\right)}{\log_q\phi(r)}\leq \max\left\{\limsup_{r\to 1^-}\frac{\log_p N\left(\frac{1+r}{2},\frac{1}{f-a}\right)}{\log_q\phi(r)},\limsup_{r\to 1^-}\frac{\log_p\left(\frac{2r}{1-r}\right)}{\log_q\phi(r)}\right\}.$$

By Eq. (5.23), we can obtain that

if $p>q\geq 1$, then $\lambda_{[p,q]}^{n}(f,\phi,a)\leq\lambda_{[p,q]}^{N}(f,\phi,a)$;
if $p=q=1$, then $\lambda_{[p,q]}^{n}(f,\phi,a)\leq\lambda_{[p,q]}^{N}(f,\phi,a)+1$;
if $p=q\geq 2$, then $\lambda_{[p,q]}^{n}(f,\phi,a)\leq\max\left\{\lambda_{[p,q]}^{N}(f,\phi,a),1\right\}$.

Also,
(5.24)

$$N\left(r,\frac{1}{f-a}\right)=\int_{r_0}^r\frac{n\left(t,\frac{1}{f-a}\right)}{t}dt+N\left(r_0,\frac{1}{f-a}\right)\leq n\left(r,\frac{1}{f-a}\right)\log\left(\frac{r}{r_0}\right)+O(1)$$

where $0<r_0<r<1$. Using Eq. (5.24), we can get that

if $p>q\geq 1$, then $\lambda_{[p,q]}^{N}(f,\phi,a)\leq\lambda_{[p,q]}^{n}(f,\phi,a)$;
if $p=q=1$, then $\lambda_{[p,q]}^{N}(f,\phi,a)\leq\lambda_{[p,q]}^{n}(f,\phi,a)$;
if $p=q\geq 2$, then $\lambda_{[p,q]}^{N}(f,\phi,a)\leq\lambda_{[p,q]}^{n}(f,\phi,a)$.

Therefore, the results hold.

□

6. Future Aspects

Keeping in mind the results already established, one may apply this results to study the growth of solutions of linear differential equation having meromorphic coefficient of $[p,q]-\phi$ order in unit disc. This is still a virgin domain for the new researchers and therefore it may be posed as an open problem to the future workers of this branch.

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