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ON GENERALIZED W_3 RECURRENT RIEMANNIAN MANIFOLDS

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Abstract. The object of the present work is to study a generalized W_3 recurrent manifold. We obtain a necessary and sufficient condition for the scalar curvature to be constant in such a manifold. Also, sufficient condition for generalized W_3 recurrent manifold to be special quasi-Einstein manifold are given. Ricci symmetric and decomposable generalized W_3 recurrent manifold are studied. Finally, the existence of such a manifold is ensured by a non-trivial example.

1. Introduction

Let (M^n, g) be an n-dimensional smooth Riemannian manifold and ∇ be the covariant differentiation with respect to the metric tensor g. Symmetric spaces play a significant role in the study of differential geometry. Cartan [3] studied Riemanian symmetric spaces and obtain its classification. A Riemannian manifold is said to be a locally symmetric manifold [3] if $\nabla K = 0$.

Generalized recurrent Riemannian manifolds have been studied by several authors in different context such as Singh and Khan [11], De and Pal [6], De and Gazi [4], Arslan et. al. [2], De and Guha [5] etc. Semi-generalized W_3 recurrent manifolds has been studied by K. Lalnunsiami and J. P. Singh [8].

A Riemannian manifold (M^n, g) (n$\geq$ 3) is said to be a generalized recurrent manifold [5] if the Riemann curvature tensor K of type $(1, 3)$ satisfies the condition

$$(1) \quad (\nabla_X K)(Y, Z, W) = A(X)K(Y, Z, W) + B(X)\Big[g(Z, W)Y - g(Y, W)Z\Big],$$

where A and B are two 1-forms in which B is non-zero defined as

$$g(X, \rho) = A(X) \qquad and \qquad g(X, \sigma) = B(X),$$

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for every vector field X. Here the vector fields ρ and σ are called the basic vector fields of the manifold corresponding to the associated 1-forms A and B respectively. Such a manifold has been denoted by GK_n. If $B = 0$, then GK_n reduces to a recurrent manifold [7] denoted by K_n.

Contracting (1) with respect to Y, we get

$$(\nabla_X \mathrm{Ric})(Z,W) = A(X)\mathrm{Ric}(Z,W) + (n-1)B(X)g(Z,W). \tag{2}$$

In this case, the Riemannian manifold (M^n, g) is called a generalized Ricci recurrent manifold [1]. If the 1-form $B(X)$ becomes zero in (2), then the generalized Ricci recurrent manifold reduces to a Ricci-recurrent manifold.

A non-flat Riemannian manifold (M^n, g) $(n \geq 3)$ is defined to be a quasi-Einstein manifold [12] if its Ricci tensor is not identically zero and satisfies the condition

$$\mathrm{Ric}(X,Y) = ag(X,Y) + bE(X)E(Y),$$

where $a, b \neq 0$ are scalars and E is a non-zero 1-form such that

$$g(X,U) = E(X),$$

for all vector fields X; U being a unit vector field. If a and b are constants, we call such a manifold is a special quasi-Einstein manifold.

In 1973, Pokhariyal [9] introduced a new curvature tensor of type $(1,3)$ in an n-dimensional Riemannian manifold (M^n, g), $(n > 2)$ denoted by W_3 and defined by

$$W_3(Y,Z,U) = K(Y,Z,U) + \frac{1}{n-1}\Big[g(Z,U)R(Y) - Ric(Y,U)Z\Big], \tag{3}$$

where K denotes the Riemannian curvature tensor of type $(1,3)$ and R is the Ricci tensor of type $(1,1)$, defined as

$$g(R(X),Y) = \mathrm{Ric}(X,Y), \tag{4}$$

for every differentiable vector fields X, Y.

From (3) we can define a $(0,4)$ type W_3 curvature tensor $\tilde{W_3}$ as follows

$$\begin{aligned}\tilde{W_3}(Y,Z,U,V) = \tilde{K}(Y,Z,U,V) + \frac{1}{n-1}\Big[&g(Z,U)\mathrm{Ric}(Y,V) \\ &- g(Z,V)\mathrm{Ric}(Y,U)\Big],\end{aligned} \tag{5}$$

where $\tilde{K}$ denotes the Riemannian curvature tensor of type $(0,4)$ defined by

$$\tilde{K}(Y,Z,U,V) = g(K(Y,Z,U),V),$$

and

$$\tilde{W_3}(Y,Z,U,V) = g(W_3(Y,Z,U),V).$$

From (5), we have

$$\sum_{i=1}^{n} \tilde{W_3}(e_i,Z,U,e_i) = \frac{1}{n-1}\Big[(n-2)Ric(Z,U) + rg(Z,U)\Big], \tag{6}$$

$$\sum_{i=1}^{n} \tilde{W}_3(Y, e_i, e_i, V) = 2Ric(Y, V), \tag{7}$$

and

$$\sum_{i=1}^{n} \tilde{W}_3(Y, Z, e_i, e_i) = 0 = \sum_{i=1}^{n} \tilde{W}_3(e_i, e_i, U, V). \tag{8}$$

Also,

$$\begin{cases} \tilde{W}_3(Y, Z, U, V) \neq -\tilde{W}_3(Z, Y, U, V), \\ \tilde{W}_3(Y, Z, U, V) = -\tilde{W}_3(Y, Z, V, U), \\ \tilde{W}_3(Y, Z, U, V) \neq \tilde{W}_3(U, V, Y, Z), \\ \tilde{W}_3(Y, Z, U, V) + \tilde{W}_3(Z, U, Y, V) + \tilde{W}_3(U, Y, Z, V) \neq 0. \end{cases} \tag{9}$$

In this paper, we have considered a non-flat n-dimensional Riemannian manifold in which the $\tilde{W}_3$ curvature tensor satisfies the condition

$$\begin{aligned} (\nabla_X \tilde{W}_3)(Y, Z, U, V) &= A(X)\tilde{W}_3(Y, Z, U, V) \\ &\quad + B(X)\Big[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\Big], \end{aligned} \tag{10}$$

where A and B are 1-forms. Such an n-dimensional Riemannian manifold will be called a generalized W_3 recurrent manifold. If the 1-form B is zero, then the manifold reduces to W_3 recurrent manifold.

The paper is presented as follows: After introduction in Section 2, we obtain a necessary and sufficient condition for the scalar curvature to be constant in a generalized W_3 recurrent manifold. In Section 3, Ricci symmetric generalized W_3 recurrent manifolds are studied. In the next section, sufficient condition for a generalized W_3 recurrent manifold to be a special quasi-Einstein manifold are given. Section 5 is on the study of decomposable generalized W_3 recurrent manifold. Finally, the existence of such a manifold is ensured by a non-trivial example.

2. Generalized W_3 recurrent manifold with constant scalar curvature

In this section, we obtain a necessary and sufficient condition for the scalar curvature to be constant in a generalized W_3 recurrent manifold.

Taking covariant derivative of (5) with respect to X and then using (10), we get

$$\begin{aligned}(\nabla_X\tilde{K})(Y,Z,U,V) &= A(X)\tilde{K}(Y,Z,U,V) + B(X)\Big[g(Z,U)g(Y,V)\\ &- g(Y,U)g(Z,V)\Big] + \frac{1}{n-1}\Big[A(X)\Big\{g(Z,U)\text{Ric}(Y,V)\\ &- g(Z,V)\text{Ric}(Y,U)\Big\} - \Big\{g(Z,U)(\nabla_X\text{Ric})(Y,V)\\ &- g(Z,V)(\nabla_X\text{Ric})(Y,U)\Big\}\Big].\end{aligned}\tag{11}$$

Permuting (11) over X, Y, Z and then using Bianchi's second identity, we have

$$\begin{aligned}&A(X)\tilde{K}(Y,Z,U,V) + A(Y)\tilde{K}(Z,X,U,V) + A(Z)\tilde{K}(X,Y,U,V)\\ &+ B(X)\Big[g(Z,U)g(Y,V) - g(Y,U)g(Z,V)\Big] + B(Y)\Big[g(X,U)g(Z,V)\\ &- g(Z,U)g(X,V)\Big] + B(Z)\Big[g(Y,U)g(X,V) - g(X,U)g(Y,V)\Big]\\ &+ \frac{1}{n-1}\Big[A(X)\{g(Z,U)\text{Ric}(Y,V) - g(Z,V)\text{Ric}(Y,U)\}\\ &+ A(Y)\{g(X,U)\text{Ric}(Z,V) - g(X,V)\text{Ric}(Z,U)\}\\ &+ A(Z)\{g(Y,U)\text{Ric}(X,V) - g(Y,V)\text{Ric}(X,U)\}\\ &- \{g(Z,U)(\nabla_X\text{Ric})(Y,V) - g(Z,V)(\nabla_X\text{Ric})(Y,U)\}\\ &- \{g(X,U)(\nabla_Y\text{Ric})(Z,V) - g(X,V)(\nabla_Y\text{Ric})(Z,U)\}\\ &- \{g(Y,U)(\nabla_Z\text{Ric})(X,V) - g(Y,V)(\nabla_Z\text{Ric})(X,U)\}\Big] = 0.\end{aligned}\tag{12}$$

Setting $Y = V = e_i$ in the equation (12) and using (6), (7) and (8), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and then taking summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}&A(X)\text{Ric}(Z,U) + A(K(Z,X,U)) - A(Z)\text{Ric}(X,U)\\ &+ (n-2)B(X)g(Z,U) - (n-2)B(Z)g(X,U)\\ &+ \frac{1}{n-1}\Big[A(X)\{rg(Z,U) - 2\text{Ric}(Z,U)\} + A(R(Z))g(X,U)\\ &- (n-1)A(Z)\text{Ric}(X,U) + 2(\nabla_X\text{Ric})(U,Z) + (n-1)(\nabla_Z\text{Ric}(X,U))\\ &- \{dr(X)g(Z,U) + \frac{dr(Z)}{2}g(X,U)\}\Big] = 0.\end{aligned}\tag{13}$$

Contracting the equation (13) over Z and U, we obtain

$$\Big(\frac{2n-3}{n-1}\Big)rA(X) - \Big(\frac{3n-4}{n-1}\Big)A(R(X)) + (n-1)(n-2)B(X) - \frac{(n-4)}{2(n-1)}dr(X) = 0,$$

which implies that

$$rA(X) = \frac{(3n-4)}{(2n-3)}A(R(X)) - \frac{(n-1)^2(n-2)}{(2n-3)}B(X) + \frac{(n-4)}{2(2n-3)}dr(X).\tag{14}$$

Thus, we can state the following result:

Theorem 2.1. *Necessary and sufficient condition for a generalized W_3 recurrent manifold is that the scalar curvature r is constant if and only if*

$$rA(X) = \frac{(3n-4)}{(2n-3)}A(R(X)) - \frac{(n-1)^2(n-2)}{(2n-3)}B(X)$$

for all vector fields X.

Now, we consider that the scalar curvature r in a generalized W_3 recurrent manifold is constant. Then the relation (14) reduces to

$$rA(X) = \frac{(3n-4)}{(2n-3)}A(R(X)) - \frac{(n-1)^2(n-2)}{(2n-3)}B(X). \tag{15}$$

Contracting the equation (11) over Y and V, we get

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Z,U) &= A(X)\mathrm{Ric}(Z,U) + (n-1)B(X)g(Z,U)\\ &\quad + \frac{1}{n-1}\Big[A(X)\Big\{rg(Z,U) - \mathrm{Ric}(Z,U)\Big\}\\ &\quad - \Big\{dr(X)g(Z,U) - (\nabla_X \mathrm{Ric})(Z,U)\Big\}\Big],\end{aligned} \tag{16}$$

which in view of (15), the equation (16) gives

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Z,U) &= A(X)\mathrm{Ric}(Z,U)\\ &\quad + \Big[\frac{(3n-4)}{(2n-3)(n-2)}A(R(X)) - \frac{(n-1)^3}{(2n-3)(n-2)}B(X)\Big]g(Z,U).\end{aligned}$$

The above expression can be written as

$$(\nabla_X \mathrm{Ric})(Z,U) = A(X)\mathrm{Ric}(Z,U) + (n-1)D(X)g(Z,U), \tag{17}$$

where $D(X) = \frac{1}{n-1}\Big[\frac{(3n-4)}{(2n-3)(n-2)}A(R(X)) - \frac{(n-1)^3}{(2n-3)(n-2)}B(X)\Big]$. The Relation (17) is of the type (2). Thus, the considered manifold is a generalized Ricci-recurrent manifold. Hence, we can state the following theorem:

Theorem 2.2. *A generalized W_3 recurrent manifold with constant scalar curvature is generalized Ricci recurrent manifold.*

3. Ricci symmetric generalized W_3 recurrent manifold

Assume that the generalized W_3 recurrent manifold is Ricci symmetric. Then, $\nabla\mathrm{Ric} = 0$, i.e., $\nabla R = 0$. This implies that r is constant and $dr = 0$. Then, from the equation (16), we have

$$\Big(\frac{n-2}{n-1}\Big)A(X)\mathrm{Ric}(Z,U) + \Big\{\frac{r}{n-1}A(X) + (n-1)B(X)\Big\}g(Z,U) = 0. \tag{18}$$

From the equation (15), we have

$$B(X) = -\frac{(2n-3)}{(n-2)(n-1)^2}A(X) + \frac{(3n-4)}{(n-2)(n-1)^2}A(R(X)), \tag{19}$$

which in view of (19), the relation (18) becomes

$$\text{Ric}(Z,U) = \frac{1}{(n-2)^2}\Big[(n-1)r - (3n-4)\frac{A(R(X))}{A(X)}\Big]g(Z,U),$$

where we take X so that (at least locally) $A(X) \neq 0$. In order to guarantee that $A \neq 0$ we have to assume that M is not locally symmetric. Assume $\lambda = \frac{1}{(n-2)^2}\Big[(n-1)r - (3n-4)\frac{A(R(X))}{A(X)}\Big]$ is a scalar. Then the above relation takes the form

$$\text{Ric}(Z,U) = \lambda g(Z,U),$$

which shows that the manifold is an Einstein manifold. Therefore, we have the following result:

Theorem 3.1. *A Ricci symmetric generalized W_3 recurrent manifold is an Einstein manifold.*

4. Sufficient condition for a generalized W_3 recurrent manifold to be a quasi Einstein manifold

In this section, we would like to obtain a sufficient condition for a generalized W_3 recurrent manifold to be a quasi-Einstein manifold.

Now, from the equation (16), we have

$$\begin{aligned}(\nabla_X \text{Ric})(Z,U) &= A(X)\text{Ric}(Z,U) \\ &+ \frac{n}{n-2}\Big[\frac{(n-1)^2}{n}B(X) + \frac{rA(X)}{n} - \frac{dr(X)}{n}\Big]g(Z,U).\end{aligned} \tag{20}$$

A vector field P defined by $g(X,P) = A(X)$ is said to be a concircular vector field [10] if

$$(\nabla_X A)(Y) = \lambda g(X,Y) + \omega(X)A(Y), \tag{21}$$

where λ is a smooth function and ω is a closed 1-form. If P is unit, then the equation (21) can be written as

$$(\nabla_X A)(Y) = \lambda\Big[g(X,Y) - A(X)A(Y)\Big]. \tag{22}$$

Suppose a generalized W_3 recurrent manifold admits a unit concircular vector field P with a non-zero constant λ. Using Ricci identity in the equation (22), we have

$$A(K(X,Y,Z)) = -\lambda^2\Big[g(X,Z)A(Y) - g(Y,Z)A(X)\Big]. \tag{23}$$

Contraction of (23) with respect to Y and Z gives

$$A(R(X)) = (n-1)\lambda^2 A(X). \tag{24}$$

From (4), we have

$$\mathrm{Ric}(X,P)=(n-1)\lambda^2 A(X). \tag{25}$$

We know that

$$(\nabla_X \mathrm{Ric})(Y,P)=\nabla_X \mathrm{Ric}(Y,P)-\mathrm{Ric}(\nabla_X Y,P)-\mathrm{Ric}(Y,\nabla_X P). \tag{26}$$

Using (25) in (26), we obtain

$$(\nabla_X \mathrm{Ric})(Y,P)=(n-1)\lambda^2\nabla_X A(Y)-(n-1)\lambda^2 A(\nabla_X Y)-\mathrm{Ric}(Y,\nabla_X P),$$

or

$$(\nabla_X \mathrm{Ric})(Y,P)=(n-1)\lambda^2(\nabla_X A)(Y)-\mathrm{Ric}(Y,\nabla_X P)$$

which in view of (22) gives

$$(\nabla_X \mathrm{Ric})(Y,P)=(n-1)\lambda^2\Big[g(X,Y)-A(X)A(Y)\Big]-\mathrm{Ric}(Y,\nabla_X P). \tag{27}$$

Now,

$$\begin{aligned}(\nabla_X A)(Y)&=\nabla_X A(Y)-A(\nabla_X Y)=\nabla_X g(Y,P)-g(\nabla_X Y,P)\\&=g(Y,\nabla_X P),\quad \textit{since}\quad (\nabla_X g)(Y,P)=0.\end{aligned}$$

By virtue of the equation (22), implies that

$$\lambda\Big[g(X,Y)-A(X)A(Y)\Big]=g(Y,\nabla_X P),$$

$\Rightarrow g(\lambda X,Y)-g(\lambda A(X)P,Y)=g(\nabla_X P,Y)$, or $\nabla_X P=\lambda\Big[X-A(X)P\Big]$.

Therefore,

$$\mathrm{Ric}(Y,\nabla_X P)=\mathrm{Ric}(Y,\lambda X)-\mathrm{Ric}(Y,\lambda A(X)P),$$

which implies

$$\mathrm{Ric}(Y,\nabla_X P)=\lambda\Big[\mathrm{Ric}(X,Y)-A(X)\mathrm{Ric}(Y,P)\Big]. \tag{28}$$

Making use of (28) in (27), we have

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Y,P)&=(n-1)\lambda^3\Big[g(X,Y)-A(X)A(Y)\Big]\\&\quad-\lambda\Big[\mathrm{Ric}(X,Y)-A(X)\mathrm{Ric}(Y,P)\Big].\end{aligned} \tag{29}$$

Using (25) in (29), we obtain

$$(\nabla_X \mathrm{Ric})(Y,P)=(n-1)\lambda^3 g(X,Y)-\lambda\mathrm{Ric}(X,Y). \tag{30}$$

From the equation (20), we have

$$\begin{aligned}(\nabla_X \mathrm{Ric})(Y,P)&=A(X)\mathrm{Ric}(Y,P)+\frac{n}{n-2}\Big[\frac{(n-1)^2}{n}B(X)\\&\quad+\frac{rA(X)}{n}-\frac{dr(X)}{n}\Big]g(Y,P).\end{aligned} \tag{31}$$

Using (25) and (30) in (31), we have

$$(n-1)\lambda^3 g(X,Y) - \lambda \mathrm{Ric}(X,Y) = (n-1)\lambda^2 A(X)A(Y) + (n-1)\Big[\frac{(n-1)}{(n-2)}B(X) - \frac{r}{(n-1)(n-2)}A(X) + \frac{dr(X)}{(n-1)(n-2)}\Big]A(Y). \tag{32}$$

From (15) and (32), we find

$$\mathrm{Ric}(X,Y) = (n-1)\lambda^2 g(X,Y) + \Big[\frac{r}{\lambda(n-2)} - \frac{(3n-4)}{(n-2)^2}\Big\{(n-1)\lambda - \frac{r(2n-3)}{\lambda}\Big\} - (n-1)\lambda\Big]A(X)A(Y),$$

which can be written as

$$\mathrm{Ric}(X,Y) = ag(X,Y) + bA(X)A(Y),$$

where $a = (n-1)\lambda^2$ and $b = \Big[\frac{r}{\lambda(n-2)} - \frac{(3n-4)}{(n-2)^2}\Big\{(n-1)\lambda - \frac{r(2n-3)}{\lambda}\Big\} - (n-1)\lambda\Big]$ are two non-zero constants. Which is a special quasi-Einstein manifold. Thus, we are in the position to state the following theorem:

Theorem 4.1. *If the scalar curvature in a generalized W_3 recurrent manifold is constant and the associated unit vector field P is a unit concircular vector field whose associated scalar is a non-zero constant, then the manifold reduces to a special quasi-Einstein manifold.*

5. Decomposable generalized W_3 recurrent manifold

A Riemannian manifold (M^n, g) $(n > 2)$ is said to be a decomposable Riemannian manifold [10] if it can be expressed in the form $M^n = M_1^p \times M_2^{n-p}$ for some p, $2 \le p \le (n-2)$, i.e., in some coordinate neighbourhood of M^n, the metric g can be written as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{33}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$, $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^*, $a, b, c...$ runs from 1 to p and $\alpha, \beta, \gamma, ...$ runs from $p+1$ to n. M_1^p and M_2^{n-p} are called the components of M^n.

Suppose a generalized W_3 recurrent manifold (M^n, g) $(n > 2)$ is decomposable. Then, $M^n = M_1^p \times M_2^{n-p}$ for some p, $2 \le p \le (n-2)$. Let $\bar{X}, \bar{Y}, \bar{Z}, \bar{U}, \bar{V} \in \mathfrak{X}(M_1)$, $X^*, Y^*, Z^*, U^*, V^* \in \mathfrak{X}(M_2)$. Since M^n is decomposable, we have

$$Ric(\bar{X}, \bar{Y}) = \overline{Ric}(\bar{X}, \bar{Y}),$$
$$Ric(X^*, Y^*) = Ric^*(X^*, Y^*),$$
$$(\nabla_{\bar{X}} Ric)(\bar{Y}, \bar{Z}) = (\bar{\nabla}_{\bar{X}} Ric)(\bar{Y}, \bar{Z}),$$

$$(\nabla_{X^*}Ric)(Y^*,Z^*) = (\nabla^*_{X^*}Ric)(Y^*,Z^*)$$

and $r = \bar{r} + r^*$. From (5), we have

$$\tilde{W}_3(\bar{X},\bar{Y},\bar{Z},\bar{U}) = \bar{W}_3(\bar{X},\bar{Y},\bar{Z},\bar{U}), \tag{34}$$

$$\tilde{W}_3(X^*,Y^*,Z^*,U^*) = W_3^*(X^*,Y^*,Z^*,U^*),$$

$$\tilde{W}_3(Y^*,\bar{Z},\bar{U},\bar{V}) = 0 = \tilde{W}_3(\bar{Y},Z^*,U^*,V^*) = \tilde{W}_3(\bar{Y},Z^*,\bar{U},\bar{V}) = \tilde{W}_3(\bar{Y},\bar{Z},U^*,V^*),$$

$$\tilde{W}_3(\bar{Y},Z^*,U^*,\bar{V}) = \frac{1}{(n-1)}g(Z^*,U^*)Ric(\bar{Y},\bar{V}),$$

$$\tilde{W}_3(Y^*,\bar{Z},\bar{U},V^*) = \frac{1}{(n-1)}g(\bar{Z},\bar{U})Ric(Y^*,V^*),$$

$$\tilde{W}_3(Y^*,\bar{Z},U^*,\bar{V}) = -\frac{1}{(n-1)}g(\bar{Z},\bar{V})Ric(Y^*,U^*), \tag{35}$$

$$\tilde{W}_3(\bar{Y},Z^*,\bar{U},V^*) = -\frac{1}{(n-1)}g(Z^*,V^*)Ric(\bar{Y},\bar{U}), \tag{36}$$

$$(\nabla_{X^*}\tilde{W}_3)(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0 = (\nabla_{\bar{X}}\tilde{W}_3)(Y^*,Z^*,U^*,V^*).$$

From (10), we get

$$\begin{aligned}(\nabla_{\bar{X}}\tilde{W}_3)(\bar{Y},\bar{Z},\bar{U},\bar{V}) &= A(\bar{X})\tilde{W}_3(\bar{Y},\bar{Z},\bar{U},\bar{V})\\ &\quad + B(\bar{X})\Big[g(\bar{Z},\bar{U})g(\bar{Y},\bar{V}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})\Big],\end{aligned}$$

$$A(X^*)\tilde{W}_3(\bar{Y},\bar{Z},\bar{U},\bar{V}) + B(X^*)\Big[g(\bar{Z},\bar{U})g(\bar{Y},\bar{V}) - g(\bar{Y},\bar{U})g(\bar{Z},\bar{V})\Big] = 0, \tag{37}$$

and

$$B_{(\bar{p},p^*)}(0 \oplus v) = 0$$

for every $\bar{p} \in M_1, p^* \in M_2$ and $v \in T_{p^*}M_2$. Also for every $(\bar{p},p^*) \in M$ from (10), we obtain

$$(\nabla_{X^*}\tilde{W}_3)_{(\bar{p},p^*)}(Y^*,Z^*,U^*,V^*) = (\nabla^*_{X^*}{\tilde{W}_3}^{\,*})_{p^*}(Y^*,Z^*,U^*,V^*), \tag{38}$$

and the R. H. S. does not depend on $\bar{p} \in M_1$.

If possible let $B(X^*) = 0$ for all $X^* \in \mathfrak{X}(M_2)$, then from (37) we get

$$A(X^*)\tilde{W}_3(\bar{Y},\bar{Z},\bar{U},\bar{V}) = 0. \tag{39}$$

Using (34) in above equation, we get

$$A(X^*)\bar{W}_3(\bar{X},\bar{Y},\bar{Z},\bar{U}) = 0. \tag{40}$$

If M_1 is not W_3 flat, that is, $(\bar{W}_3)_{\bar{p}_0} \neq 0$ for some $\bar{p}_0 \in M_1$, then from (39) and (40), it follows that

$$A_{(\bar{p},p^*)}(0 \oplus v) = 0 \tag{41}$$

for every $\bar{p} \in M_1$, $p^* \in M_2$ and for every $v \in T_{p^*}M_2$. Hence (10) yields

$$(\nabla_{X^*}\tilde{W}_3)_{(\bar{p},p^*)}(Y^*,Z^*,U^*,V^*) = 0$$

for every $\bar{p} \in M_1$ and $p^* \in M_2$. It follows that if M_1 is not W_3 flat, then

$$A_{(\bar{p},p^*)}(X^*)(\tilde{W_3}^*)_{p^*}(Y^*, Z^*, U^*, V^*) = 0 \tag{42}$$

for all $\bar{p} \in M_1$ and $p^* \in M_2$.

Now. we assume that

$$\begin{aligned}(\nabla_X \tilde{W_3})(Y, Z, U, V) &= \bar{A}(X)\tilde{W_3}(Y, Z, U, V) \\ &\quad + \bar{B}(X)\Big[g(Z, U)g(Y, V) - g(Y, U)g(Z, V)\Big],\end{aligned} \tag{43}$$

where $\bar{A}$ and $\bar{B}$ are 1-forms. Putting (43) in (10), we get

$$\begin{aligned}[A(X) - \bar{A}(X)]\tilde{W_3}(Y, Z, U, V) + [B(X) - \bar{B}(X)][g(Z, U)g(Y, V) \\ -g(Y, U)g(Z, V)] = 0.\end{aligned} \tag{44}$$

Contraction of (44) over Y and V, gives

$$\begin{aligned}[A(X) - \bar{A}(X)]\Big[Ric(Z, U) - \frac{1}{(n-1)}\{rg(Z, U) - Ric(Z, U)\}\Big] \\ +(n-1)[B(X) - \bar{B}(X)]g(Z, U) = 0.\end{aligned} \tag{45}$$

Again, contracting the equation (45) over Z and U, we have

$$B(X) = \bar{B}(X), \tag{46}$$

In view of (46), the relation (44) becomes

$$A(X) = \bar{A}(X),$$

for all $X \in M^n$ provided $W_3 \neq 0$, i.e., the manifold is not W_3 flat. Thus, the 1-forms A and B are uniquely determined provided that the manifold is not W_3 flat. So, from equation (42) we obtain

$$A_{(\bar{p},p^*)}(X^*) = 0 \tag{47}$$

for all $\bar{p} \in M_1$ and $p^* \in M_2$.

From (40) we conclude that either

(i) $A(X^*) = 0$ for all $X^* \in \mathfrak{X}(M_2)$, or
(ii) M_1 is W_3 flat.

Also, from the equation (10), we have

$$\begin{aligned}(\nabla_{X^*} \tilde{W_3})(Y^*, \bar{Z}, \bar{U}, V^*) &= A(X^*)\tilde{W_3}(Y^*, \bar{Z}, \bar{U}, V^*) \\ &\quad + B(X^*)\Big[g(\bar{Z}, \bar{U})g(Y^*, V^*) - g(Y^*, \bar{U})g(\bar{Z}, V^*)\Big].\end{aligned} \tag{48}$$

Now, we consider the case (i). From (48), it follows that

$$(\nabla_{X^*} \tilde{W_3})(Y^*, \bar{Z}, \bar{U}, V^*) = 0,$$

which by virtue of (36) gives

$$(\nabla_{X^*} Ric)(Y^*, V^*) = 0. \tag{49}$$

Hence, the component M_2 is Ricci symmetric. Using (36), (38), (41), (42) and (47) and $A(X^*) = 0$, $B(X^*) = 0$ for all $X^* \in \mathfrak{X}(M_2)$, from (10), we have

$$(\nabla_{X^*}\tilde{W}_3)(Y^*, Z^*, U^*, V^*) = 0$$

and hence

$$(\nabla_{X^*}\tilde{K})(Y^*, Z^*, U^*V^*) + \frac{1}{(n-1)}\Big[g(Z^*, U^*)(\nabla_{X^*}Ric)(Y^*, V^*) - g(Z^*, V^*)(\nabla_{X^*}Ric)(Y^*, U^*)\Big] = 0,$$

which by virtue of the equation (49) yields

$$(\nabla_{X^*}\tilde{K})(Y^*, Z^*, U^*V^*) = 0,$$

that is, the component M_2 is locally symmetric. A similar result can be proved for M_1. Thus we have the following result:

Theorem 5.1. *Let M^n be a decomposable generalized W_3 recurrent manifold which is not W_3 flat such that $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq (n-2)$. If $B(X^*) = 0$ for all $X \in M_2$,(respectively $B(\bar{X}) = 0$ for all $\bar{X} \in M_1$), then either (i) or (ii) holds.*

(i) *$A(X^*) = 0$ for all $X \in \mathfrak{X}(M_2)$, (respectively $A(\bar{X}) = 0$, for all $X \in \mathfrak{X}(M_1)$), and hence M_2(respectively M_1) is Ricci symmetric as well as locally symmetric.*

(ii) *M_2(respectively M_1) is W_3 flat.*

Also, from the equation (10), we have

$$(\nabla_{\bar{X}}\tilde{W}_3)(\bar{Y}, Z^*, U^*, V^*) = A(\bar{X})\tilde{W}_3(\bar{Y}, Z^*, U^*, V^*) + B(\bar{X})\Big[g(Z^*, U^*)g(\bar{Y}, \bar{V}) - g(\bar{Y}, U^*)g(Z^*, \bar{V})\Big]. \tag{50}$$

Using (35) in (50), we get

$$\frac{1}{(n-1)}g(Z^*, U^*)(\nabla_{\bar{X}}Ric)(\bar{Y}, \bar{V}) = \frac{A(\bar{X})}{(n-1)}g(Z^*, U^*)Ric(\bar{Y}, \bar{V}) + B(\bar{X})g(Z^*, U^*)g(\bar{Y}, \bar{V}). \tag{51}$$

Now, we assume that $Ric(Z^*, U^*) = 0$ and $g(Z^*, U^*) \neq 0$. Then from (51) we get

$$(\nabla_{\bar{X}}Ric)(\bar{Y}, \bar{V}) = A(\bar{X})Ric(\bar{Y}, \bar{V}) + (n-1)B(\bar{X})g(\bar{Y}, \bar{V}).$$

Therefore, we have the following theorem:

Theorem 5.2. *Let M^n be a decomposable generalized W_3 recurrent manifold which is not W_3 flat such that $M^n = M_1^p \times M_2^{n-p}$, $2 \leq p \leq (n-2)$. Then M_1(respectively) M_2 is generalized Ricci recurrent.*

6. Example of a generalized W_3 recurrent manifold

In this section, we construct an example of a generalized W_3 recurrent manifold and shown that the existence of such a manifold by considering the following metric.

We define a Riemannian metric g on the 4-dimensional real number space $\mathbb{R}^4$ by the relation

$$ds^2 = g_{ij}dx^i dx^j = (1-4p)\Big[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2\Big], \tag{52}$$

where $p = \frac{e^{x^1}}{k^2}$, for a non-zero constant k and $x^1 \neq 0$. Then the non-vanishing components of covariant and contravariant metric tensor in (52) are

$$g_{11} = g_{22} = g_{33} = g_{44} = 1-4p$$

and

$$g^{11} = g^{22} = g^{33} = g^{44} = \frac{1}{1-4p}.$$

In the metric considered the only non-vanishing components of the Christoffel symbols are

$$\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{2p}{1-4p}, \tag{53}$$

$$\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = -\frac{2p}{1-4p}. \tag{54}$$

The non-zero derivatives of equations (53) and (54) as follows:

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{2p}{(1-4p)^2},$$

$$\frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = -\frac{2p}{(1-4p)^2}.$$

The Riemannian curvature tensor as follows

$$K^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial x^j} & \frac{\partial}{\partial x^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}. \tag{55}$$

The non-zero components of (I) in (55) goes as follows:

$$K^1_{212} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{2p}{(1-4p)^2},$$

$$K^1_{313} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{2p}{(1-4p)^2},$$

$$K^1_{414} = \frac{\partial}{\partial x^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{2p}{(1-4p)^2},$$

and the non-zero components of (II) in (55) goes as follows:

$$K^1_{313} = \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ m3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} = -\frac{4p^2}{(1-4p)^2},$$

$$K^1_{414} = \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 1 \\ m1 \end{Bmatrix} - \begin{Bmatrix} m \\ 41 \end{Bmatrix}\begin{Bmatrix} 1 \\ m4 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} - \begin{Bmatrix} 1 \\ 31 \end{Bmatrix}\begin{Bmatrix} 1 \\ 13 \end{Bmatrix} = -\frac{4p^2}{(1-4p)^2},$$

Now, using these components in (55), we get

$$K^1_{212} = \frac{2p}{(1-4p)^2}, \quad K^1_{313} = K^1_{414} = \frac{2p-4p^2}{(1-4p)^2}.$$

Thus, the non-vanishing components of the Riemannian curvature tensor of type $(0,4)$ up to symmetry are:

$$\tilde{K}_{1212} = \frac{2p}{1-4p}, \quad \tilde{K}_{1313} = \tilde{K}_{1414} = \frac{2p-4p^2}{1-4p},$$

and the Ricci tensor of type $(0,2)$ goes as follows:

$$\mathrm{Ric}_{11} = -\frac{6p}{(1-4p)^2}, \qquad \mathrm{Ric}_{22} = \mathrm{Ric}_{33} = -\frac{2p}{(1-4p)^2}.$$

Now, using $r = g^{ij}\mathrm{Ric}_{ij}$, we get $r = \frac{12p}{(1-4p)^3}$, which is non-zero. By virtue of the equation (5), we get the non-zero components of the $\tilde{W_3}$ curvature tensor goes as follows:

$$(\tilde{W_3})_{1212} = \frac{4p}{1-4p}, \qquad (\tilde{W_3})_{1313} = (\tilde{W_3})_{1414} = \frac{8p}{3(1-4p)},$$

whose non-zero covariant derivatives are

$$(\tilde{W_3})_{1212,1} = \frac{4p}{(1-4p)^2}, \qquad (\tilde{W_3})_{1313,1} = (\tilde{W_3})_{1414,1} = \frac{8p}{3(1-4p)^2}$$

where ',' denotes the covariant derivative with respect to the metric tensor.

To show that $(\mathbb{R}^4, g)$ is a generalized W_3 recurrent manifold, let us consider the associated 1-forms as follows

$$A_i = \begin{cases} \dfrac{16p^2-32p+5}{1-4p}, & \text{if i=1} \\ 0, & \text{otherwise} \end{cases}, \quad and \quad B_i = \begin{cases} \dfrac{16p}{(1-4p)^2}, & \text{if i=1} \\ 0, & \text{otherwise} \end{cases}.$$

To verify the relation (10), it is sufficient to check the following relations

$$(\tilde{W_3})_{1212,1} = A_1(\tilde{W_3})_{1212} + B_1\Big[g_{21}g_{12} - g_{11}g_{22}\Big], \tag{56}$$

$$(\tilde{W_3})_{1313,1} = A_1(\tilde{W_3})_{1313} + B_1\Big[g_{31}g_{13} - g_{11}g_{33}\Big], \tag{57}$$

and

$$(\tilde{W_3})_{1414,1} = A_1(\tilde{W_3})_{1414} + B_1\Big[g_{41}g_{14} - g_{11}g_{44}\Big]. \tag{58}$$

Since for the other cases the relation (10) holds trivially.

$$\begin{aligned} R.H.S. \quad of \quad (56) &= A_1(\tilde{W}_3)_{1212} + B_1\Big[g_{21}g_{12} - g_{11}g_{22}\Big] \\ &= \frac{(16p^2 - 32p + 5)}{(1-4p)} \times (\frac{4p}{1-4p}) + \frac{16p}{(1-4p)^2}\Big[0 - (1-4p)^2\Big] \\ &= \frac{4p}{(1-4p)^2} \\ &= (\tilde{W}_3)_{1212,1} \\ &= L.H.S. \quad of \quad (56). \end{aligned}$$

By a similar argument, it can be shown that (57) and (58) are also true. Therefore the manifold ($\mathbb{R}^4$, g) is a generalized W_3 recurrent Riemannian manifold.

As a consequence of the above, one can say that

Theorem 6.1. *There exists a manifold* $(\mathbb{R}^4, g)$ *which is a generalized* W_3 *recurrent Riemannian manifold with the above mentioned choice of the* 1*-forms.*

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