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LAPLACE-BELTRAMI MINIMALITY OF TRANSLATION HYPERSURFACES IN E^4

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Abstract. In the present paper, we study translation hypersurfaces in E^4. In this context, firstly we obtain first, second and third Laplace-Beltrami ($\mathrm{LB^I}$, $\mathrm{LB^{II}}$ and $\mathrm{LB^{III}}$) operators of the translation hypersurfaces in E^4. By solving second and third order nonlinear ordinary differential equations, we prove theorems that contain $\mathrm{LB^I}$-minimal, $\mathrm{LB^{II}}$-minimal and $\mathrm{LB^{III}}$-minimal translation hypersurfaces in E^4.

1. General Information and Basic Concepts

The Laplacian is formed in differential equations that define a lot of physical phenomena like quantum mechanics, the diffusion equation for fluid flow and heat, electric potential, gravitational potential, and wave propagation. Using the Laplace equation, the Laplace operator naturally occurs in the mathematical definition of equilibrium in the physical theory of diffusion. Also, the Laplacian indicates the flux density of a function's gradient flow. For example, the proportionality of chemical concentration at a point to Laplacian can be thought of as the net velocity at which a chemical dissolved in a liquid moves away from a point or towards a point, and this equation obtained when considered symbolically is the diffusion equation. As it is in a sense confirmed by the diffusion equation, the Laplace operator itself has a physical interpretation for out-of-equilibrium diffusion to the extent that a point represents a chemical concentration source or collapse. Another reason why Laplacian is useful in physics is that solutions for $\Delta f = 0$ in the U region make Dirichlet energy functional stationary: $E(f) = \frac{1}{2}\int_U \|\nabla f\|^2 \, dx$.

Furthermore, lots of geometry processing applications (for instance, including mesh filtering, parameterization, pose transfer, segmentation, reconstruction, re-meshing, compression, simulation, and interpolation via barycentric coordinates) can be characterized by discrete Laplace operators on triangular surface meshes ([26], [68], [72], [81]). Structural properties of discrete Laplacians

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such as symmetry, sparsity, linear precision, positivity, convergence requirements that are motivated by an attempt to keep properties of the continuous case can be used in many application areas. In [75], an important theoretical limitation "discrete Laplacians cannot satisfy all natural properties; retroactively, this explains the diversity of existing discrete Laplace operators" has been proved with the aid of an old theorem stated by Cremona and Maxwell in [22] and [46]. Also, the behaviour of nodal lines for eigenfunctions of Laplacians has been first investigated by E. F. Chladni, a German physicist ([17], [74]). In [78]-[80], S. T. Yau has considered questions about the number and the distribution of critical points of eigenfunctions of the Laplacian on Riemannian manifolds. An overview of some new and old results on geometric properties of eigenfunctions of Laplacians on Riemannian manifolds has been given and some properties of nodal sets and critical points, the number of nodal domains, and asymptotic properties of eigenfunctions in the high-energy limit have been discussed in [35].

As it is known that, Laplace-Beltrami (LB) operator that defined on a Riemannian manifold is a generalization of Laplacian and LB operator plays a central role in many areas, such as image processing (see [14], [41], [58], [76]), signal processing (see [73]), surface processing (see [12], [20], [23], [48], [60], [61]), and the study of geometric partial differential equations (PDE) (see [14], [47], [55], [58]). For instance, the mathematical formulation of the mean curvature flow, surface diffusion flow ([47]) and Willmore flow (see [66]) etc. involves the first and second order LB operators. Also, LB operators are for solving geometric partial differential equations, such as numerical simulation of various geometric flows (mean curvature flow, surface diffusion flow, Willmore flow etc.), surface smoothing, surface construction and surface image processing. Also, the convergence property of the discrete LB operators is the foundation of convergence analysis of the numerical simulation process of some geometric partial differential equations which involve the operator. In [77], several simple discretization schemes of LB operators over triangulated surfaces have been proposed. Convergence results for these discrete LB operators have been established under various conditions. Numerical results that support the theoretical analysis have been given. Application examples of the proposed discrete LB operators in surface processing and modelling have also been presented.

On the other hand, lots of spectral methods have been used recently in the computing science areas such as graph theory, computer vision, machine learning, visualization, graph drawing, high performance computing, and computer graphics ([29]-[31]). The main purpose of spectral graph theory is to derive relationships between the eigenvalues of the Laplacian or adjacency matrices of a graph and various fundamental properties of the graph, e.g., its diameter and connectivity [18]. It has long been known that the graph Laplacian can be seen as a combinatorial version of the LB operator from Riemannian geometry [49]. Thus the interplay between spectral Riemannian geometry [16] and spectral graph theory has also been a subject of study [18]. In this context, spectral methods for mesh processing and analysis have been examined in [82].

Furthermore, since the LB operator completely specifies the behavior of diffusion processes, it can be seen as a generalization of the analysis of spectral dimensional and by analyzing the eigenvectors and eigenvalues of the LB operator, a novel set of observables for CDT configurations, based on spectral method has been investigated in [21]. In this context, the authors want to emphasis that, even if CDT configurations are defined by means of triangulations, the ultimate goal of the approach is to perform a continuum limit in order to obtain results describing continuum physics of quantum gravity. Here we must note that, spectral methods have many application areas in different sciences such as shape analysis in computer aided design and medical physics. The use of the surface-based LB and the volumetric Laplace eigenvalues and eigenfunctions as shape descriptors for the analysis and comparison of shapes has been introduced and it has been stated that these spectral measures are isometry invariant and therefore allow for shape comparisons with minimal shape pre-processing in [53]. [54] introduces a method to extract 'Shape-DNA', a numerical fingerprint or signature, of any 2D or 3D manifold by taking the eigenvalues (i.e. the spectrum) of its LB operator. A feature that distinguishes different stages in many different fields of physics is the presence or absence of a gap in the spectrum: think for example of Quantum Chromodynamics, where the absence/presence of a gap in the spectrum of the Dirac operator distinguishes between the phases with spontaneously broken/unbroken chiral symmetry [21]. Besides, in [57], a deformation invariant representation of surfaces, the GPS embedding, is introduced using the eigenvalues and eigenfunctions of the LB differential operator.

Also, the LB operator is approximated by the weighted Laplacian of the adjacency graph with weights chosen appropriately. The key role of the LB operator in the heat equation enables us to use the heat kernel to choose the weight decay function in a principled manner. Thus, the embedding maps for the data approximate the eigenmaps of the LB operator, which are maps intrinsically defined on the entire manifold [13].

As another application areas of Laplacian and LB operator, the graph Laplacian has been widely used for different clustering and partition problems ([52], [64], [65]). Although the connections between the LB operator and the graph Laplacian are well known to geometers and specialists in spectral graph theory [18], [19], so far we are not aware of any application to dimensionality reduction or data representation. We note, however, recent work on using diffusion kernels on graphs and other discrete structures [42]. In [13], the problem of constructing a representation for data lying on a low-dimensional manifold embedded in a high-dimensional space has been considered. Drawing on the correspondence between the graph Laplacian, the LB operator on the manifold, and the connections to the heat equation, we propose a geometrically motivated algorithm for representing the high-dimensional data. In [25], a finite element method for elliptic differential equations on arbitrary two-dimensional surfaces has been developed and the most important point in this method is that the LB operator in terms of the tangential gradient has been written down.

Also, 3-dimensional space (3D) can be thought of as the space used to describe the dimensions or positions of objects in the simplest sense. A 4-dimensional space (4D) can be thought as a geometrical extension of the 3D. Jean le Rond d'Alembert has added a fourth dimension to 3D space firstly in his article titled "Dimensions" published in 1754, then J-L. Lagrange has developed this space and this concept of 4-dimensional space has been fully defined by B. Riemann about 100 years later. The notion of 4D space has become more popular after Charles Howard Hinton's paper entitled "What is the Fourth Dimension" in 1880. Thus, spaces more than 3-dimensions have become one of the fundamental concepts in expressing modern physics and mathematics. Although the 4-dimensional Minkowski space has a more complex structure than the 4-dimensional Euclidean space, Einstein's concept of spacetime also has begun to be used. For more details about 4D, we refer to [1], [15], [27], [33], [34], [56], [59], and etc.

On the other hand, translation surfaces arise naturally in the theory of billiards, physics, dynamics and Teichmüller theory.

In 1991, the notion of a translation hypersurface in the Euclidean space E^{n+1} is given as the graph of a

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (x_1, x_2, ..., x_n) \longmapsto \sum_{i=1}^{n} f_i(x_i),$$

where f_i, $i = 1, 2, ..., n$, are smooth functions depending on one variable [24]. After Dillen has defined this notion, lots of studies about translation (hyper)surfaces have been done by mathematicians in Euclidean, Minkowskian, Galilean and pseudo-Galilean spaces. For instance, some results for the mean and Gauss–Kronecker curvatures of generalized translation graphs to be constant have been obtained and a complete description of all translation hypersurfaces with constant r-curvature S_r in Euclidean space have been given in [43] and [44], respectively. Minimal translation graphs, by imposing natural conditions on two independent functions ψ and φ, like eikonality, minimality and harmonicity have been studied in [51]. In [10], translation hypersurfaces (M^n, f) whose Allen's matrices of f are singular in $\mathbb{R}^{n+1}$ have been classified completely; the translation hypersurfaces satisfying the Tzitzeica condition are only hyperplanes in $\mathbb{R}^{n+1}$ has been stated and an application of such hypersurfaces to production functions in microeconomics has been given. In [50], the authors have studied on hypersurfaces in Euclidean 4-space E^4 defined as the sum of a curve and a surface with zero mean curvature and given a classification of these hypersurfaces. The theorem of "A translation surface is flat in E^4 if and only if it is either a hyperplane or a hypercylinder" has been proved and a necessary and sufficient condition for a quadratic triangular Bezier surface in

E^4 to become a translation surface has been given in [8]. A construction and classification of translation surfaces with zero mean curvature in E^3 have been given in [32].

Furthermore, for more studies about translation surfaces or different types of (hyper)surfaces in different spaces such as Euclidean, Minkowski and Isotropic spaces, we refer to [2]-[7], [9], [11], [36]-[40], [45], [62], [67], [69]-[71], and etc.

Since we will study translation hypersurface in E^4 and give some characterizations about these hypersurfaces in this paper, let us recall some fundamental notions for E^4.

Let $\overrightarrow{u} = (u_1, u_2, u_3, u_4)$, $\overrightarrow{v} = (v_1, v_2, v_3, v_4)$ and $\overrightarrow{w} = (w_1, w_2, w_3, w_4)$ be three vectors in E^4. The inner product, norm of a vector and vector product are given by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4, \tag{1}$$
$$\|\overrightarrow{u}\| = \sqrt{\langle \overrightarrow{u}, \overrightarrow{u} \rangle}$$

and

$$\overrightarrow{u} \times \overrightarrow{v} \times \overrightarrow{w} = \det \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{bmatrix}, \tag{2}$$

respectively.

If

(3)

$\Gamma : E^3 \longrightarrow E^4,$

$\Gamma(x_1, x_2, x_3) = (\Gamma_1(x_1, x_2, x_3), \Gamma_2(x_1, x_2, x_3), \Gamma_3(x_1, x_2, x_3), \Gamma_4(x_1, x_2, x_3))$

is a hypersurface in E^4, then the unit normal vector field, the matrix forms of the first and second fundamental forms are

$$N_\Gamma = \frac{\Gamma_{x_1} \times \Gamma_{x_2} \times \Gamma_{x_3}}{\|\Gamma_{x_1} \times \Gamma_{x_2} \times \Gamma_{x_3}\|}, \tag{4}$$

$$[g_{ij}] = \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \tag{5}$$

and

$$[h_{ij}] = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \tag{6}$$

respectively. Here $g_{ij} = \left\langle \Gamma_{x_i}, \Gamma_{x_j} \right\rangle$, $h_{ij} = \left\langle \Gamma_{x_i x_j}, N_\Gamma \right\rangle$, $\Gamma_{x_i} = \frac{\partial \Gamma(x_1,x_2,x_3)}{\partial x_i}$, $\Gamma_{x_i x_j} = \frac{\partial^2 \Gamma(x_1,x_2,x_3)}{\partial x_i x_j}$, $i, j \in \{1, 2, 3\}$.

Also, the shape operator of the hypersurface (3) is

$$S = [g_{ij}]^{-1}.[h_{ij}], \tag{7}$$

where $[g_{ij}]^{-1}$ is the inverse matrix of $[g_{ij}]$.

Using (4)-(7), the Gaussian and mean curvatures of a hypersurface in E^4 are defined by

$$K = \det(S) = \frac{\det[h_{ij}]}{\det[g_{ij}]} \tag{8}$$

and

$$3H = tr(S), \tag{9}$$

respectively [28].

Moreover, the inverse of an arbitrary matrix

$$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{10}$$

is

(11)

$$[A^{ij}] = \frac{1}{\det[A_{ij}]} \begin{bmatrix} A_{22}A_{33} - A_{23}A_{32} & A_{13}A_{32} - A_{12}A_{33} & A_{12}A_{23} - A_{13}A_{22} \\ A_{23}A_{31} - A_{21}A_{33} & A_{11}A_{33} - A_{13}A_{31} & A_{13}A_{21} - A_{11}A_{23} \\ A_{21}A_{32} - A_{22}A_{31} & A_{12}A_{31} - A_{11}A_{32} & A_{11}A_{22} - A_{12}A_{21} \end{bmatrix},$$

where

$$\begin{aligned} \det[A_{ij}] = & -A_{13}A_{22}A_{31} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{11}A_{23}A_{32} \\ & - A_{12}A_{21}A_{33} + A_{11}A_{22}A_{33}. \end{aligned} \tag{12}$$

2. Translation Hypersurfaces in E^4

In this section, we give the Gaussian and mean curvatures of translation hypersurfaces in E^4.

Let M be an immersion in E^4 given by

$$\begin{aligned} & \Gamma : \mathbb{R}^3 \longrightarrow \mathbb{R}^4 \\ & (x, y, z) \longrightarrow (x, y, z, f(x) + g(y) + h(z)), \end{aligned} \tag{13}$$

where f, g and h are smooth functions. Then M is called a translation hypersurface in E^4.

From (4) and (13) the unit normal vector field of Γ in E^4 is

$$N_\Gamma = \frac{1}{W}\left(f'(x), g'(y), h'(z), -1\right), \tag{14}$$

where $W = \sqrt{1 + f'^2(x) + g'^2(y) + h'^2(z)}$.

Also, from (5), (6) and (14), the first fundamental form, second fundamental form and their determinants are obtained by

$$[g_{ij}] = \begin{bmatrix} 1+f'^2(x) & f'(x)g'(y) & f'(x)h'(z) \\ f'(x)g'(y) & 1+g'^2(y) & g'(y)h'(z) \\ f'(x)h'(z) & g'(y)h'(z) & 1+h'^2(z) \end{bmatrix}, \tag{15}$$

$$[h_{ij}] = \frac{1}{W}\begin{bmatrix} -f''(x) & 0 & 0 \\ 0 & -g''(y) & 0 \\ 0 & 0 & -h''(z) \end{bmatrix}, \tag{16}$$

$$\det[g_{ij}] = W^2 \tag{17}$$

and

$$\det[h_{ij}] = -\frac{f''(x)g''(y)h''(z)}{W^3}, \tag{18}$$

respectively.

So, from (8), (17) and (18), we have

Theorem 2.1. *The Gaussian curvature of the translation hypersurface (13) in Euclidean 4-space is*

$$K = -\frac{f''(x)g''(y)h''(z)}{W^5}. \tag{19}$$

Also, the inverse of the first fundamental form is

$$[g^{ij}] = \begin{bmatrix} g^{11} & g^{12} & g^{13} \\ g^{21} & g^{22} & g^{23} \\ g^{31} & g^{32} & g^{33} \end{bmatrix}, \tag{20}$$

where

$$g^{11} = \frac{1+g'^2(y)+h'^2(z)}{W^2},\ g^{22} = \frac{1+f'^2(x)+h'^2(z)}{W^2},\ g^{33} = \frac{1+f'^2(x)+g'^2(y)}{W^2},$$
$$g^{12} = g^{21} = -\frac{f'(x)g'(y)}{W^2},\ g^{13} = g^{31} = -\frac{f'(x)h'(z)}{W^2},\ g^{23} = g^{32} = -\frac{g'(y)h'(z)}{W^2}.$$

So, using (16) and (20) in (7), the shape operator of the translation hypersurface (13) is obtained by

$$S = \frac{1}{W^3}\begin{bmatrix} -\left(1+g'^2(y)+h'^2(z)\right)f''(x) & f'(x)g'(y)g''(y) & f'(x)h'(z)h''(z) \\ f'(x)g'(y)f''(x) & -\left(1+f'^2(x)+h'^2(z)\right)g''(y) & g'(y)h'(z)h''(z) \\ f'(x)h'(z)f''(x) & g'(y)h'(z)g''(y) & -\left(1+f'^2(x)+g'^2(y)\right)h''(z) \end{bmatrix}. \tag{21}$$

Hence from (9) and (21), we get

Theorem 2.2. *The mean curvature of the translation hypersurface (13) in Euclidean 4-space is*

$$H=-\frac{\begin{pmatrix}\left(1+g'^2(y)+h'^2(z)\right)f''(x)+\left(1+f'^2(x)+h'^2(z)\right)g''(y)\\+\left(1+f'^2(x)+g'^2(y)\right)h''(z)\end{pmatrix}}{3W^3}. \tag{22}$$

Here, we must note that the curvatures of translation hypersurfaces in Euclidean n-space has been obtained in [24] and [63].

3. Laplace-Beltrami Operators of Translation Hypersurfaces in E^4

In this section, we obtain the first, second and third Laplace-Beltrami (LB^{I}, $\mathrm{LB}^{\mathrm{II}}$ and $\mathrm{LB}^{\mathrm{III}}$) operators of the translation hypersurface (13) in E^4 and prove theorems that contain LB^{I}-minimality, $\mathrm{LB}^{\mathrm{II}}$-minimality and $\mathrm{LB}^{\mathrm{III}}$-minimality of translation hypersurfaces in E^4.

3.1. LB^{I}-Minimal Translation Hypersurfaces in E^4

The first LB (LB^{I}) operator of a smooth function $\varphi=\varphi(x^1,x^2,x^3)|_D$, ($D\subset\mathbb{R}^3$) of class C^3 with respect to (or wrt) the first fundamental form of a hypersurface is defined as follows:

$$\Delta^I\varphi=\frac{1}{\sqrt{\det[g_{ij}]}}\sum_{i,j=1}^{3}\frac{\partial}{\partial x^i}\left(\sqrt{\det[g_{ij}]}g^{ij}\frac{\partial\varphi}{\partial x^j}\right). \tag{23}$$

Thus, using (11), (12) and (23), the LB^{I} operator of $\varphi=\varphi(x,y,z)$ can be written as

(24)
$$\Delta^I\varphi=\frac{1}{\sqrt{\det[g_{ij}]}}\left\{\begin{array}{l}\frac{\partial}{\partial x}\left(\frac{(g_{22}g_{33}-g_{23}^2)\varphi_x+(g_{13}g_{23}-g_{12}g_{33})\varphi_y+(g_{12}g_{23}-g_{13}g_{22})\varphi_z}{\sqrt{\det[g_{ij}]}}\right)\\+\frac{\partial}{\partial y}\left(\frac{(g_{13}g_{23}-g_{12}g_{33})\varphi_x+(g_{11}g_{33}-g_{13}^2)\varphi_y+(g_{12}g_{13}-g_{11}g_{23})\varphi_z}{\sqrt{\det[g_{ij}]}}\right)\\+\frac{\partial}{\partial z}\left(\frac{(g_{12}g_{23}-g_{13}g_{22})\varphi_x+(g_{12}g_{13}-g_{11}g_{23})\varphi_y+(g_{11}g_{22}-g_{12}^2)\varphi_z}{\sqrt{\det[g_{ij}]}}\right)\end{array}\right\}.$$

Here, if we denote the LB^{I} operator of the translation hypersurface (13) in E^4 as $\Delta^I\Gamma$, then from (17) and (24), we have

$$\begin{aligned}\Delta^I\Gamma&=((\Delta^I\Gamma)_1,(\Delta^I\Gamma)_2,(\Delta^I\Gamma)_3,(\Delta^I\Gamma)_4)\\&=\frac{1}{W}\begin{pmatrix}(\mathcal{U}_1)_x+(\mathcal{V}_1)_y+(\mathcal{W}_1)_z,(\mathcal{U}_2)_x+(\mathcal{V}_2)_y+(\mathcal{W}_2)_z,\\(\mathcal{U}_3)_x+(\mathcal{V}_3)_y+(\mathcal{W}_3)_z,(\mathcal{U}_4)_x+(\mathcal{V}_4)_y+(\mathcal{W}_4)_z\end{pmatrix},\end{aligned} \tag{25}$$

where

(26)
$$\begin{cases} \mathcal{U}_i = \frac{1}{W}\left\{(g_{22}g_{33}-g_{23}^2)(\Gamma_i)_x+(g_{13}g_{23}-g_{12}g_{33})(\Gamma_i)_y+(g_{12}g_{23}-g_{13}g_{22})(\Gamma_i)_z\right\}, \\ \mathcal{V}_i = \frac{1}{W}\left\{(g_{13}g_{23}-g_{12}g_{33})(\Gamma_i)_x+(g_{11}g_{33}-g_{13}^2)(\Gamma_i)_y+(g_{12}g_{13}-g_{11}g_{23})(\Gamma_i)_z\right\}, \\ \mathcal{W}_i = \frac{1}{W}\left\{(g_{12}g_{23}-g_{13}g_{22})(\Gamma_i)_x+(g_{12}g_{13}-g_{11}g_{23})(\Gamma_i)_y+(g_{11}g_{22}-g_{12}^2)(\Gamma_i)_z\right\}. \end{cases}$$

Now, using (13), (15) and (17), we have

$$\begin{cases} \mathcal{U}_1 = \frac{1+g'^2+h'^2}{W},\ \mathcal{U}_2 = -\frac{f'g'}{W},\ \mathcal{U}_3 = -\frac{f'h'}{W},\ \mathcal{U}_4 = \frac{f'}{W}, \\ \mathcal{V}_1 = -\frac{f'g'}{W},\ \mathcal{V}_2 = \frac{1+f'^2+h'^2}{W},\ \mathcal{V}_3 = -\frac{g'h'}{W},\ \mathcal{V}_4 = \frac{g'}{W}, \\ \mathcal{W}_1 = -\frac{f'h'}{W},\ \mathcal{W}_2 = -\frac{g'h'}{W},\ \mathcal{W}_3 = \frac{1+f'^2+g'^2}{W},\ \mathcal{W}_4 = \frac{h'}{W}. \end{cases} \tag{27}$$

So, using (27) in (25), the LB$^{\mathrm{I}}$ operator of the translation hypersurface (13) is obtained as follows:

Theorem 3.1. *The LB^I operator of the translation hypersurface (13) in E^4 is*

$$\Delta^I\Gamma = (-f'Q, -g'Q, -h'Q, Q), \tag{28}$$

where $Q = \frac{(1+g'^2+h'^2)f''+(1+f'^2+h'^2)g''+(1+f'^2+g'^2)h''}{W^4}$.

We say that a hypersurface Γ is called LB$^{\mathrm{I}}$-minimal, if it satisfies $\Delta^I\Gamma = 0$. Thus, from Theorem 3.1, we can prove the following theorem:

Theorem 3.2. *LB^I-minimal translation hypersurfaces (13) in E^4 can be parametrized by*

$$\Gamma(x,y,z) = (x,y,z,a_1x+a_2y+a_3z+d_1), \tag{29}$$

$$\Gamma(x,y,z) = \left(x,y,z,a_4x+c_1\ln\left[\frac{\cos\left(\frac{\sqrt{1+(a_4)^2}(y-c_1c_3)}{c_1}\right)}{\cos\left(\frac{\sqrt{1+(a_4)^2}(z+c_1c_2)}{c_1}\right)}\right]+d_2\right), \tag{30}$$

$$\Gamma(x,y,z) = \left(x,y,z,a_5y+c_4\ln\left[\frac{\cos\left(\frac{\sqrt{1+(a_5)^2}(x-c_4c_6)}{c_4}\right)}{\cos\left(\frac{\sqrt{1+(a_5)^2}(z+c_4c_5)}{c_4}\right)}\right]+d_3\right), \tag{31}$$

$$\Gamma(x,y,z) = \left(x,y,z,a_6z+c_7\ln\left[\frac{\cos\left(\frac{\sqrt{1+(a_6)^2}(x-c_7c_9)}{c_7}\right)}{\cos\left(\frac{\sqrt{1+(a_6)^2}(y+c_7c_8)}{c_7}\right)}\right]+d_4\right), \tag{32}$$

where a_i, c_i, d_i are constants and $c_i \neq 0$.

Proof. If the components of the LB^{I} operator of the translation hypersurface (13) in E^4 is zero, then from (28) we have $Q = 0$, that is we get the following second order nonlinear ordinary differential equation:

$$(1 + g'^2 + h'^2)f'' + (1 + f'^2 + h'^2)g'' + (1 + f'^2 + g'^2)h'' = 0. \tag{33}$$

Here, let we solve the last equation according to the following cases:

Case 1. *Let $f''(x) = g''(y) = h''(z) = 0$.*
In this case, one can easily see that LB^I-minimal translation hypersurface is parametrized by (29).

Case 2. *Let $f''(x) = g''(y) = 0$ and $h''(z) \neq 0$.*
If we take $f(x) = m_1x+n_1$ and $g(y) = m_2y+n_2$, where $m_1, m_2, n_1, n_2 \in \mathbb{R}$, then the equation (33) becomes

$$(1 + (m_1)^2 + (m_2)^2)h''(z) = 0. \tag{34}$$

Since $1 + (m_1)^2 + (m_2)^2 \neq 0$, it must be $h''(z) = 0$ and this is a contradiction. So, there aren't any I-minimal translation hypersurfaces in this case.

For the cases of $f''(x) = h''(z) = 0$, $g''(y) \neq 0$ and $g''(y) = h''(z) = 0$, $f''(x) \neq 0$, we get similar contradictions. So, there aren't any LB^I-minimal translation hypersurfaces for these cases, too.

Case 3. *Let $f''(x) = 0$ and $g''(y) \neq 0 \neq h''(z)$.*
If we take $f(x) = a_4x+b_1$, where $a_4, b_1 \in \mathbb{R}$, then the equation (33) becomes

$$(1 + (a_4)^2 + (h'(z))^2)g''(y) + (1 + (a_4)^2 + (g'(y))^2)h''(z) = 0. \tag{35}$$

From $g''(y) \neq 0 \neq h''(z)$ and (35) we get

$$\frac{(1 + (a_4)^2 + (h'(z))^2)}{h''(z)} = -\frac{(1 + (a_4)^2 + (g'(y))^2)}{g''(y)}. \tag{36}$$

Since y and z are independent variables, each side of the equation (36) must be constant, i.e.

$$\frac{(1 + (a_4)^2 + (h'(z))^2)}{h''(z)} = -\frac{(1 + (a_4)^2 + (g'(y))^2)}{g''(y)} = c_1, \ c_1 \in \mathbb{R}. \tag{37}$$

If we take $c_1 = 0$ in (37), then there is no real solutions for $g(y)$ and $h(z)$.
If we solve the second order nonlinear ordinary differential equation (37) by taking $c_1 \neq 0$, then we have

$$\begin{cases} h(z) = b_2 - c_1 \ln\left[\cos\left(\frac{\sqrt{1+(a_4)^2}(z+c_1c_2)}{c_1}\right)\right], \\ \\ g(y) = b_3 + c_1 \ln\left[\cos\left(\frac{\sqrt{1+(a_4)^2}(y-c_1c_3)}{c_1}\right)\right]. \end{cases} \tag{38}$$

Taking $b_1 + b_2 + b_3 = d_2$ and using $f(x) = a_4x + b_1$ and (38) in (13), we obtain the LB^I-minimal translation hypersurface as (30).

With similar calculations for the cases of $g''(y) = 0$, $f''(x) \neq 0 \neq h''(z)$ and $h''(z) = 0$, $f''(x) \neq 0 \neq g''(y)$, we get the LB^I-minimal hypersurfaces as (31) and (32), respectively.

Case 4. *Let $f''(x) \neq 0$, $g''(y) \neq 0$ and $h''(z) \neq 0$.*
Differentiating (33) with respect to x and y, we get

$$2g'(y)g''(y)f'''(x) + 2f'(x)f''(x)g'''(y) = 0. \tag{39}$$

Since $f''(x) \neq 0 \neq g''(y)$, from (39) we have

$$\frac{g'''(y)}{g'(y)g''(y)} = -\frac{f'''(x)}{f'(x)f''(x)}. \tag{40}$$

Also, since x and y are independent variables, each side of the equation (40) must be constant, i.e.

$$\frac{g'''(y)}{g'(y)g''(y)} = -\frac{f'''(x)}{f'(x)f''(x)} = c_8, \ c_8 \in \mathbb{R}. \tag{41}$$

Now, differentiating (33) with respect to x and z, since $f''(x) \neq 0 \neq h'''(z)$ we have

$$\frac{h'''(z)}{h'(z)h''(z)} = -\frac{f'''(x)}{f'(x)f''(x)} = c_9, \ c_9 \in \mathbb{R} \tag{42}$$

and differentiating (33) with respect to y and z, since $g''(y) \neq 0 \neq h'''(z)$ we have

$$\frac{h'''(z)}{h'(z)h''(z)} = -\frac{g'''(y)}{g'(y)g''(y)} = c_{10}, \ c_{10} \in \mathbb{R}. \tag{43}$$

From (41)-(43), we get

$$c_8 = c_9 = c_{10} = 0. \tag{44}$$

If we solve the equations (41)-(43) by considering (44), then we obtain that

$$\left\{ \begin{array}{l} f(x) = c_{11}x^2 + c_{12}x + c_{13}, \\ \\ g(y) = c_{14}y^2 + c_{15}y + c_{16}, \\ \\ h(z) = c_{17}z^2 + c_{18}z + c_{19}. \end{array} \right\}, \ c_i \in \mathbb{R}, \ i = 11, 12, ..., 19. \tag{45}$$

Finally, putting (45) in (33), we get

$$\begin{aligned} 0 &= 4xc_{11}(xc_{11} + c_{12})(c_{14} + c_{17}) + 4zc_{17}(zc_{17} + c_{18})(c_{11} + c_{14}) \\ &\quad + 4yc_{14}(yc_{14} + c_{15})(c_{11} + c_{17}) + (1 + (c_{12})^2 + (c_{15})^2)c_{17} \\ &\quad + (1 + (c_{12})^2 + (c_{18})^2)c_{14} + (1 + (c_{15})^2 + (c_{18})^2)c_{11} \end{aligned}$$

and from the last equation we have $c_{11} = c_{14} = c_{17} = 0$. From (45), this contradicts with our assumption that $f''(x) \neq 0$, $g''(y) \neq 0$ and $h''(z) \neq 0$. So, there aren't any LB^I-minimal translation hypersurfaces for this case.

□

Also it is known that, a hypersurface is minimal if and only if the first Laplace-Beltrami operator of this hypersurface vanishes. So, for other versions of the above proof (in E^n), one can see [24] or [63].

3.2. $\mathrm{LB}^{\mathrm{II}}$-Minimal Translation Hypersurfaces in E^4

The second LB ($\mathrm{LB}^{\mathrm{II}}$) operator of the smooth function $\varphi = \varphi(x^1, x^2, x^3)|_D$, ($D \subset \mathbb{R}^3$) with respect to the second fundamental form of a hypersurface is defined as follows:

$$\Delta^{II}\varphi = \frac{1}{\sqrt{\det[h_{ij}]}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i}\left(\sqrt{\det[h_{ij}]}\,h^{ij}\frac{\partial \varphi}{\partial x^j}\right). \tag{46}$$

Thus, using (11), (12) and (46), the $\mathrm{LB}^{\mathrm{II}}$ operator $\varphi = \varphi(x, y, z)$ can be written as

(47)

$$\Delta^{II}\varphi = \frac{1}{\sqrt{\det[h_{ij}]}}\left\{\begin{array}{l} \frac{\partial}{\partial x}\left(\frac{(h_{22}h_{33}-h_{23}^2)\varphi_x+(h_{13}h_{23}-h_{12}h_{33})\varphi_y+(h_{12}h_{23}-h_{13}h_{22})\varphi_z}{\sqrt{\det[h_{ij}]}}\right) \\ +\frac{\partial}{\partial y}\left(\frac{(h_{13}h_{23}-h_{12}h_{33})\varphi_x+(h_{11}h_{33}-h_{13}^2)\varphi_y+(h_{12}h_{13}-h_{11}h_{23})\varphi_z}{\sqrt{\det[h_{ij}]}}\right) \\ +\frac{\partial}{\partial z}\left(\frac{(h_{12}h_{23}-h_{13}h_{22})\varphi_x+(h_{12}h_{13}-h_{11}h_{23})\varphi_y+(h_{11}h_{22}-h_{12}^2)\varphi_z}{\sqrt{\det[h_{ij}]}}\right) \end{array}\right\}.$$

Here, if we denote the $\mathrm{LB}^{\mathrm{II}}$ operator of the translation hypersurface (13) in E^4 as $\Delta^{II}\Gamma$, then from (18) and (47), we have

$$\begin{aligned} \Delta^{II}\Gamma &= ((\Delta^{II}\Gamma)_1, (\Delta^{II}\Gamma)_2, (\Delta^{II}\Gamma)_3, (\Delta^{II}\Gamma)_4) \\ &= \frac{1}{\sqrt{\det[h_{ij}]}}\left(\begin{array}{c} (U_1)_x + (V_1)_y + (W_1)_z, (U_2)_x + (V_2)_y + (W_2)_z, \\ (U_3)_x + (V_3)_y + (W_3)_z, (U_4)_x + (V_4)_y + (W_4)_z \end{array}\right), \end{aligned} \tag{48}$$

where

(49)

$$\left\{\begin{array}{l} U_i = \frac{1}{\sqrt{\det[h_{ij}]}}\left\{(h_{22}h_{33}-h_{23}^2)(\Gamma_i)_x + (h_{13}h_{23}-h_{12}h_{33})(\Gamma_i)_y + (h_{12}h_{23}-h_{13}h_{22})(\Gamma_i)_z\right\}, \\ V_i = \frac{1}{\sqrt{\det[h_{ij}]}}\left\{(h_{13}h_{23}-h_{12}h_{33})(\Gamma_i)_x + (h_{11}h_{33}-h_{13}^2)(\Gamma_i)_y + (h_{12}h_{13}-h_{11}h_{23})(\Gamma_i)_z\right\}, \\ W_i = \frac{1}{\sqrt{\det[h_{ij}]}}\left\{(h_{12}h_{23}-h_{13}h_{22})(\Gamma_i)_x + (h_{12}h_{13}-h_{11}h_{23})(\Gamma_i)_y + (h_{11}h_{22}-h_{12}^2)(\Gamma_i)_z\right\}. \end{array}\right.$$

Now, using (13), (16) and (18), we have

(50)
$$\begin{cases} U_1 = \frac{g''(y)h''(z)}{\sqrt{-f''(x)g''(y)h''(z)W}},\ U_2 = 0,\ U_3 = 0,\ U_4 = \frac{f'(x)g''(y)h''(z)}{\sqrt{-f''(x)g''(y)h''(z)W}}, \\ V_1 = 0,\ V_2 = \frac{f''(x)h''(z)}{\sqrt{-f''(x)g''(y)h''(z)W}},\ V_3 = 0,\ V_4 = \frac{g'(y)f''(x)h''(z)}{\sqrt{-f''(x)g''(y)h''(z)W}}, \\ W_1 = 0,\ W_2 = 0,\ W_3 = \frac{f''(x)g''(y)}{\sqrt{-f''(x)g''(y)h''(z)W}},\ W_4 = \frac{h'(z)f''(x)g''(y)}{\sqrt{-f''(x)g''(y)h''(z)W}}. \end{cases}$$

So, using (50) in (48), the LB$^{\mathrm{II}}$ operator of the translation hypersurface (13) is obtained as follows:

Theorem 3.3. *The LB^{II} operator of the translation hypersurface (13) in E^4 is*

$$\Delta^{II}\Gamma = \begin{pmatrix} \frac{f'f''^2+W^2f'''}{2Wf''^2},\ \frac{g'g''^2+W^2g'''}{2Wg''^2},\ \frac{h'h''^2+W^2h'''}{2Wh''^2}, \\ \frac{1+5W^2}{-2W} + \frac{W}{2}\left(\frac{f'f'''}{f''^2} + \frac{g'g'''}{g''^2} + \frac{h'h'''}{h''^2}\right) \end{pmatrix}. \tag{51}$$

We say that a hypersurface Γ is called LB$^{\mathrm{II}}$-minimal, if it satisfies $\Delta^{II}\Gamma = 0$. Thus, from Theorem 3.3, we can prove the following theorem:

Theorem 3.4. *The translation hypersurface (13) is never LB^{II}-minimal in E^4.*

Proof. Let us suppose that the translation hypersurface (13) is LB$^{\mathrm{II}}$-minimal in E^4. Then, all components of $\Delta^{II}\Gamma$ obtained as (51) must be zero. For instance, let the first component of (51) vanishes, i.e.

$$f'f''^2 + W^2f''' = 0. \tag{52}$$

Using $W = \sqrt{1 + f'^2(x) + g'^2(y) + h'^2(z)} > 0$ in the third order nonlinear ordinary differential equation (52), either W is a function which doesn't depend on y and z, or $f'''(x) = 0$. If W doesn't depend on y and z, then the functions $g(y)$ and $h(z)$ must be linear. And if $f'''(x) = 0$, then from (52), we have $f'f''^2 = 0$. Hence, from (51), these two statements are contradictions. So, the proof completes. (If the other components of $\Delta^{II}\Gamma$ are zero, this proof can be done similarly.) □

3.3. LB$^{\mathrm{III}}$-Minimal Translation Hypersurfaces in E^4

The third LB (LB$^{\mathrm{III}}$) operator of the smooth function $\varphi = \varphi(x^1, x^2, x^3)|_D$, $(D \subset R^3)$ with respect to the third fundamental form of a hypersurface is defined as follows:

$$\Delta^{III}\varphi = \frac{1}{\sqrt{\det[m_{ij}]}} \sum_{i,j=1}^{3} \frac{\partial}{\partial x^i}\left(\sqrt{\det[m_{ij}]}m^{ij}\frac{\partial\varphi}{\partial x^j}\right), \tag{53}$$

where $[m_{ij}]$ is the third fundamental form matrix and m^{ij} are the components of inverse matrix $[m_{ij}]^{-1}$. Here, the third fundamental form matrix, the inverse matrix and the determinant of the third fundamental form matrix of (13) are obtained by

(54)
$$\begin{aligned}[m_{ij}] &= \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \\ &= \frac{1}{W^4}\begin{bmatrix} (1+g'^2+h'^2)f''^2 & -f'g'f''g'' & -f'h'f''h'' \\ -f'g'f''g'' & (1+f'^2+h'^2)g''^2 & -g'h'g''h'' \\ -f'h'f''h'' & -g'h'g''h'' & (1+f'^2+g'^2)h''^2 \end{bmatrix},\end{aligned}$$

(55)
$$[m^{ij}] = W^2 \begin{bmatrix} \frac{1+f'^2}{f''^2} & \frac{f'g'}{f''g''} & \frac{f'h'}{f''h''} \\ \frac{f'g'}{f''g''} & \frac{1+g'^2}{g''^2} & \frac{g'h'}{g''h''} \\ \frac{f'h'}{f''h''} & \frac{g'h'}{g''h''} & \frac{1+h'^2}{h''^2} \end{bmatrix}$$

and

(56)
$$\det[m_{ij}] = \frac{f''^2 g''^2 h''^2}{W^8},$$

respectively. Here, $m_{ij} = \left\langle (N_\Gamma)_{x_i}, (N_\Gamma)_{x_j} \right\rangle$. Thus, using (11), (12) and (53) the LB$^{\text{III}}$ operator of $\varphi = \varphi(x,y,z)$ can be written as

(57)
$$\Delta^{III}\varphi = \frac{1}{\sqrt{\det[m_{ij}]}} \left\{ \begin{array}{l} \frac{\partial}{\partial x}\left(\frac{(m_{22}m_{33}-m_{23}^2)\varphi_x + (m_{13}m_{23}-m_{12}m_{33})\varphi_y + (m_{12}m_{23}-m_{13}m_{22})\varphi_z}{\sqrt{\det[m_{ij}]}} \right) \\ +\frac{\partial}{\partial y}\left(\frac{(m_{13}m_{23}-m_{12}m_{33})\varphi_x + (m_{11}m_{33}-m_{13}^2)\varphi_y + (m_{12}m_{13}-m_{11}m_{23})\varphi_z}{\sqrt{\det[m_{ij}]}} \right) \\ +\frac{\partial}{\partial z}\left(\frac{(m_{12}m_{23}-m_{13}m_{22})\varphi_x + (m_{12}m_{13}-m_{11}m_{23})\varphi_y + (m_{11}m_{22}-m_{12}^2)\varphi_z}{\sqrt{\det[m_{ij}]}} \right) \end{array} \right\}.$$

Here, if we denote the LB$^{\text{III}}$ operator of the translation hypersurface (13) in E^4 as $\Delta^{III}\Gamma$, then from (13) and (57), we get

(58)
$$\begin{aligned}\Delta^{III}\Gamma &= ((\Delta^{III}\Gamma)_1, (\Delta^{III}\Gamma)_2, (\Delta^{III}\Gamma)_3, (\Delta^{III}\Gamma)_4) \\ &= \frac{1}{\sqrt{\det[m_{ij}]}} \left(\begin{array}{c} (\mathfrak{U}_1)_x + (\mathfrak{V}_1)_y + (\mathfrak{W}_1)_z, (\mathfrak{U}_2)_x + (\mathfrak{V}_2)_y + (\mathfrak{W}_2)_z, \\ (\mathfrak{U}_3)_x + (\mathfrak{V}_3)_y + (\mathfrak{W}_3)_z, (\mathfrak{U}_4)_x + (\mathfrak{V}_4)_y + (\mathfrak{W}_4)_z \end{array} \right),\end{aligned}$$

where

(59)
$$\left\{ \begin{array}{l} \mathfrak{U}_i = \frac{1}{\sqrt{\det[m_{ij}]}} \left((m_{22}m_{33} - m_{23}^2)(\Gamma_i)_x + (m_{13}m_{23} - m_{12}m_{33})(\Gamma_i)_y + (m_{12}m_{23} - m_{13}m_{22})(\Gamma_i)_z \right), \\ \mathfrak{V}_i = \frac{1}{\sqrt{\det[m_{ij}]}} \left((m_{13}m_{23} - m_{12}m_{33})(\Gamma_i)_x + (m_{11}m_{33} - m_{13}^2)(\Gamma_i)_y + (m_{12}m_{13} - m_{11}m_{23})(\Gamma_i)_z \right), \\ \mathfrak{W}_i = \frac{1}{\sqrt{\det[m_{ij}]}} \left((m_{12}m_{23} - m_{13}m_{22})(\Gamma_i)_x + (m_{12}m_{13} - m_{11}m_{23})(\Gamma_i)_y + (m_{11}m_{22} - m_{12}^2)(\Gamma_i)_z \right). \end{array} \right.$$

Now, taking $i = 1,2,3,4$ and using (13), (54), (56), we have

(60)
$$\begin{cases} \mathfrak{U}_1 = \frac{(1+f'^2)g''h''}{W^2 f''},\ \mathfrak{U}_2 = \frac{f'g'h''}{W^2}, \\ \mathfrak{U}_3 = \frac{f'h'g''}{W^2},\ \mathfrak{U}_4 = \frac{f'\left(h'^2 f''g''+\left(g'^2 f''+(1+f'^2)g''\right)h''\right)}{W^2 f''}; \\ \\ \mathfrak{V}_1 = \frac{f'g'h''}{W^2},\ \mathfrak{V}_2 = \frac{(1+g'^2)f''h''}{W^2 g''}, \\ \mathfrak{V}_3 = \frac{g'h'f''}{W^2},\ \mathfrak{V}_4 = \frac{g'\left(h'^2 f''g''+\left(f'^2 g''+(1+g'^2)f''\right)h''\right)}{W^2 g''}; \\ \\ \mathfrak{W}_1 = \frac{f'h'g''}{W^2},\ \mathfrak{W}_2 = \frac{g'h'f''}{W^2}, \\ \mathfrak{W}_3 = \frac{(1+h'^2)f''g''}{W^2 h''},\ \mathfrak{W}_4 = \frac{h'\left(f'^2 g''h''+\left(g'^2 h''+(1+h'^2)g''\right)f''\right)}{W^2 h''}. \end{cases}$$

Thus, using (60) in (58), we obtain the LB$^{\text{III}}$ operator of the translation hypersurface (13) as follows:

Theorem 3.5. *The LB^{III} operator of the translation hypersurface (13) in E^4 is*

(61)
$$\Delta^{III}\Gamma = \left(\begin{array}{c} \frac{W^2\left(2f'f''^2 - f''' - f'^2 f'''\right)}{f''^3}, \frac{W^2\left(2g'g''^2 - g''' - g'^2 g'''\right)}{g''^3}, \frac{W^2\left(2h'h''^2 - h''' - h'^2 h'''\right)}{h''^3}, \\ \\ \frac{W^2}{f''^3 g''^3 h''^3} \left(\begin{array}{c} \left(1+3f'^2\right) f''^2 g''^3 h''^3 - f'\left(1+f'^2\right) g''^3 h''^3 f''' \\ -f''^3 \left(\begin{array}{c} g'\left(1+g'^2\right) h''^3 g''' - \left(1+3g'^2\right) g''^2 h''^3 \\ +g''^3\left(h'\left(1+h'^2\right)h''' - \left(1+3h'^2\right)h''^2\right) \end{array} \right) \end{array} \right) \end{array} \right).$$

We say that a hypersurface Γ is called LB$^{\text{III}}$-minimal, if it satisfies $\Delta^{III}\Gamma = 0$. Thus, from Theorem 3.5, we can prove the following theorem:

Theorem 3.6. *LB^{III}-minimal translation hypersurfaces (13) in E^4 can be parametrized by*

(62)
$$\Gamma(x,y,z) = \left(\begin{array}{l} x, y, z, \\ -\left(\frac{\ln[\cos(k_1(x+k_2))]}{k_1} + \frac{\ln[\cos(k_3(y+k_4))]}{k_3} + \frac{\ln[\cos(k_5(z+k_6))]}{k_5}\right) + k_7 \end{array} \right),$$

where $k_1 = \frac{-k_3 k_5}{k_3+k_5}$, $k_i \in \mathbb{R}$.

Proof. Let us suppose that the translation hypersurface (13) is LB$^{\text{III}}$-minimal in E^4. Then all components of $\Delta^{III}\Gamma$ obtained as (61) must be zero. So, for the first three components of (61) to be zero, the following third order nonlinear ordinary differential equations must hold:

(63) $$2f'f''^2 - f''' - f'^2 f''' = 0,$$

(64) $$2g'g''^2 - g''' - g'^2 g''' = 0,$$

(65) $$2h'h''^2 - h''' - h'^2 h''' = 0.$$

The solutions of the third order nonlinear ordinary differential equations (63)-(65) are

$$f(x) = -\frac{\ln[\cos(k_1(x+k_2))]}{k_1} + k_8, \tag{66}$$

$$g(y) = -\frac{\ln[\cos(k_3(y+k_4))]}{k_3} + k_9, \tag{67}$$

$$h(z) = -\frac{\ln[\cos(k_5(z+k_6))]}{k_5} + k_{10}, \tag{68}$$

respectively. For the last component of (61) to be zero, it must be

$$\begin{aligned} &\left(1+3f'^2\right) f''^2 g''^3 h''^3 - f'\left(1+f'^2\right) g''^3 h''^3 f''' \\ &- f''^3 \left(\begin{array}{c} g'\left(1+g'^2\right) h''^3 g''' - \left(1+3g'^2\right) g''^2 h''^3 \\ +g''^3 \left(h'\left(1+h'^2\right) h''' - \left(1+3h'^2\right) h''^2\right) \end{array} \right) = 0. \end{aligned} \tag{69}$$

If we use (66)-(68) in (69), then we have

$$k_1^2 k_3^2 k_5^2 \left(k_3 k_5 + k_1 k_3 + k_1 k_5\right) \left(\sec(k_1(x+k_2)) \sec(k_3(y+k_4)) \sec(k_5(z+k_6))\right)^6 = 0. \tag{70}$$

Here, one can easily see that the equation (70) holds for $k_3 k_5 + k_1(k_3 + k_5) = 0$. And the proof completes for $k_7 = k_8 + k_9 + k_{10}$. □

4. Conclusion and Future Work

As we have stated in the first section of this study, the Laplacian and also LB operator is used in many application areas, such as physics (quantum mechanics, the diffusion equation for fluid flow and heat, electric potential, gravitational potential, and wave propagation); geometry processing applications (mesh filtering, parameterization, pose transfer, segmentation, reconstruction, re-meshing, compression, simulation, and interpolation via barycentric coordinates); image processing, signal processing, surface processing; the study of geometric partial differential equations; CDT configurations; Shape-DNA; a numerical fingerprint or signature; different clustering and partition problems and etc.

In addition, by expanding the surface examinations from 3D to 4D space, more optimal results can be obtained in different applications using this geometric information. For example, images accepted as a surface in image processing will be subjected to 4D surface analysis and will contribute to the solution of different engineering problems such as fingerprint recognition, face recognition, object detection and classification of radar images. With the help of derivative, eigenvalue-eigenvector calculations, Gaussian and mean curvature calculations and the calculation of the LB operator, which are widely used in these fields, distinctive pixel behaviors in the image can be analyzed by moving them to 4D instead of 3D. Thus, the negativities caused by the situations such as translation, rotation and illumination changes, which are frequently encountered in the images, will be eliminated and the change information of the neighboring pixels will be obtained in a more detailed way in 4-dimensional form.

Also, in the near future LB^{I}, $\mathrm{LB}^{\mathrm{II}}$ and $\mathrm{LB}^{\mathrm{III}}$ operators can be handled for different hypersurfaces in different four dimensional spaces, such as Lorentz-Minkowski 4-space, Galilean 4-space or pseudo-Galilean 4-space. And we hope that, this study will bring a new viewpoint and break fresh ground to scientists from different areas who are dealing with the application areas of Laplacian or LB operator.

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