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BEST PROXIMITY POINTS FOR CONTRACTIVE MAPPINGS IN GENERALIZED MODULAR METRIC SPACES

V. Anbukkarasi*, M. Marudai, and R. Theivaraman

Abstract. In this paper, we prove existence of best proximity points for 2-convex contraction, 2-sided contraction, and M-weakly cyclic 2-convex contraction mappings in the setting of complete strongly regular generalized modular metric spaces that generalize many results in the literature.

1. Introduction

Istratescu [6] introduced the concepts of 2-convex contraction and 2-sided convex contraction mappings and proved the fixed point theorem for 2-convex contraction and 2-sided convex contraction mappings. M. Menaka [2] proved the fixed point theorem for M-weakly cyclic 2-convex contraction. In this paper, we prove best proxomity point theorems for 2-convex contraction, 2-sided convex contraction and M-weakly cyclic 2 -convex contraction mappings in complete strongly regular generalized modular metric spaces. C. Alaca et al. (refer, [1], [2]) have proved some new fixed-point theorems on modular metric spaces and modular ultrametric spaces. Also, they have found some innovative applications in homotopy. The authors, M. E. Ege et al. (see, [5], [6]) have discussed some properties of both the modular S-metric spaces and the modular b-metric spaces and proved fixed point theorems from them. In [8] & [9] , the authors invented the Meir-Keeler type contractions in extended modular b-metric spaces with an application and Meir-Keeler type contractive mappings in modular and partial modular metric spaces. Moreover M. Ramezani [14] proved a new version of Schauder and Petryshyn type fixed point theorems in s-modular function spaces. In 2020,(see in [1], [12]), the authours dicussed and explained the concept about Some fixed point theorems for mappings satisfying rational inequality in modular metric spaces with applications and fixed points of Kannan maps in modular metric spaces. Our results generalize many results in the literature.

This manuscript is laid out as follows. The very next section compiles the preliminaries and essential definitions. The third section presents the main results, which includes Best Proximity Points for Contractive Mappings in Generalized Modular Metric Spaces . Finally, we come to the end of the manuscript with a conclusion.

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$*$ Corresponding author.

2. Preliminaries

Let X be a nonempty set and let R^m be the set of all $m \times 1$ matrices with real entries. If $\alpha, \beta \in R^m$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)^T$, $\beta = (\beta_1, \beta_2, ..., \beta_m)^T$ and $c \in R$, then define $\leq$ on R^M by $\alpha \leq \beta$ ($\alpha < \beta$) iff $\alpha_i \leq \beta_i$ $(\alpha_i < \beta_i)$ for each $i \in \{1, 2, ..., m\}$ and by $\alpha \leq c$, iff $\alpha_i \leq c$ for each $i \in \{1, 2, ..., m\}$.

A mapping $d : X \times X \to R^m$ is said to be a vector-valued / generalized metric on X if the following properties are satisfied :

1. $d(x, y) \geq 0$ for all $x, y \in X$; if $d(x, y) = 0$, then $x = y$, vice-versa ;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

A set X together with a vector-valued / generalized metric d is called a vector-valued / generalized metric space and it is denoted by (X,d).
Throughout this paper we denote

1. $R_+ = [0, \infty)$;
2. $M_{m\times m}(R_+) =$ The set of all $m \times m$ matrices with non-negative elements
3. $D_{m\times m}([0, 1)) =$ The set of all $m \times m$ diagonal matrix with elements $\in [0, 1)$
4. Zero $m \times m$ matrix $= \overline{0}$
5. Identity $m \times m$ matrix =I
6. $A^0 = I$ if $A \neq \overline{0}$.

A matrix A is said to be convergent to zero if and only if $A^n \to \overline{0}$ as $n \to \infty$.
Following are some matrices which converges to zero :

1. Any matrix $A = \begin{pmatrix} b & b \\ a & a \end{pmatrix}$, where $a, b \in R_+$ and $a + b < 1$.
2. Any matrix $A = \begin{pmatrix} b & a \\ b & a \end{pmatrix}$, where $a, b \in R_+$ and $a + b < 1$.
3. Any matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, where $a, b, c \in R_+$ and $\max\{a, c\} < 1$.
4. If $A \in D_{m\times m}([0, 1))$, then A converges to $\overline{0}$.

If $A \in D_{m\times m}([0, 1))$, then $A \leq I$ and A converges to zero.

Theorem 2.1. *Let $A \in M_{m\times m}(R_+)$. The following statements are equivalent.*

1. *A is convergent to zero.*
2. *$A^n \to \overline{0}$ as $n \to \infty$.*
3. *The eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in C$ with $det(A - \lambda I) = 0$.*
4. *The matrix I-A is nonsingular and $(I - A)^{-1} = I + A + ... + A^n + ...$*

Definition 2.2. A function $\omega : (0, \infty) \times X \times X \to [0, \infty)$ is said to be a modular metric on X if the following axioms hold :

1. $\omega(\lambda, x, y) = 0$ for every $\lambda > 0$ if and only if $x = y$;
2. for each $x, y \in X$, $\omega(\lambda, x, y) = \omega(\lambda, y, x)$ for all $\lambda > 0$;
3. for each $x, y, z \in X$, $\omega(\lambda + \mu, x, z) \leq \omega(\lambda, x, y) + \omega(\mu, y, z)$ for all $\lambda, \mu > 0$.

A modular metric on X is said to be regular if (1) is replaced with the following axiom :

$x = y$ if and only if $\omega(\lambda, x, y) = 0$ for some $\lambda > 0$.

Definition 2.3. A modular generalized metric ω is on X is said to be strongly regular if the following conditions hold:

1. condtion (1) of modular generalized metric ω is replaced with $x = y$ iff $\omega(1, x, y) = 0$.
2. $\lim_{n\to\infty} \omega(1, x_n, x) = 0$ and $\lim_{n\to\infty} \omega(1, x_n, y) = 0$ implies $x = y$.

Muhammed Usman Ali et.al[3] introduced the following definitions.

Definition 2.4. Let (X, ω) be a modular generalized metric space and let $\{x_n\}$ be a sequence in X.

1. The sequence $\{x_n\}$ is $\omega - convergent$ to $s \in X_\omega$ if and only if $\omega(1, x_n, x) \to 0$ as $n \to \infty$.
2. The sequence $\{x_n\}$ is $\omega - Cauchy$ if $\omega(1, x_m, x_n) \to 0$ as $m, n \to \infty$.
3. A subset D of X_ω is $\omega - complete$ if any $\omega - Cauchy$ sequence in D is a $\omega - convergent$ in D.
4. A subset D of X_ω is $\omega - closed$ if $\omega - limit$ of each $\omega - convergent$ sequence of D always belongs to D.
5. A subset D of X_ω is $\omega - bounded$ if we have $\delta_\omega(D) = sup\{\omega(1, x, y); x, y \in D\} < \infty$.
6. A subset D of X_ω is $\omega - compact$ if for any $\{x_n\}$ in D, there exists a subsequence $\{x_{n_k}\}$ and $x \in D$ such that $\omega(1, x_{n_k}, x) \to 0$ as $k \to \infty$.

Definition 2.5. Let A, B be two nonempty subsets of a metric space (X, d). Let $T : A \to B$ be a nonself mapping. A point $x \in X$ is said to be a best proxomity point of T if $d(x, Tx) = d(A, B)$ where
$d(A, B) = \inf\{d(a, b) : a \in A \text{ and } b \in B\}$.

Definition 2.6. Let A, B be two nonempty subsets of a metric space (X, d). $T : A \cup B \to A \cup B$ is said to be a cyclic mapping if $T(A) \subset B$ and $T(B) \subset A$.

3. Best proxomity points for cyclic 2-convex contraction mappings

In this section we prove existence of proximity point for cyclic 2-convex contraction mappings.

Definition 3.1. Let X_ω be a generalized modular metric space. Let $T : X_\omega \to X_\omega$. We say T is ω - continuous if and only if whenever $\{x_n\}$ is a sequence in X_ω that converges to x $\omega(1, Tx_n, Tx) \to 0$ as $n \to \infty$.

Definition 3.2. Let A, B be two nonempty subsets of a generalized modular metric space X_ω. A cyclic ω - continuous mapping $T : A \cup B \to A \cup B$ is said to be a cyclic 2 - Convex Contraction if there exists $C, D \in D_{m,m}(R_+)$ such that for every $x \in A$, $y \in B$

$$\omega(1, T^2x, T^2y) \leq C\,\omega(1, x, y) + D\,\omega(1, Tx, Ty) + (I - (C + D)\omega(1, A, B)$$

where $C + D < I$ and $\omega(1, A, B) = \inf\{\omega(1, a, b) : a \in A \text{ and } b \in B\}$.

Theorem 3.3. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a cyclic 2 - Convex Contraction on $A \cup B$. Then foe any $x_0 \in A \cup B$, the sequence $\omega(1, T^n x_0, T^{n+1} x_0)$ converges to $\omega(1, A, B)$.*

Proof. Let $T : A \cup B \to A \cup B$ be a cyclic 2 - Convex Contraction mapping. Fix $x_0 \in X_\omega$.
Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2.....$
Let $k = \max\{\omega(1, x_0, x_1), \omega(1, x_1, x_2)\}$.
Now

$$\begin{aligned}
\omega(1, x_2, x_3) &= \omega(1, Tx_1, Tx_2) \leq C\,\omega(1, x_1, x_2) + D\,\omega(1, x_0, x_1) + (I - (C + D)\,\omega(1, A, B) \\
&\leq k(C + D) + \omega(1, A, B) \\
\omega(1, x_3, x_4) &= \omega(1, Tx_2, Tx_3) \leq C\,\omega(1, x_2, x_3) + D\,\omega(1, x_1, x_2) + (I - (C + D)\,\omega(1, A, B) \\
&\leq k(C + D) + \omega(1, A, B) \\
\omega(1, x_4, x_5) &= \omega(1, Tx_3, Tx_4) \leq C\,\omega(1, x_3, x_4) + D\,\omega(1, x_2, x_3) + (I - (C + D)\,\omega(1, A, B) \\
&\leq k(C + D)^2 + \omega(1, A, B) \\
\omega(1, x_5, x_6) &= \omega(1, Tx_4, Tx_5) \leq C\,\omega(1, x_4, x_5) + D\,\omega(1, x_3, x_4) + (I - (C + D)\,\omega(1, A, B) \\
&\leq k(A + B)^2 + \omega(1, A, B)
\end{aligned}$$

By induction,

$$\begin{aligned}
\omega(1, x_{2m-1}, x_{2m}) &\leq k(C + D)^m + \omega(1, A, B) \\
\textit{and}\ \omega(1, x_{2m}, x_{2m+1}) &\leq k(C + D)^m + \omega(1, A, B).
\end{aligned}$$

Since $C+D < I$, $\omega(1, x_{2m}, x_{2m+1}) \leq k(C+D)^m + \omega(1, A, B) \to \omega(1, A, B)$ as $m \to \infty$.
Hence $\lim_{m\to\infty} \omega(1, x_{2m}, x_{2m+1}) \leq \omega(1, A, B)$. But for each m, $\omega(1, x_{2m}, x_{2m+1}) \geq \omega(1, A, B)$.
Therefore $\lim_{m\to\infty} \omega(1, x_{2m}, x_{2m+1}) = \omega(1, A, B)$. Let $2m = n$. Thus $\omega(1, T^n x_0, T^{n+1} x_0)$ converges to $\omega(1, A, B)$. □

Theorem 3.4. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a cyclic 2 - convex contraction on $A \cup B$. If there exists a subsequence $\{T^{n_i} x_0\}$ of $\{T^n x_0\}$ such that $\lim_{i\to\infty} \omega(1, T^{n_i} x_0, p) = 0$, then p is the best proxomity of T.*

Proof. By the previous theorem, $\lim_{i\to\infty} \omega(1, T^{n_i} x_0, T^{n_i+1} x_0) = \omega(1, A, B)$.
Since T is continuous,

$$\begin{aligned}
\omega(1, p, Tp) &= \lim_{i\to\infty} \omega(1, T^{n_i} x_0, T^{n_i+1} x_0) \\
&= \omega(1, A, B).
\end{aligned}$$

□

Theorem 3.5. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a cyclic- 2 convex contraction on $A \cup B$. If either A or B is $\omega-$compact, them T has a best proximity point.*

Proof. Let $T : A \cup B \to A \cup B$ be a cyclic-2 convex contraction mapping. Fix $x_0 \in A$. Assume that A is $\omega-$compact. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2.....$
Since A is $\omega-$compact, $\{x_n\}$ has a convergent subsequence. Hence there exists $p \in$

A such that $x_{n_i} \to p$ as $i \to \infty$. By the previous theorem, $\lim_{n\to\infty} \omega(1, x_{n+1}, x_n) = \omega(1, A, B)$. Since, T is $\omega-$continuous,

$$\omega(1, p, Tp) = \lim_{i\to\infty} \omega(1, x_{n_i}, Tx_{n_i}).$$

Hence T has a best proximity point. □

3.1. Best proxomity points for cyclic 2-sided convex contraction mappings. In this section by defining cyclic 2-sided convex contraction, we prove existence of proxomity points for cyclic 2-sided convex contraction mappings.

Definition 3.6. Let A, B be two nonempty subsets of a generalized modular metric space X_ω. A cyclic ω - continuous mapping $T : A \cup B \to A \cup B$ is said to be a cyclic 2 - Sided Convex Contraction if there exists $C, D, E, F \in D_{m,m}([0, 1))$ such that for every $x \in A$, $y \in B$

$$\begin{aligned}\omega(1, T^2x, T^2y) \leq C\,\omega(1, x, Tx) + D\,\omega(1, Tx, T^2x) + E\,\omega(1, y, Ty) + F\omega(1, Ty, T^2y)\\ + (I - (C + D + E + F))\,\omega(1, A, B)\end{aligned}$$

where $C + D + E + F < I$ and $\omega(1, A, B) = \inf\{\omega(1, a, b) : a \in A \text{ and } b \in B\}$.

Theorem 3.7. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a cyclic 2 - sided convex contraction on $A\cup B$. Then foe any $x_0 \in A\cup B$, the sequence $\omega(1, T^n x_0, T^{n+1}x_0)$ Let $T : A\cup B \to A \cup B$ be a cyclic 2 -sided convex contraction mapping. Fix $x_0 \in X_\omega$.*
Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2.....$
Let $k = \max\{\omega(1, x_0, x_1), \omega(1, x_1, x_2)\}$.
converges to $\omega(1, A, B)$.

Proof. Let $T : A \cup B \to A \cup B$ be a cyclic 2 -sided convex contraction mapping. Fix $x_0 \in X_\omega$.
Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2.....$
Let $k = \max\{\omega(1, x_0, x_1), \omega(1, x_1, x_2)\}$.
Now

$$\begin{aligned}\omega(1, x_2, x_3) &= \omega(1, T^2x_0, T^2x_1)\\ &\leq C\,\omega(1, x_0, x_1) + D\,\omega(1, x_1, x_2) + E\,\omega(1, x_2, x_1) + F\,\omega(1, x_2, x_3)\\ &\quad + (I - (C + D + E + F))\,\omega(1, A, B)\\ &\leq k((C + D + E) + F\,\omega(1, x_2, x_3) + (I - (C + D + E))\,\omega(1, A, B)\end{aligned}$$

Hence

$$(I - F)\,\omega(1, x_2, x_3) \leq k((C + D + E) + (I - (C + D + E))\,\omega(1, A, B)$$

Therefore

$$\begin{aligned}\omega(1, x_2, x_3) &\leq k((C + D + E)(I - F)^{-1} + (I - (C + D + E))(I - F)^{-1}\,\omega(1, A, B)\\ &\leq k((C + D + E)(I - F)^{-1} + \omega(1, A, B)\end{aligned}$$

Similarly,

$$\omega(1, x_3, x_4) \leq k((C + D + E)(I - F)^{-1} + \omega(1, A, B)$$

In general, by induction

$$\omega(1, x_{2n-1}, x_{2n}) \leq k((C+D+E)/(I-F)^{-1})^n + \omega(1, A, B)$$
$$and\ \omega(1, x_{2n}, x_{2n+1}) \leq k((C+D+E)/(I-F)^{-1})^n + \omega(1, A, B)$$

Since $(I-(C+D+E))(I-F)^{-1} < I$,

$$\omega(1, x_{2n}, x_{2n+1}) \leq k((C+D+E)/(I-F)^{-1})^n + \omega(1, A, B) \to \omega(1, A, B)\, \text{as}\, n \to \infty.$$

Hence $\lim_{n\to\infty} \omega(1, x_{2n}, x_{2n+1}) \leq \omega(1, A, B)$. But for each n, $\omega(1, x_{2n}, x_{2n+1}) \geq \omega(1, A, B)$.

Therefore $\lim_{m\to\infty} \omega(1, x_{2n}, x_{2n+1}) = \omega(1, A, B)$. Let $2n = n$. Thus $\omega(1, T^m x_0, T^{m+1} x_0)$ converges to $\omega(1, A, B)$. □

Theorem 3.8. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a cyclic 2 -sided convex contraction on $A \cup B$. If either A or B is $\omega-$compact, them T has a best proximity point.*

Proof. Let $T : A \cup B \to A \cup B$ be a cyclic 2-sided convex contraction mapping. Fix $x_0 \in A$. Assume that A is $\omega-$compact. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2.....$
Since A is $\omega-$compact, $\{x_n\}$ has a convergent subsequence. Hence there exists $p \in A$ such that $x_{n_i} \to p$ as $i \to \infty$. By the previous theorem, $\lim_{n\to\infty} \omega(1, x_{n+1}, x_n) = \omega(1, A, B)$. Since, T is $\omega-$continuous,

$$\omega(1, p, Tp) = \lim_{i\to\infty} \omega(1, x_{n_i}, Tx_{n_i}).$$

Hence T has a best proximity point. □

3.2. Best proxomity points for M-weakly cyclic-2 convex contraction mappings. In this section we prove best proximity point theorem for M-weakly cyclic 2-convex contraction mappings

Definition 3.9. Let A, B be two nonempty subsets of a generalized modular metric space X_ω. A cyclic ω - continuous mapping $T : A \cup B \to A \cup B$ is said to be a M- weakly cyclic-2 Convex Contraction if there exists $C, D, E, F \in D_{m,m}([0, 1))$ such that for every $x \in A$, $y \in B$.

$$\begin{aligned}\omega(1, T^2x, T^2y) \leq\, & C\,\{\omega(1, x, Tx) + \omega(1, y, Ty)\} + D\,\omega(1, x, y)\\ & + E\,\{\omega(1, x, Ty) + \omega(1, y, Tx)\} + (I-(2C+D+2E))\,\omega(1, A, B)\end{aligned}$$

where $2C + D + 2E < I$ and $\omega(1, A, B) = \inf\{\omega(1, a, b) : a \in A \text{ and } b \in B\}$.

Theorem 3.10. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a M- weakly cyclic- 2 convex contraction on $A \cup B$. Then foe any $x_0 \in A \cup B$. the sequence $\omega(1, T^n x_0, T^{n+1} x_0)$ converges to $\omega(1, A, B)$.*

Proof. Let $T : A \cup B \to A \cup B$ be a M- weakly cyclic-2 convex contraction mapping. Fix $x_0 \in X_\omega$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2......$ Let $k = \max\{\omega(1, x_0, x_1), \omega(1, x_1, x_2)\}$. Since T is M-weakly cyclic 2-convex contraction on

$A \cup B$, then we have,

$$\begin{aligned}
\omega(1, x_3, x_2) &\le C[\omega(1, x_0, x_1) + \omega(1, x_1, x_2)] + D\omega(1, x_1, x_0) \\
&\quad + E[\omega(1, x_0, x_2) + \omega(1, x_1, x_1)] \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&\le (C + E)\omega(1, x_1, x_2) + (C + D + E)\omega(1, x_0, x_1) \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&\le (2C + D + 2E)k + \omega(1, A, B) \\
\omega(1, x_4, x_3) &\le (C + E)\omega(1, x_3, x_2) + (C + D + E)\omega(1, x_2, x_1) \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&\le (C + E)[(2C + D + 2E)k + \omega(1, A, B)] + (C + D + E)k \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&\le (C + E)k + (C + E)\omega(1, A, B) + (C + D + E)k \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&= (2C + D + 2E)k + \omega(1, A, B) \\
\omega(1, x_5, x_4) &\le (C + E)\omega(1, x_4, x_3) + (C + D + E)\omega(1, x_3, x_2) \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&\le (C + E)\big[(2C + D + 2E)k + \omega(1, A, B)\big] \\
&\quad + (C + D + E)[(2C + D + 2E)k + \omega(1, A, B)] \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&= (2C + D + 2E)^2 k + \omega(1, A, B)
\end{aligned}$$

$$\begin{aligned}
\omega(1, x_6, x_5) &\le (C + E)\omega(1, x_5, x_4) + (C + D + E)\omega(1, x_4, x_3) \\
&\quad + (I - (2C + D + 2E))\omega(1, A, B) \\
&\le (C + E)(2C + D + 2E)^2 k + \omega(1, A, B) \\
&\quad + (C + D + E)(2C + D + 2E)k + \omega(1, A, B) \\
&\quad + (2C + D + 2E)k + \omega(1, A, B) \\
&\le (C + E)(2C + D + 2E)k \\
&\quad + \omega(1, A, B) + (C + D + E)(2C + D + 2E)k + \omega(1, A, B) \\
&\quad + (2C + D + 2E)k + \omega(1, A, B) \\
&= (2C + D + 2E)^2 k + \omega(1, A, B).
\end{aligned}$$

By the induction, □

THEOREM 3.11. *Let A, B be two nonempty subsets of a complete strongly regular generalized modular metric space X_ω and let T be a M- weakly cyclic- 2 convex contraction on $A \cup B$. If either A or B is $\omega-$compact, them T has a best proximity point.*

Proof. Let $T : A \cup B \to A \cup B$ be a M- weakly cyclic-2 convex contraction mapping. Fix $x_0 \in A$. Assume that A is $\omega-$compact. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$, $n = 0, 1, 2.....$

Since A is $\omega-$compact, $\{x_n\}$ has a convergent subsequence. Hence there exists $p \in A$ such that $x_{n_i} \to p$ as $i \to \infty$. By the previous theorem, $\lim_{n\to\infty} \omega(1, x_{n+1}, x_n) = \omega(1, A, B)$. Since, T is $\omega-$continuous,

$$\omega(1, p, Tp) = \lim_{i\to\infty} \omega(1, x_{n_i}, Tx_{n_i}).$$

Hence T has a best proximity point. □

4. Conclusion

In this paper, some best proximity points for cyclic 2-convex contraction mapping theorem and 2-sided convex contraction theorem are established in generalized modular metric spaces by using various types of contraction mappings. It is worth observing that all the results established in the present paper produces more restricted fixed points. Since different findings delivered in the future might be shown in a smaller setting to ensure the existence of the fixed points.

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V. Anbukkarasi
Department of Mathematics, Bharathidasan University,
Tiruchirappalli-620 024, Tamilnadu, India
E-mail: anbuvelmath@gmail.com

M. Marudai
Department of Mathematics, Bharathidasan University,
Tiruchirappalli-620 024, Tamilnadu, India
E-mail: mmarudai@yahoo.com

R. Theivaraman
Department of Mathematics, Bharathidasan University,
Tiruchirappalli-620 024, Tamilnadu, India
E-mail: deivaraman@gmail.com