Honam Mathematical J. **45** (2023), No. 2, pp. 300–315
https://doi.org/10.5831/HMJ.2023.45.2.300

ON THE BIHARMONICITY OF VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

Amina Alem, Bouazza Kacimi, and Mustafa Özkan*

Abstract. In this article, we deal with the biharmonicity of a vector field X viewed as a map from a pseudo-Riemannian manifold (M, g) into its tangent bundle TM endowed with the Sasaki metric g_S. Precisely, we characterize those vector fields which are biharmonic maps, and find the relationship between them and biharmonic vector fields. Afterwards, we study the biharmonicity of left-invariant vector fields on the three dimensional Heisenberg group endowed with a left-invariant Lorentzian metric. Finally, we give examples of vector fields which are proper biharmonic maps on the Gödel universe.

1. Introduction

Let (M, g) and (N, h) be smooth pseudo-Riemannian manifolds of dimensions m and n respectively, and let $\varphi : (M, g) \to (N, h)$ be a smooth map between them. The energy functional or the Dirichlet energy of φ over a compact domain D of M is defined by

$$E(\varphi, D) = \frac{1}{2}\int_D \sum_{i=1}^{m} \varepsilon_i h(d\varphi(e_i), d\varphi(e_i))v_g, \tag{1}$$

where $\{e_i\}_{i=1}^m$ a local pseudo-orthonormal frame field of (M, g) with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices $i = 1, 2, \cdots, m$. If M is compact, we write $E(\varphi) = E(\varphi, M)$. The map φ is called harmonic if it is a critical point of the energy functional (1). The Euler-Lagrange equation of (1) is [2, 8]

$$\tau(\varphi) = \mathrm{Tr}_g(\nabla d\varphi) = \sum_{i=1}^{m} \varepsilon_i\{\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)\} = 0.$$

Here $\tau(\varphi)$ is the tension field of φ and ∇^{φ} denotes the connection on the vector bundle $\varphi^{-1}TN \to M$ induced from the Levi-Civita connection ∇^N of (N, h) and ∇ the Levi-Civita connection of (M, g).

Received November 05, 2022. Accepted January 16, 2023.
2020 Mathematics Subject Classification. 58E20, 53C20.
Key words and phrases. tangent bundle, Sasaki metric, biharmonic maps, Gödel universe.
*Corresponding author

Now, denote by $\mathfrak{X}(M)$ the set of all smooth vector fields on M and by g_S the Sasaki metric on the tangent bundle TM. Any $X \in \mathfrak{X}(M)$ determines a smooth map from (M, g) to (TM, g_S). The energy of X is, by definition, the energy of the corresponding map. When M is compact and g is positive definite, it was proved in [11, 16] that $X : (M, g) \to (TM, g_S)$ is an harmonic map if and only if X is parallel, moreover this results remain true if X is a harmonic vector field i.e. X is a critical point of the energy functional E restricted to the set $\mathfrak{X}(M)$ see [9]. In contrast to the Riemannian case, it was shown in [3] the existence of non-parallel left-invariant vector fields which define harmonic maps on three dimensional unimodular and non-unimodular Lorentzian Lie groups.

One of the first generalizations of harmonic maps is the notion of polyharmonic maps of order k between Riemannian manifolds introduced by Eells and Lemaire in [7]. For $k = 2$, they defined the bienergy of φ as the functional

$$E_2(\varphi) = \frac{1}{2}\int_D \|\tau(\varphi)\|^2 \, v_g,$$

and a smooth map φ is biharmonic if and only if it is a critical point of E_2. The associated Euler-Lagrange equation is established in [12]. By definition, it can be seen that every harmonic map is biharmonic. However, a biharmonic map can be non-harmonic in which case it is called proper biharmonic. We refer to [17, 19] for more information on results concerning the theory of biharmonic maps. The notion of biharmonic map between Riemannian manifolds has been extended to the case of pseudo-Riemannian manifolds. The corresponding critical point condition has been derived in [5] as follows

$$\tau_2(\varphi) = \sum_{i=1}^{m} \varepsilon_i \Big(\big(\nabla^\varphi_{e_i} \nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla^M_{e_i} e_i}\big)\tau(\varphi) - R^N(d\varphi(e_i), \tau(\varphi))d\varphi(e_i) \Big) = 0,$$

where $\tau_2(\varphi)$ is the bitension field of φ and R^N is the curvature tensor of N.

On the other hand, when (M, g) is the pseudo-Riemannian manifold, Markellos and Urakawa [15] defined the bienergy of $X \in \mathfrak{X}(M)$ as the bienergy of the corresponding map (see [14] for the Riemannian case) and obtained the critical point of the bienergy functional E_2 restricted to the set $\mathfrak{X}(M)$ (equivalently, X is a biharmonic vector field, see [14] for the Riemannian case), further in [14] they proved that if g is positive definite and M is compact then X is biharmonic vector field (resp. biharmonic map) if and only if X is parallel. In [1], when g is positive definite we established the formula of the bitension field of $X : (M, g) \to (TM, g_S)$ and obtained characterization theorem for a vector field to be biharmonic map, furthermore we constructed an example of non-parallel vector field which is biharmonic map on the solvable Lie group Sol_3 and we have shown that a left-invariant vector field X on three dimensional unimodular Lie group is biharmonic vector field (resp. biharmonic map) if and only if X is parallel.

Let (M, g) be a pseudo-Riemannian manifold. In this note, we will study the biharmonicity of $X \in \mathfrak{X}(M)$ viewed as a map $X : (M, g) \to (TM, g_S)$ in pseudo-Riemannian settings. More precisely, we address the problem of characterizing those vector fields which are biharmonic maps, and we examine the relationship between vector fields X that are critical points of the functional E_2 restricted to variations through vector fields (equivalently, X are biharmonic vector fields, see [15]) and vector fields which are biharmonic maps. This paper is organized as follows. In Section 2, we collect some basic facts that will be needed later. In section 3, we compute the formula of the bitension field of $X : (M, g) \to (TM, g_S)$ (see Theorem 3.1) and we characterize those vector fields which are biharmonic maps (see Theorem 3.2). On making use of the formula of the bitension field of X, we give a simple proof of the first variational formula associated to the bienergy functional E_2 restricted to the space $\mathfrak{X}(M)$ (see Theorem 3.4). As a corollary, we obtain the critical point condition characterizes biharmonic vector fields (see Corollary 3.5), consequently we get the relationship between biharmonic vector fields and vector fields which are biharmonic maps (see Corollary 3.7). In section 4, we entirely determine the set of left-invariant vector fields which are biharmonic (resp. biharmonic maps) on the three dimensional Heisenberg group endowed with a left-invariant Lorentzian metric. Finally, in section 5, we give examples of vector fields which are proper biharmonic maps on the Gödel universe.

2. Preliminaries

We recall here some basic facts on the geometry of tangent bundle. We refer the reader to [4, 13, 20] and references therein for further details. Let (M, g) be an m-dimensional pseudo-Riemannian manifold and (TM, π, M) be its tangent bundle, ∇ denotes the associated Levi-Civita connection and R the corresponding Riemannian curvature tensor taken with the sign convention

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all vector fields X, Y and Z on M. A local chart $(U, x^i)_{1 \leq i \leq m}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{1 \leq i \leq m}$ on TM, denotes the Christoffel symbols of g by Γ^i_{jk}. The tangent space $T_{(x,u)}TM$ at a point (x, u) in TM is a direct sum of the vertical subspace $\mathcal{V}_{(x,u)} = \ker(d\pi \mid_{(x,u)})$ and the horizontal subspace $\mathcal{H}_{(x,u)}$, with respect to the Levi-Civita connection ∇ of M:

$$T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}.$$

Let $X|_U = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined respectively by:

$$X^v|_{\pi^{-1}(U)} = (X^i \circ \pi) \frac{\partial}{\partial y^i},$$

and

$$X^h|_{\pi^{-1}(U)} = (X^i \circ \pi) \frac{\partial}{\partial x^i} - (\Gamma^i_{jk} \circ \pi)(X^j \circ \pi) y^k \frac{\partial}{\partial y^i}.$$

The Sasaki metric on TM is the pseudo-Riemannian metric g_S defined by

$$g_S(X^h, Y^h) = g_S(X^v, Y^v) = g(X,Y) \circ \pi, \; g_S(X^v, Y^h) = 0,$$

for any $X, Y \in \Gamma(TM)$. Denoting by $\widetilde{\nabla}$ the Levi-Civita connection of g_S, one has the following formulas [13]

$$\begin{aligned}
(\widetilde{\nabla}_{X^h} Y^h)_{(x,u)} &= (\nabla_X Y)^h_{(x,u)} - \frac{1}{2}(R(X,Y)Z)^v_{(x,u)}, \\
(\widetilde{\nabla}_{X^h} Y^v)_{(x,u)} &= (\nabla_X Y)^v_{(x,u)} + \frac{1}{2}(R(Z,Y)X)^h_{(x,u)}, \\
(\widetilde{\nabla}_{X^v} Y^h)_{(x,u)} &= \frac{1}{2}(R(Z,X)Y)^h_{(x,u)}, \\
(\widetilde{\nabla}_{X^v} Y^v)_{(x,u)} &= 0,
\end{aligned} \tag{2}$$

for any $X, Y \in \Gamma(TM)$ and any $(x,u) \in TM$, where $Z \in \Gamma(TM)$ such that $Z_{\pi(x,u)} = (x,u)$.

A vector field X on (M,g) can be viewed as the immersion $X : (M,g) \to (TM, g_S)$; $x \mapsto (x, X_x) \in TM$ into its tangent bundle TM equipped with the Sasaki metric g_S. The energy of X is, by definition, the energy of the corresponding map $X : (M,g) \to (TM, g_S)$, that is [10]

$$E(X) = \frac{1}{2}\int_M \|dX\|^2 v_g = \frac{m}{2} Vol(M) + \frac{1}{2}\int_M \|\nabla X\|^2 v_g \tag{3}$$

(assuming M compact; in the non-compact case, one works over compact domain). Moreover, the tension field $\tau(X)$ is given by [9]

$$\tau(X) = \Big(-\sum_{i=1}^m \varepsilon_i R(\nabla_{e_i} X, X) e_i \Big)^h + \Big(\sum_{i=1}^m \varepsilon_i (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X) \Big)^v.$$

We can rewrite $\tau(X)$ as follows [15]:

$$\tau(X) = (-S(X))^h + (\nabla^* \nabla X)^v, \tag{4}$$

where

$$S(X) = \sum_{i=1}^m \varepsilon_i R(\nabla_{e_i} X, X) e_i,$$

and $\nabla^*\nabla X$ is the rough Laplacian given by

$$\nabla^* \nabla X = \sum_{i=1}^m \varepsilon_i (\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X).$$

A vector field X defines a harmonic map from (M,g) to (TM, g_S) if and only if $\tau(X) = 0$, equivalently $\nabla^*\nabla X = 0$ and $S(X) = 0$, X is called harmonic vector field if it is a critical point of the energy functional (3), only considering variations among maps defined by vector fields. The corresponding Euler-Lagrange equation is given by $\nabla^*\nabla X = 0$, so X is a harmonic map if and only if X is a harmonic vector field and $S(X) = 0$.

3. Biharmonicity of vector fields

In the next Theorem, we compute the bitension field $\tau_2(X)$ of X.

Theorem 3.1. *Let (M,g) be an m-dimensional pseudo-Riemannian manifold and (TM,g_S) its tangent bundle equipped with the Sasaki metric. If $X:(M,g)\to(TM,g_S)$ is a smooth vector field then the bitension field of X is given by*

$$
\begin{aligned}
\tau_2(X) =&\Big\{(\nabla^*\nabla)^2X+\sum_{i=1}^m\varepsilon_i[(\nabla_{e_i}R)(e_i,S(X))X+R(e_i,\nabla_{e_i}S(X))X\\
&+2R(e_i,S(X))\nabla_{e_i}X]\Big\}^v+\Big\{-\nabla^*\nabla S(X)-R(X,\nabla^*\nabla X)S(X)\\
&+\sum_{i=1}^m\varepsilon_i[R(X,\nabla_{e_i}\nabla^*\nabla X)e_i-R(\nabla_{e_i}X,\nabla^*\nabla X)e_i+R(e_i,S(X))e_i\\
&-(\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i-R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)\\
&+R(X,R(e_i,S(X))X)e_i]\Big\}^h. \qquad (5)
\end{aligned}
$$

Proof. Let $(x,u)\in TM$ and $\{e_i\}_{i=1}^m$ be a local pseudo-orthonormal frame on M such that $\nabla_{e_i}e_i=0$ at $x\in M$ and $u=X_x$. If $Y\in\Gamma(TM)$ then, we have (see [6, pp. 50])

$$dX(Y)=\{Y^h+(\nabla_YX)^v\}\circ X, \qquad (6)$$

using (2), (4) and (6), one has

$$
\begin{aligned}
\sum_{i=1}^m\nabla^X_{e_i}\tau(X)\big|_{(x,X_x)} =&\sum_{i=1}^m\big[\widetilde{\nabla}_{e_i^h+(\nabla_{e_i}X)^v}(-S(X)^h+\nabla^*\nabla X^v)\big]\big|_{(x,X_x)}\\
=&\Big\{\sum_{i=1}^m\big[-\nabla_{e_i}S(X)-\frac{1}{2}R(X,\nabla_{e_i}X)S(X)\\
&+\frac{1}{2}R(X,\nabla^*\nabla X)e_i\big]\Big\}^h_{(x,X_x)}+\Big\{\sum_{i=1}^m\big[\nabla_{e_i}\nabla^*\nabla X\\
&+\frac{1}{2}R(e_i,S(X))X\big]\Big\}^v_{(x,X_x)}
\end{aligned}
$$

and

$$\sum_{i=1}^{m} \varepsilon_i \nabla^X_{e_i} \nabla^X_{e_i} \tau(X)\big|_{(x,X_x)} = \Big\{ \sum_{i=1}^{m} \varepsilon_i \big[-\nabla_{e_i}\nabla_{e_i} S(X) - \frac{1}{2}\nabla_{e_i} R(X,\nabla_{e_i}X)S(X)$$
$$+ \frac{1}{2}\nabla_{e_i} R(X,\nabla^*\nabla X)e_i - \frac{1}{2}R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)$$
$$- \frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla_{e_i}X)S(X) + \frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla^*\nabla X)e_i$$
$$+ \frac{1}{2}R(X,\nabla_{e_i}\nabla^*\nabla X)e_i + \frac{1}{4}R(X,R(e_i,S(X))X)e_i\big]\Big\}^h_{(x,X_x)}$$
$$+ \Big\{ \sum_{i=1}^{m} \varepsilon_i \big[\nabla_{e_i}\nabla_{e_i}\nabla^*\nabla X + \frac{1}{2}\nabla_{e_i} R(e_i,S(X))X$$
$$+ \frac{1}{2}R(e_i,\nabla_{e_i}S(X))X + \frac{1}{4}R(e_i,R(X,\nabla_{e_i}X)S(X))X$$
$$- \frac{1}{4}R(e_i,R(X,\nabla^*\nabla X)e_i)X\big]\Big\}^v_{(x,X_x)}. \tag{7}$$

Let $\widetilde{R}$ the curvature tensor field of $\widetilde{\nabla}$. On making use of Theorem 1 in [13], we find

$$-\sum_{i=1}^{m} \varepsilon_i \widetilde{R}(dX(e_i),\tau(X))dX(e_i)\big|_{(x,X_x)} = \Big\{ -\frac{1}{2}R(X,\nabla^*\nabla X)S(X)$$
$$+ \sum_{i=1}^{m} \varepsilon_i \big[R(e_i,S(X))e_i + \frac{3}{4}R(X,R(e_i,S(X))X)e_i$$
$$- (\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i + \frac{1}{2}(\nabla_{e_i}R)(X,\nabla_{e_i}X)S(X)$$
$$+ \frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla_{e_i}X)S(X) - \frac{1}{2}(\nabla_{e_i}R)(X,\nabla^*\nabla X)e_i$$
$$+ \frac{3}{2}R(\nabla^*\nabla X,\nabla_{e_i}X)e_i - \frac{1}{4}R(X,\nabla_{e_i}X)R(X,\nabla^*\nabla X)e_i\big]\Big\}^h_{(x,X_x)}$$
$$+ \Big\{ \sum_{i=1}^{m} \varepsilon_i \big[\frac{1}{2}(\nabla_{e_i}R)(e_i,S(X))X - \frac{3}{2}R(S(X),e_i)\nabla_{e_i}X$$
$$+ \frac{1}{4}R(R(X,\nabla_{e_i}X)S(X),e_i)X - \frac{1}{4}R(R(X,\nabla^*\nabla X)e_i,e_i)X\big]\Big\}^v_{(x,X_x)}. \tag{8}$$

On the other hand, we have the following formulae

$$\sum_{i=1}^{m} \varepsilon_i \nabla_{e_i} R(X,\nabla_{e_i}X)S(X) = \sum_{i=1}^{m} \varepsilon_i \big[(\nabla_{e_i}R)(X,\nabla_{e_i}X)S(X)$$
$$+ R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)\big] + R(X,\nabla^*\nabla X)S(X), \tag{9}$$

$$\begin{aligned}\sum_{i=1}^{m}\varepsilon_i\nabla_{e_i}R(X,\nabla^*\nabla X)e_i =&\sum_{i=1}^{m}\varepsilon_i\big[(\nabla_{e_i}R)(X,\nabla^*\nabla X)e_i+R(\nabla_{e_i}X,\nabla^*\nabla X)e_i\\ &+R(X,\nabla_{e_i}\nabla^*\nabla X)e_i\big],\end{aligned}\tag{10}$$

$$\begin{aligned}\sum_{i=1}^{m}\varepsilon_i\nabla_{e_i}R(e_i,S(X))X =&\sum_{i=1}^{m}\varepsilon_i\big[(\nabla_{e_i}R)(e_i,S(X))X+R(e_i,\nabla_{e_i}S(X))X\\ &+R(e_i,S(X))\nabla_{e_i}X\big].\end{aligned}\tag{11}$$

One can calculate $\tau_2(X)$ by summing up (7) and (8) and using the formulae (9)-(11). $\square$

Then, we give the following characterization theorem.

Theorem 3.2. *Let (M,g) be an m-dimensional pseudo-Riemannian manifold and $X\in\mathfrak{X}(M)$, then $X:(M,g)\to(TM,g_S)$ is a biharmonic map if and only if*

$$\begin{aligned}(\nabla^*\nabla)^2X+\sum_{i=1}^{m}\varepsilon_i[(\nabla_{e_i}R)(e_i,S(X))X\\ +R(e_i,\nabla_{e_i}S(X))X+2R(e_i,S(X))\nabla_{e_i}X]=0,\end{aligned}$$

and

$$\begin{aligned}&-\nabla^*\nabla S(X)-R(X,\nabla^*\nabla X)S(X)+\sum_{i=1}^{m}\varepsilon_i[R(X,\nabla_{e_i}\nabla^*\nabla X)e_i\\ &\quad-R(\nabla_{e_i}X,\nabla^*\nabla X)e_i+R(e_i,S(X))e_i-(\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i\\ &\qquad-R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)+R(X,R(e_i,S(X))X)e_i]=0,\end{aligned}$$

where $\{e_i\}_{i=1}^m$ is a local pseudo-orthonormal frame field of (M,g).

Definition 3.3 ([15])**.** *Let (M,g) be a pseudo-Riemannian manifold. A vector field $X\in\mathfrak{X}(M)$ is called biharmonic if the corresponding map $X:(M,g)\longrightarrow(TM,g_S)$ is a critical point for the bienergy functional E_2, only considering variations among maps defined by vector fields.*

By virtue of the formula (5), one obtain another proof of the next Theorem given in [14].

Theorem 3.4. *Let (M,g) be a compact oriented m-dimensional pseudo-Riemannian manifold, $\{e_i\}_{i=1}^m$ a local pseudo-orthonormal frame field of (M,g), X a tangent vector field on M and $E_2:\mathfrak{X}(M)\longrightarrow[0,+\infty)$ the bienergy functional restricted to the space of all vector fields. Then*

$$\begin{aligned}\frac{d}{dt}E_2(X_t)\Big|_{t=0}=&\int_M\Big\{g((\nabla^*\nabla)^2X+\sum_{i=1}^{m}\varepsilon_i\big[(\nabla_{e_i}R)(e_i,S(X))X\\ &+R(e_i,\nabla_{e_i}S(X))X+2R(e_i,S(X))\nabla_{e_i}X\big],V)\Big\}v_g\end{aligned}$$

for any smooth 1*-parameter variation* $U : M \times (-\epsilon, \epsilon) \to TM$ *of* X *through vector fields i.e.,* $X_t(z) = U(z,t) \in T_zM$ *for any* $|t| < \epsilon$ *and* $z \in M$*, or equivalently* $X_t \in \mathfrak{X}(M)$ *for any* $|t| < \epsilon$*. Also,* V *is the tangent vector field on* M *given by*

$$V(z) = \frac{d}{dt} X_z(0), \quad z \in M,$$

where $X_z(t) = U(z,t)$*,* $(z,t) \in M \times (-\epsilon, \epsilon)$*.*

Proof. Let $U : M \times (-\epsilon, \epsilon) \to TM$ be a smooth variation of X(i.e., $U(z,0) = X(z)$ for any $z \in M$) such that $X_t(z) = U(z,t) \in T_zM$ for any $z \in M$ and any $|t| < \epsilon$. We have

$$E_2(X_t) = \frac{1}{2} \int_M \|\tau(X_t)\|^2 \, v_g.$$

As in the Riemannian case [12], we can write

$$\frac{d}{dt} E_2(X_t)\Big|_{t=0} = \int_M g_S(\mathcal{V}, \tau_2(X)) v_g,$$

where $\mathcal{V}(z) = \frac{d}{dt} X_t(z)\big|_{t=0}$, $z \in M$, however from [6, pp. 58], we have

$$\mathcal{V} = V^v \circ X. \tag{12}$$

On making use of the expression of $\tau_2(X)$ given by (5) and (12), we find

$$\begin{aligned} \frac{d}{dt} E_2(X_t)\Big|_{t=0} &= \int_M g_S(V^v, \tau_2(X)) v_g \\ &= \int_M \Big\{ g(V, (\nabla^*\nabla)^2 X + \sum_{i=1}^m \varepsilon_i \big[(\nabla_{e_i} R)(e_i, S(X))X \\ &\qquad + R(e_i, \nabla_{e_i} S(X))X + 2R(e_i, S(X))\nabla_{e_i} X \big]) \Big\} v_g, \end{aligned}$$

which completes the proof. □

Then, we deduce the following [15].

Corollary 3.5. *A vector field* X *of an* m*-dimensional pseudo-Riemannian manifold* (M, g) *is biharmonic if and only if*

$$\begin{aligned} (\nabla^*\nabla)^2 X + \sum_{i=1}^m \varepsilon_i [(\nabla_{e_i} R)(e_i, S(X))X \\ + R(e_i, \nabla_{e_i} S(X))X + 2R(e_i, S(X))\nabla_{e_i} X] = 0, \end{aligned}$$

where $\{e_i\}_{i=1}^m$ *is a local pseudo-orthonormal frame field of* (M, g)*.*

Remark 3.6. *Theorem 3.4 holds if* (M, g) *is a non-compact pseudo-Riemannian manifold see* [15].

A reformulation of Theorem 3.2 is then

Corollary 3.7. *Let (M,g) be an m-dimensional pseudo-Riemannian manifold and $X \in \mathfrak{X}(M)$. Then X is a biharmonic map if and only if X is biharmonic vector field and*

$$\begin{aligned}
&-\nabla^*\nabla S(X) - R(X,\nabla^*\nabla X)S(X) + \sum_{i=1}^{m} \varepsilon_i [R(X,\nabla_{e_i}\nabla^*\nabla X)e_i \\
&\quad - R(\nabla_{e_i}X,\nabla^*\nabla X)e_i + R(e_i,S(X))e_i - (\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i \\
&\qquad\qquad - R(X,\nabla_{e_i}X)\nabla_{e_i}S(X) + R(X,R(e_i,S(X))X)e_i] = 0.
\end{aligned}$$

4. Biharmonicity of left-invariant vector fields of Heisenberg group

The Heisenberg group H_3 can be seen as the Cartesian 3-space $\mathbb{R}^3(x,y,z)$ endowed with multiplication

$$(x,y,z)(\bar{x},\bar{y},\bar{z}) = (x+\bar{x}, y+\bar{y}, z+\bar{z}-x\bar{y}).$$

H_3 is three-dimensional Lie group. In [18], the authors proved that any left-invariant Lorentzian metric on H_3, is isometric to one of the subsequent metrics

$$\begin{aligned}
g_1 &= -dx^2 + dy^2 + (xdy+dz)^2, \\
g_2 &= dx^2 + dy^2 - (xdy+dz)^2, \\
g_3 &= dx^2 + (xdy+dz)^2 - ((1-x)dy-dz)^2.
\end{aligned}$$

In this section we investigate biharmonicity of left-invariant vector fields on H_3 endowed with g_1, g_2 and g_3 respectively.

4.1. Biharmonicity of left-invariant vector fields on (H_3, g_1)

The aim of this subsection is to completely determine the set of left-invariant vector fields on (H_3, g_1) which are harmonic and biharmonic maps, respectively. The left-invariant vector fields

$$e_1 = \frac{\partial}{\partial z}, \;\; e_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \;\; e_3 = \frac{\partial}{\partial x},$$

constitute an orthonormal basis of the Lie algebra of H_3 with

$$g_1(e_1,e_1) = g_1(e_2,e_2) = 1, \;\; g_1(e_3,e_3) = -1,$$

for which, we have the Lie brackets:

$$[e_2,e_3] = e_1, \;\; [e_1,e_2] = 0, \;\; [e_1,e_3] = 0.$$

The components of the Levi-Civita connection of (H_3, g_1) are determined by [18]

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0, && \nabla_{e_1} e_2 = \frac{1}{2} e_3, && \nabla_{e_1} e_3 = \frac{1}{2} e_2,\\
&\nabla_{e_2} e_1 = \frac{1}{2} e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_2} e_3 = \frac{1}{2} e_1,\\
&\nabla_{e_3} e_1 = \frac{1}{2} e_2, && \nabla_{e_3} e_2 = -\frac{1}{2} e_1, && \nabla_{e_3} e_3 = 0.
\end{aligned} \tag{13}$$

Also the curvature components are given by

$$\begin{aligned}
&R(e_1, e_2)e_1 = \frac{1}{4} e_2, && R(e_1, e_2)e_2 = -\frac{1}{4} e_1, && R(e_1, e_2)e_3 = 0,\\
&R(e_2, e_3)e_1 = 0, && R(e_2, e_3)e_2 = -\frac{3}{4} e_3, && R(e_2, e_3)e_3 = -\frac{3}{4} e_2,\\
&R(e_3, e_1)e_1 = -\frac{1}{4} e_3, && R(e_3, e_1)e_2 = 0, && R(e_3, e_1)e_3 = -\frac{1}{4} e_1.
\end{aligned} \tag{14}$$

Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be an arbitrary left-invariant vector field on (H_3, g_1) where α, β and γ are constants. By using (13) and (14), one has

$$\nabla^* \nabla X = \frac{\alpha}{2} e_1 + \frac{\beta}{2} e_2 + \frac{\gamma}{2} e_3, \tag{15}$$

$$(\nabla^* \nabla)^2 X = \frac{\alpha}{4} e_1 + \frac{\beta}{4} e_2 + \frac{\gamma}{4} e_3, \tag{16}$$

$$S(X) = \frac{\alpha\gamma}{4} e_2 + \frac{\alpha\beta}{4} e_3. \tag{17}$$

By virtue of (13)-(17), a long but straightforward calculation we get

Proposition 4.1. *Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be a left-invariant vector field on the Lorentzian Lie group (H_3, g_1). Then,*

$$\begin{aligned}
&(\nabla^* \nabla)^2 X + \sum_{i=1}^{3} \varepsilon_i [(\nabla_{e_i} R)(e_i, S(X))X + R(e_i, \nabla_{e_i} S(X))X\\
&+ 2R(e_i, S(X))\nabla_{e_i} X] = \frac{\alpha(4 - (\beta^2 - \gamma^2))}{16} e_1 + \frac{\beta(4 - \alpha^2)}{16} e_2 + \frac{\gamma(4 - \alpha^2)}{16} e_3,
\end{aligned}$$

and

$$\begin{aligned}
&-\nabla^* \nabla S(X) - R(X, \nabla^* \nabla X)S(X) + \sum_{i=1}^{3} \varepsilon_i [R(X, \nabla_{e_i} \nabla^* \nabla X)e_i\\
&\quad - R(\nabla_{e_i} X, \nabla^* \nabla X)e_i + R(e_i, S(X))e_i - (\nabla_{S(X)} R)(X, \nabla_{e_i} X)e_i\\
&\quad - R(X, \nabla_{e_i} X)\nabla_{e_i} S(X) + R(X, R(e_i, S(X))X)e_i]\\
&= \frac{\alpha\gamma(-8 - 2(\gamma^2 - \beta^2) - \alpha^2)}{16} e_2 + \frac{\alpha\beta(-8 - 2(\gamma^2 - \beta^2) - \alpha^2)}{16} e_3.
\end{aligned}$$

From Proposition 4.1, we easily conclude that the vector field $X = \alpha e_1 + \beta e_2 + \gamma e_3$ is biharmonic map if and only if

$$\begin{cases} \alpha(4-(\beta^2-\gamma^2)) = 0, \\ \beta(4-\alpha^2) = 0, \\ \gamma(4-\alpha^2) = 0, \end{cases} \tag{18}$$

and

$$\begin{cases} \alpha\gamma(-8-2(\gamma^2-\beta^2)-\alpha^2) = 0, \\ \alpha\beta(-8-2(\gamma^2-\beta^2)-\alpha^2) = 0. \end{cases}$$

In particular, X is a biharmonic vector field if and only if (18) holds. From the system (18), we obtain that the coordinates of X satisfy the equations of hyperbolas: $C_1 = \{\alpha = 2,\ \beta^2 - \gamma^2 = 4\}$ and $C_2 = \{\alpha = -2,\ \beta^2 - \gamma^2 = 4\}$. Summarizing, we yield

Theorem 4.2. *Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be a left-invariant vector field on the Lorentzian Lie group (H_3, g_1). Then,*

1. *$X = \alpha e_1 + \beta e_2 + \gamma e_3$ is a biharmonic vector field which does not define biharmonic map if and only if the coordinates of X satisfy the equations of the equilateral hyperbolas C_1 and C_2.*
2. *The set of left-invariant vector fields which are proper biharmonic maps into TH_3 is empty.*

4.2. Biharmonicity of left-invariant vector fields on (H_3, g_2)

This subsection is devoted to the determination of the set of left-invariant vector fields on (H_3, g_2) which are harmonic and biharmonic maps, respectively. The left-invariant vector fields

$$e_1 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},\ \ e_2 = \frac{\partial}{\partial x},\ \ e_3 = \frac{\partial}{\partial z},$$

constitute an orthonormal basis of the Lie algebra of H_3 with

$$g_2(e_1, e_1) = g_2(e_2, e_2) = 1,\ \ g_2(e_3, e_3) = -1,$$

for which, we have the Lie brackets:

$$[e_1, e_2] = e_3,\ \ [e_1, e_3] = 0,\ \ [e_2, e_3] = 0.$$

The components of the Levi-Civita connection of (H_3, g_2) are determined by [18]

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = \frac{1}{2} e_3, & \nabla_{e_1} e_3 = \frac{1}{2} e_2, \\ \nabla_{e_2} e_1 = -\frac{1}{2} e_3, & \nabla_{e_2} e_2 = 0, & \nabla_{e_2} e_3 = -\frac{1}{2} e_1, \\ \nabla_{e_3} e_1 = \frac{1}{2} e_2, & \nabla_{e_3} e_2 = -\frac{1}{2} e_1, & \nabla_{e_3} e_3 = 0. \end{array} \tag{19}$$

Also the curvature components are given by

$$\begin{array}{llll} & R(e_1,e_2)e_1=-\frac{3}{4}e_2, & R(e_1,e_2)e_2=\frac{3}{4}e_1, & R(e_1,e_2)e_3=0,\\ (20) & R(e_2,e_3)e_1=0, & R(e_2,e_3)e_2=\frac{1}{4}e_3, & R(e_2,e_3)e_3=\frac{1}{4}e_2,\\ & R(e_3,e_1)e_1=-\frac{1}{4}e_3, & R(e_3,e_1)e_2=0, & R(e_3,e_1)e_3=-\frac{1}{4}e_1.\end{array}$$

Let $X=\alpha e_1+\beta e_2+\gamma e_3$ be an arbitrary left-invariant vector field on (H_3,g_2). By using (19) and (20), then one obtains

$$\nabla^*\nabla X=\frac{\alpha}{2}e_1+\frac{\beta}{2}e_2+\frac{\gamma}{2}e_3, \tag{21}$$

$$(\nabla^*\nabla)^2X=\frac{\alpha}{4}e_1+\frac{\beta}{4}e_2+\frac{\gamma}{4}e_3, \tag{22}$$

$$S(X)=\frac{-\beta\gamma}{4}e_1+\frac{\alpha\gamma}{4}e_2. \tag{23}$$

By virtue of (19)-(23), long but direct and easy calculations we get

Proposition 4.3. *Let $X=\alpha e_1+\beta e_2+\gamma e_3$ be a left-invariant vector field on the Lorentzian Lie group (H_3,g_2). Then,*

$$\begin{aligned}&(\nabla^*\nabla)^2X+\sum_{i=1}^{3}\varepsilon_i[(\nabla_{e_i}R)(e_i,S(X))X+R(e_i,\nabla_{e_i}S(X))X\\ &+2R(e_i,S(X))\nabla_{e_i}X]=\frac{\alpha(4+\gamma^2)}{16}e_1+\frac{\beta(4+\gamma^2)}{16}e_2+\frac{\gamma(4-(\alpha^2+\beta^2))}{16}e_3,\end{aligned}$$

and

$$\begin{aligned}&-\nabla^*\nabla S(X)-R(X,\nabla^*\nabla X)S(X)+\sum_{i=1}^{3}\varepsilon_i[R(X,\nabla_{e_i}\nabla^*\nabla X)e_i\\ &\quad-R(\nabla_{e_i}X,\nabla^*\nabla X)e_i+R(e_i,S(X))e_i-(\nabla_{S(X)}R)(X,\nabla_{e_i}X)e_i\\ &\quad-R(X,\nabla_{e_i}X)\nabla_{e_i}S(X)+R(X,R(e_i,S(X))X)e_i]\\ &=\frac{\beta\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2)}{32}e_1+\frac{-\alpha\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2)}{32}e_2.\end{aligned}$$

From Proposition 4.3, one conclude that the vector field $X=\alpha e_1+\beta e_2+\gamma e_3$ is biharmonic map if and only if

$$\begin{cases}\alpha(4+\gamma^2)=0,\\ \beta(4+\gamma^2)=0,\\ \gamma(4-(\alpha^2+\beta^2))=0,\end{cases} \tag{24}$$

and

$$\begin{cases}\beta\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2)=0,\\ \alpha\gamma(16+5(\alpha^2+\beta^2)-3\gamma^2)=0.\end{cases} \tag{25}$$

In particular, X is biharmonic vector field if and only if (24) holds. From (24) and (25), one has

Theorem 4.4. *We have the following statements on the Lorentzian Lie group (H_3, g_2):*

1. *The set of left-invariant biharmonic vector fields which do not define harmonic maps into TH_3 is empty.*
2. *The set of left-invariant vector fields which are proper biharmonic maps into TH_3 is empty.*

4.3. Biharmonicity of left-invariant vector fields on (H_3, g_3)

In this subsection we aim to completely determine the set of left-invariant vector fields on (H_3, g_3) which are harmonic and biharmonic maps, respectively. The left-invariant vector fields

$$e_1 = \frac{\partial}{\partial x}, \ \ e_2 = \frac{\partial}{\partial y} + (1-x)\frac{\partial}{\partial z}, \ \ e_3 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z},$$

constitute an orthonormal basis of the Lie algebra of H_3 with

$$g_3(e_1, e_1) = g_3(e_2, e_2) = 1, \ \ g_3(e_3, e_3) = -1,$$

for which, we have the Lie brackets:

$$[e_2, e_3] = 0, \ \ [e_3, e_1] = e_2 - e_3, \ \ [e_2, e_1] = e_2 - e_3.$$

The components of the Levi-Civita connection of (H_3, g_3) are determined by [18]

$$\begin{array}{lll} \nabla_{e_1} e_1 = 0, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = e_2 - e_3, & \nabla_{e_2} e_2 = -e_1, & \nabla_{e_2} e_3 = -e_1, \\ \nabla_{e_3} e_1 = e_2 - e_3, & \nabla_{e_3} e_2 = -e_1, & \nabla_{e_3} e_3 = -e_1. \end{array} \tag{26}$$

Let $X = \alpha e_1 + \beta e_2 + \gamma e_3$ be an arbitrary left-invariant vector field on (H_3, g_3). By using (26), we get that $\nabla^*\nabla X = 0$ and since g_3 is flat we deduce that $S(X) = 0$. Then, we yield

Theorem 4.5. *On the Lorentzian Lie group (H_3, g_3), every left-invariant vector field is biharmonic map.*

5. Gödel universe

An interesting space-time in general relativity is the classical Gödel universe [10]. This model is $\mathbb{R}^4$ endowed with the metric

$$\langle \cdot, \cdot \rangle_L = dx_1^2 + dx_2^2 - \frac{1}{2} e^{2\alpha x_1} dy^2 - 2e^{\alpha x_1} dy dt - dt^2,$$

where α is a positive constant. We denote by $\partial_{\bar{y}} = \sqrt{2}(e^{-\alpha x_1}\partial_y - \partial_t)$. The Levi-Civita connection in the pseudo-orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ where $e_1 = \partial_{x_1}$, $e_2 = \partial_{x_2}$, $e_3 = \partial_{\bar{y}}$ and $e_4 = \partial_t$, is given by [10]

$$\begin{aligned}
&\nabla_{e_1} e_4 = -\frac{\alpha}{\sqrt{2}} e_3, && \nabla_{e_2} e_4 = 0, && \nabla_{e_3} e_4 = \frac{\alpha}{\sqrt{2}} e_1, \\
&\nabla_{e_4} e_4 = 0, && \nabla_{e_1} e_1 = 0, && \nabla_{e_2} e_1 = 0, \\
&\nabla_{e_3} e_1 = \frac{\alpha}{\sqrt{2}} e_4 + \alpha e_3, && \nabla_{e_2} e_2 = 0, && \nabla_{e_3} e_2 = 0, \\
&\nabla_{e_1} e_3 = -\frac{\alpha}{\sqrt{2}} e_4, && \nabla_{e_3} e_3 = -\alpha e_1.
\end{aligned} \tag{27}$$

Taking the vector field $X = f(x_2)e_4$, where $f(x_2)$ is a smooth real function depending of the variable x_2. From [15] we have

$$R(e_1, e_4)e_3 = R(e_3, e_4)e_1 = 0, \tag{28}$$

$$\nabla^*\nabla X = (f'' + \alpha^2 f)e_4, \tag{29}$$

$$(\nabla^*\nabla)^2 X = (f'''' + 2\alpha^2 f'' + \alpha^4 f)e_4,$$

and

$$S(X) = 0,$$

where $f' = \frac{df}{dz}$, $f'' = \frac{d^2 f}{dz^2}$ etc. By virtue of relations (27), (28) and (29), we get

$$\sum_{i=1}^{3} \varepsilon_i R(X, \nabla_{e_i}\nabla^*\nabla X)e_i = 0, \text{ and } \sum_{i=1}^{3} \varepsilon_i R(\nabla_{e_i} X, \nabla^*\nabla X)e_i = 0.$$

Then, from Theorem 3.2, it follows that X is biharmonic map if and only if the function f satisfies the subsequent differential equation.

$$f'''' + 2\alpha^2 f'' + \alpha^4 f = 0. \tag{30}$$

Note that (30) is homogeneous fourth order differential equation with general solution see [15]

$$f(x_2) = c_1 \cos(\alpha x_2) + c_2 \sin(\alpha x_2) + c_3 x_2 \cos(\alpha x_2) + c_4 x_2 \sin(\alpha x_2), \tag{31}$$

where c_1, c_2, c_3 and c_4 are real constants. Particularly, in [14] Markellos and Urakawa proved that $X = f(x_2)e_4$ is biharmonic vector field, where $f(x_2)$ is given by (31).

Proposition 5.1. *The vector fields* $X = x_2(c_3 \cos(\alpha x_2) + c_4 \sin(\alpha x_2))e_4$ *are proper biharmonic maps of* $(\mathbb{R}^4, \langle\cdot,\cdot\rangle_L)$.

References

[1] A. Alem, B. Kacimi, and M. Özkan, *Vector fields which are biharmonic maps*, J. Geom. **113** (2022), Article Number 14.

[2] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, In: Lond. Math. Soc. Monogr., vol. 29, Oxford University Press, Oxford, 2003.

[3] G. Calvaruso, *Harmonicity properties of invariant vector fields on three-dimensional Lorentzian Lie groups*, J. Geom. Phys. **61** (2011), 498–515.

[4] P. Dombrowski, *On the geometry of tangent bundle*, J. Reine Angew. Math. **210** (1962), 73–88.

[5] Y. Dong and Y. Lin-Ou, *Biharmonic submanifolds of pseudo-Riemannian manifolds*, J. Geom. Phys. **112** (2017), 252–262.

[6] S. Dragomir and D. Perrone, *Harmonic Vector fields: Variational Principles and Differential Geometry*, Elsevier, Amsterdam, 2011.

[7] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conference Series in Mathematics, vol. 50, Amer. Math. Soc., Providence, 1983.

[8] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), no. 1, 109–160.

[9] O. Gil-Medrano, *Relationship between volume and energy of unit vector fields*, Differ. Geo. Appl. **15** (2001), 137–152.

[10] O. Gil-Medrano and A. Hurtado, *Spacelike energy of timelike unit vector fields on a Lorentzian manifold*, J. Geom. Phys. **51** (2004), 82–100.

[11] T. Ishihara, *Harmonic sections of tangent bundles*, J. Math. Tokushima Univ. **13** (1979), 23–27.

[12] G. Jiang, *2-Harmonic maps and their first and second variational formulas*, Translated into English by Hajime Urakawa. Note Mat. **28** (2008), no. 1, 209–232.

[13] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle of a Riemannian manifold*, J. Reine Angew. Math. **250** (1971), 124–129.

[14] M. Markellos and H. Urakawa, *The biharmonicity of sections of the tangent bundle*, Monatsh. Math. **178** (2015), 389–404.

[15] M. Markellos and H. Urakawa, *Biharmonic vector fields on pseudo-Riemannian manifolds*, J. Geom. Phys. **130** (2018), 293–314.

[16] O. Nouhaud, *Applications harmoniques d'une variété Riemannienne dans son fibré tangent*, C. R. Acad. Sci. Paris Sér. A **284** (1977), 815–818.

[17] Y. L. Ou and B. Y. Chen, *Biharmonic Submanifolds and Biharmonic Maps in Riemannian Geometry*, World Scientific, 2020.

[18] N. Rahmani and S. Rahmani, *Lorentzian geometry of the Heisenberg group*, Geom. Dedicata **118** (2006), 133–140.

[19] H. Urakawa, *Geometry of Biharmonic Mappings: Differential Geometry of Variational Methods*, World Scientific, 2018.

[20] K. Yano and S. Ishihara, *Tangent and Cotangent Bundles*, Dekker, New York, 1973.

Amina Alem
Department of Mathematics, Faculty of Exact Sciences, University of Mascara,
BP 305 Route de Mamounia, Mascara 29000, Algeria.
E-mail: amina.alem@univ-mascara.dz

Bouazza Kacimi
Department of Mathematics, Faculty of Exact Sciences, University of Mascara,
BP 305 Route de Mamounia, Mascara 29000, Algeria.
E-mail: bouazza.kacimi@univ-mascara.dz

Mustafa Özkan
Department of Mathematics, Faculty of Sciences, Gazi University,
Ankara 06500, Turkey.
E-mail: ozkanm@gazi.edu.tr