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PERTURBED FRACTIONAL NEWTON-TYPE INEQUALITIES BY TWICE DIFFERENTIABLE FUNCTIONS

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Abstract. In the present paper, we establish some perturbed Newton-type inequalities in the case of twice differentiable convex functions. These inequalities are established by using the well-known Riemann-Liouville fractional integrals. With the aid of special cases of our main results, we also give some previously obtained Newton-type inequalities.

1. Introduction

The inequality theory is a favoured subject in many mathematical areas with plenty of applications. Moreover, convex functions have a considerable status in the theory of inequality. Another significant result related to convex function is the Hermite–Hadamard inequality. Many researchers have established the Hermite–Hadamard inequality with great interest to generalize and extend it to the case of different classes of functions such as s-convex functions, quasi-convex functions, log-convex, etc. In recent years, Fractional calculus has grown attention because of its indicated applications in a wide range of seemingly different areas of science. Due to the significance of fractional calculus, researchers have considered distinct fractional integral inequalities. The bounds of new formulas by using the Hermite–Hadamard and Simpson's type inequalities.

Throughout the paper, we shall give fundamental fractional integral notations and different fractional integrals needed in the sequel. In the papers [9, 15], mathematical preliminaries of fractional calculus theory will be presented as follows:

Definition 1.1. [9, 15]. *Let us define $\varpi \in L_1[\mu,\eta]$. The Riemann–Liouville integrals $\mathbb{J}_{\mu+}^{\alpha}\varpi$ and $\mathbb{J}_{\eta-}^{\alpha}\varpi$ of order $\alpha>0$ with $\mu \geq 0$ are described by*

$$\mathbb{J}_{\mu+}^{\alpha}\varpi(x)=\frac{1}{\Gamma(\alpha)}\int_{\mu}^{x}(x-\xi)^{\alpha-1}\varpi(\xi)d\xi,\quad x>\mu$$

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and

$$\mathbb{J}_{\eta-}^{\alpha}\varpi(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\eta}(\xi-x)^{\alpha-1}\varpi(\xi)d\xi,\quad x<\eta,$$

respectively Here, $\Gamma(\alpha)$ *is the Gamma function and its defined as*

$$\Gamma(\alpha)=\int_{0}^{\infty}e^{-\upsilon}\upsilon^{\alpha-1}d\upsilon.$$

Let us note that $\mathbb{J}_{\mu+}^{0}\varpi(x)=\mathbb{J}_{\eta-}^{0}\varpi(x)=\varpi(x)$.

Sarikaya et al. proved some Hermite–Hadamard type inequalities for the first time using the Riemann-Liouville fractional integrals in the paper [22]. Sarikaya and Yildirim established some new Hermite–Hadamard type inequalities and midpoint type inequalities by using Riemann-Liouville fractional integrals in the case of differentiable convex functions in the paper [21]. Khan et al. [14] presented some new guesses of Hermite–Hadamard type inequalities for Riemann-Liouville fractional integrals and conformable fractional integrals. Tunc [26] investigated some new variants of Hermite–Hadamard inequalities based on h-convex functions with the aid of the fractional integrals. Moreover, Sarikaya and Ertugral [23] presented a new class of fractional integrals, which is known as generalized fractional, and they used these integrals to establish the general version of Hermite–Hadamard type inequalities for convex functions. Budak et al. investigated several variants of Ostrowski's and Simpson's type for differentiable convex functions by generalized fractional integrals in the paper [5]. For a better understanding of refinements of fractional integral inequalities, we suggest interested readers see the papers [2, 3, 19, 14, 25, 26].

The authors [20] presented some of Simpson's type inequalities with the aid of the Riemann-Liouville fractional integrals and in the case of general convex functions. Moreover, it is investigated several fractional Simpson type inequalities for functions whose second derivatives in absolute value are convex in the paper [10]. See references [6, 27, 18], and the references therein for further information concerned with Simpson type inequalities and several properties of Riemann–Liouville fractional integrals and various fractional integral operators.

In the paper [7], some new integral inequalities of Newton-type for functions whose first derivative in modulus of certain power are arithmetically-harmonically convex are established. In the paper [11], some Newton-type inequalities via differentiable convex functions based on the well-known Riemann-Liouville fractional integrals are proved and some inequalities of Riemann-Liouville fractional Newton-type via mappings of bounded variation are obtained. In the paper [24], some Newton type inequalities are established by using Riemann-Liouville fractional integrals and the authors demonstrated some inequalities of Riemann-Liouville fractional Newton-type via mappings of bounded variation. In the paper [8], some new inequalities of Newton-type based on convexity are presented and some applications for special cases of real functions are also given. For more information about Newton-type of inequalities including convex differentiable functions, it can be referred to

[12, 13, 16, 17] and the references therein. This paper aims to establish some perturbed Newton-type inequalities in the case of twice differentiable convex functions. These inequalities are proved by using Riemann-Liouville fractional integrals. The general structure of the study consists of three chapters including an introduction. The remaining part of the paper proceeds as follows: In Section 2, we will prove integral equality. Moreover, some perturbed Newton-type inequalities will be proved in the case of differentiable convex functions with the aid of the Riemann-Liouville fractional integrals. Furthermore, we show that our main inequalities reduce to Newton-type inequalities proved in earlier published papers. In Section 3, some conclusions and further research directions are discussed at the end of the paper.

2. Main results of Perturbed Fractional Newton-type Inequalities

Lemma 2.1. *Let us assume that $\varpi : [\mu, \eta] \to \mathbb{R}$ is an absolutely continuous function (μ, η) so that $\varpi'' \in L_1([\mu, \eta])$ and $\alpha > 0$. Then, the following equality holds:*

$$\begin{aligned}
(1)\qquad & \frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right] \\
& +\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right] \\
& -\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right] \\
& =-\frac{(\eta-\mu)^{2}}{2}\left[\Upsilon_{1}+\Upsilon_{2}+\Upsilon_{3}\right].
\end{aligned}$$

Here,

$$\begin{cases}
\Upsilon_1 = \int\limits_0^{\frac{1}{3}} \left(\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1} + \frac{3\xi}{4} - \frac{1}{2}\right)\varpi''(\xi\eta+(1-\xi)\mu)\,d\xi, \\
\Upsilon_2 = \int\limits_{\frac{1}{3}}^{\frac{2}{3}} \left(\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1} - \frac{1}{4}\right)\varpi''(\xi\eta+(1-\xi)\mu)\,d\xi, \\
\Upsilon_3 = \int\limits_{\frac{2}{3}}^{1} \left(\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1} - \frac{3\xi}{4} + \frac{1}{4}\right)\varpi''(\xi\eta+(1-\xi)\mu)\,d\xi.
\end{cases}$$

Proof. By using the fundamental rules of integration by parts, it follows

(2)

$$\Upsilon_1 = \int_0^{\frac{1}{3}} \left(\frac{\xi^{\alpha+1} + (1-\xi)^{\alpha+1}}{\alpha+1} + \frac{3\xi}{4} - \frac{1}{2} \right) \varpi'' \left(\xi\eta + (1-\xi)\mu \right) d\xi$$

$$= \frac{1}{\eta-\mu} \left(\frac{\xi^{\alpha+1} + (1-\xi)^{\alpha+1}}{\alpha+1} + \frac{3\xi}{4} - \frac{1}{2} \right) \varpi' \left(\xi\eta + (1-\xi)\mu \right) \Big|_0^{\frac{1}{3}}$$

$$- \frac{1}{\eta-\mu} \int_0^{\frac{1}{3}} \left(\xi^{\alpha} - (1-\xi)^{\alpha} + \frac{3}{4} \right) \varpi' \left(\xi\eta + (1-\xi)\mu \right) d\xi$$

$$= \frac{1}{\eta-\mu} \left[\left(\left(\frac{1}{3^{\alpha+1}} + \left(\frac{2}{3}\right)^{\alpha+1} \right) \frac{1}{(\alpha+1)} - \frac{1}{4} \right) \varpi' \left(\frac{2\mu+\eta}{3} \right) + \frac{\alpha-1}{2(\alpha+1)} \varpi'(\mu) \right]$$

$$- \frac{1}{\eta-\mu} \left[\frac{1}{\eta-\mu} \left(\xi^{\alpha} - (1-\xi)^{\alpha} + \frac{3}{4} \right) \varpi \left(\xi\eta + (1-\xi)\mu \right) \Big|_0^{\frac{1}{3}} \right.$$

$$\left. - \frac{\alpha}{\eta-\mu} \int_0^{\frac{1}{3}} \left[\xi^{\alpha-1} + (1-\xi)^{\alpha-1} \right] \varpi \left(\xi\eta + (1-\xi)\mu \right) d\xi \right]$$

$$= \frac{1}{\eta-\mu} \left[\left(\left(\frac{1}{3^{\alpha+1}} + \left(\frac{2}{3}\right)^{\alpha+1} \right) \frac{1}{(\alpha+1)} - \frac{1}{4} \right) \varpi' \left(\frac{2\mu+\eta}{3} \right) + \frac{\alpha-1}{2(\alpha+1)} \varpi'(\mu) \right]$$

$$- \frac{1}{(\eta-\mu)^2} \left(\frac{1}{3^{\alpha}} - \left(\frac{2}{3}\right)^{\alpha} + \frac{3}{4} \right) \varpi \left(\frac{2\mu+\eta}{3} \right) - \frac{1}{4(\eta-\mu)^2} \varpi(\mu)$$

$$+ \frac{\alpha}{(\eta-\mu)^2} \int_0^{\frac{1}{3}} \left[\xi^{\alpha-1} + (1-\xi)^{\alpha-1} \right] \varpi \left(\xi\eta + (1-\xi)\mu \right) d\xi.$$

Similarly, we obtain

(3)

$$\Upsilon_2=\frac{1}{\eta-\mu}\left(\left(\frac{1}{3^{\alpha+1}}+\left(\frac{2}{3}\right)^{\alpha+1}\right)\frac{1}{(\alpha+1)}-\frac{1}{4}\right)\left(\varpi'\left(\frac{\mu+2\eta}{3}\right)-\varpi'\left(\frac{2\mu+\eta}{3}\right)\right)$$

$$-\frac{1}{(\eta-\mu)^2}\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{1}{3^{\alpha}}\right)\left[\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi\left(\frac{2\mu+\eta}{3}\right)\right]$$

$$+\frac{\alpha}{(\eta-\mu)^2}\int_{\frac{1}{3}}^{\frac{2}{3}}\left[\xi^{\alpha-1}+(1-\xi)^{\alpha-1}\right]\varpi\left(\xi\eta+(1-\xi)\mu\right)d\xi,$$

and

(4)

$$\Upsilon_3=\frac{1}{\eta-\mu}\left[\frac{1-\alpha}{2(\alpha+1)}\varpi'(\eta)-\left(\left(\frac{1}{3^{\alpha+1}}+\left(\frac{2}{3}\right)^{\alpha+1}\right)\frac{1}{(\alpha+1)}-\frac{1}{4}\right)\varpi'\left(\frac{\mu+2\eta}{3}\right)\right]$$

$$-\frac{1}{(\eta-\mu)^2}\left(\frac{\varpi(\eta)}{4}-\left(\left(\frac{2}{3}\right)^{\alpha}-\frac{1}{3^{\alpha}}-\frac{3}{4}\right)\varpi\left(\frac{\mu+2\eta}{3}\right)\right)$$

$$+\frac{\alpha}{(\eta-\mu)^2}\int_{\frac{2}{3}}^{1}\left[\xi^{\alpha-1}+(1-\xi)^{\alpha-1}\right]\varpi\left(\xi\eta+(1-\xi)\mu\right)d\xi.$$

If we add the equalities from (2) to (4), then we obtain the following equality

(5)

$$\Upsilon_1+\Upsilon_2+\Upsilon_3=\frac{\alpha-1}{2(\eta-\mu)(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]$$

$$-\frac{1}{4(\eta-\mu)^2}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]$$

$$+\frac{\alpha}{(\eta-\mu)^2}\int_{0}^{1}\left[\xi^{\alpha-1}+(1-\xi)^{\alpha-1}\right]\varpi\left(\xi\eta+(1-\xi)\mu\right)d\xi$$

$$= \frac{\alpha-1}{2(\eta-\mu)(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]$$

$$-\frac{1}{4(\eta-\mu)^2}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]$$

$$+\frac{\Gamma(\alpha+1)}{(\eta-\mu)^{\alpha+2}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right].$$

If we multiply the both sides of (5) by $-\frac{(\eta-\mu)^2}{2}$, then the equality (1) is obtained. Hence, the proof of Lemma 2.1 is completed. □

Remark 2.2. *If we choose $\alpha = 1$ in Lemma 2.1, then Lemma 2.1 reduces to* [8, Lemma 2.1].

Theorem 2.3. *Let's assume that the conditions of Lemma 2.1 are valid. Suppose also that the function $|\varpi''|$ is convex on $[\mu, \eta]$. Then, it follows*

(6)

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\right|$$

$$\leq \frac{(\eta-\mu)^2}{2}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)\right)\left[\left|\varpi''(\mu)\right|+\left|\varpi''(\eta)\right|\right],$$

where

$$\left\{\begin{array}{l} \Omega_1(\alpha)=\int\limits_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|\xi d\xi, \\ \Omega_2(\alpha)=\int\limits_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|\xi d\xi, \\ \Omega_3(\alpha)=\int\limits_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|\xi d\xi. \end{array}\right. \tag{7}$$

Proof. If we take modulus in Lemma 2.1, then we get

(8)

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\Bigg|$$

$$\leq \frac{(\eta-\mu)^2}{2}\left[\int_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|d\xi\right.$$

$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|d\xi$$

$$\left.+\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|d\xi\right].$$

Since we know the fact that $|\varpi''|$ is convex, it follows

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\Bigg|$$

$$\leq \frac{(\eta-\mu)^2}{2}\left[\int_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|\left[\xi\left|\varpi''(\eta)\right|+(1-\xi)\left|\varpi''(\mu)\right|\right]d\xi\right.$$

$$+\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|\left[\xi\left|\varpi''(\eta)\right|+(1-\xi)\left|\varpi''(\mu)\right|\right]d\xi$$

$$\left.+\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|\left[\xi\left|\varpi''(\eta)\right|+(1-\xi)\left|\varpi''(\mu)\right|\right]d\xi\right]$$

$$=\frac{(\eta-\mu)^2}{2}\left[\Omega_1(\alpha)\left|\varpi''(\eta)\right|+\Omega_3(\alpha)\left|\varpi''(\mu)\right|+\Omega_2(\alpha)\left|\varpi''(\eta)\right|+\Omega_2(\alpha)\left|\varpi''(\mu)\right|\right.$$

$$+\Omega_3(\alpha)\left|\varpi''(\eta)\right| + \Omega_1(\alpha)\left|\varpi''(\mu)\right|\Big].$$

This finishes the proof of Theorem 2.3. □

Remark 2.4. *Let us consider $\alpha = 1$ in Theorem 2.3. Then, the following Newton-type inequality holds:*

$$\left|\frac{1}{8}\left[\varpi(\mu) + 3\varpi\left(\frac{2\mu+\eta}{3}\right) + 3\varpi\left(\frac{\mu+2\eta}{3}\right) + \varpi(\eta)\right] - \frac{1}{\eta-\mu}\int_{\mu}^{\eta}\varpi(\xi)\,d\xi\right|$$
$$\leq \frac{(\eta-\mu)^2}{384}\left[\left|\varpi''(\mu)\right| + \left|\varpi''(\eta)\right|\right],$$

which is established by [8, Theorem 2.1].

Theorem 2.5. *Let's consider the conditions of Lemma 2.1. Let us also consider that the function $|\varpi''|^q$, $q > 1$ is convex on $[\mu, \eta]$. Then, the following inequality holds:*

$$\Bigg|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu) - \varpi'(\eta)\right] + \frac{1}{8}\left[\varpi(\mu) + 3\varpi\left(\frac{2\mu+\eta}{3}\right) + 3\varpi\left(\frac{\mu+2\eta}{3}\right) + \varpi(\eta)\right]$$
$$-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta) + \mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\Bigg|$$
$$\leq \frac{(\eta-\mu)^2}{2}\left[\varphi_1(\alpha,p)\left(\frac{5\left|\varpi''(\mu)\right|^q + \left|\varpi''(\eta)\right|^q}{18}\right)^{\frac{1}{q}} + \varphi_2(\alpha,p)\left(\frac{\left|\varpi''(\mu)\right|^q + \left|\varpi''(\eta)\right|^q}{6}\right)^{\frac{1}{q}}\right.$$
$$\left.+\varphi_3(\alpha,p)\left(\frac{\left|\varpi''(\mu)\right|^q + 5\left|\varpi''(\eta)\right|^q}{18}\right)^{\frac{1}{q}}\right].$$

Here, $\frac{1}{p} + \frac{1}{q} = 1$ *and*

$$\begin{cases} \varphi_1(\alpha,p) = \left(\int_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1} + \frac{3\xi}{4} - \frac{1}{2}\right|^p d\xi\right)^{\frac{1}{p}}, \\ \varphi_2(\alpha,p) = \left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1} - \frac{1}{4}\right|^p d\xi\right)^{\frac{1}{p}}, \\ \varphi_3(\alpha,p) = \left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1} - \frac{3\xi}{4} + \frac{1}{4}\right|^p d\xi\right)^{\frac{1}{p}}. \end{cases}$$

Proof. Let us apply the Hölder inequality to inequality (8). Then, we derive

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\right|$$

$$\leq\frac{(\eta-\mu)^{2}}{2}\left[\left(\int_{0}^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{0}^{\frac{1}{3}}\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|^{q}d\xi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|^{q}d\xi\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|^{q}d\xi\right)^{\frac{1}{q}}\right].$$

From the fact that $|\varpi''|^{q}$ is convex, we obtain

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\right|$$

$$\leq\frac{(\eta-\mu)^{2}}{2}\left[\left(\int_{0}^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|^{p}d\xi\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_{0}^{\frac{1}{3}}\xi\left|\varpi''(\eta)\right|^{q}+(1-\xi)\left|\varpi''(\mu)\right|^{q}d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\xi\left|\varpi''(\eta)\right|^{q}+(1-\xi)\left|\varpi''(\mu)\right|^{q}d\xi\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\int_{\frac{2}{3}}^{1}\xi\left|\varpi''(\eta)\right|^{q}+(1-\xi)\left|\varpi''(\mu)\right|^{q}d\xi\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\eta-\mu)^{2}}{2}\left[\left(\int_{0}^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\frac{5\left|\varpi''(\mu)\right|^{q}+\left|\varpi''(\eta)\right|^{q}}{18}\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\frac{\left|\varpi''(\mu)\right|^{q}+\left|\varpi''(\eta)\right|^{q}}{6}\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|^{p}d\xi\right)^{\frac{1}{p}}\left(\frac{\left|\varpi''(\mu)\right|^{q}+5\left|\varpi''(\eta)\right|^{q}}{18}\right)^{\frac{1}{q}}\right].$$

This is the end of the proof of Theorem 2.5. □

Theorem 2.6. *Assume that the assumptions of Lemma 2.1 hold. Assume also that the function $|\varpi''|^q$, $q \geq 1$ is convex on $[\mu,\eta]$. Then, the following inequality*

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\right|$$

$$\leq\frac{(\eta-\mu)^{2}}{2}\left[(\Omega_{4}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{1}(\alpha)\left|\varpi''(\eta)\right|^{q}+(\Omega_{4}(\alpha)-\Omega_{1}(\alpha))\left|\varpi''(\mu)\right|^{q}\right)^{\frac{1}{q}}\right]\right.$$

$$+(\Omega_{5}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{2}(\alpha)\left|\varpi''(\eta)\right|^{q}+(\Omega_{5}(\alpha)-\Omega_{2}(\alpha))\left|\varpi''(\mu)\right|^{q}\right)^{\frac{1}{q}}\right]$$

$$+\left(\Omega_6(\alpha)\right)^{1-\frac{1}{q}}\left[\left(\Omega_3(\alpha)\left|\varpi''(\eta)\right|^q+\left(\Omega_6(\alpha)-\Omega_3(\alpha)\right)\left|\varpi''(\mu)\right|^q\right)^{\frac{1}{q}}\right]\Bigg]$$

is valid. Here, $\Omega_1(\alpha)$, $\Omega_2(\alpha)$, *and* $\Omega_3(\alpha)$ *are defined in* (7) *and*

$$\begin{cases} \Omega_4(\alpha)=\int\limits_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|d\xi, \\ \Omega_5(\alpha)=\int\limits_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|d\xi, \\ \Omega_6(\alpha)=\int\limits_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|d\xi. \end{cases}$$

Proof. Let us apply the power-mean inequality to inequality (8), we get

$$\Bigg|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]$$

$$-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^\alpha}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\Bigg|$$

$$\leq\frac{(\eta-\mu)^2}{2}\left[\left(\int\limits_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|d\xi\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int\limits_0^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int\limits_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int\limits_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|\left|\varpi''(\xi\eta+(1-\xi)\mu)\right|^q d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|\left|\varpi''\left(\xi\eta+(1-\xi)\mu\right)\right|^{q}d\xi\right)^{\frac{1}{q}}\right].$$

Since $|\varpi''|^q$ is convex, it follows

$$\left|\frac{(\eta-\mu)(1-\alpha)}{4(\alpha+1)}\left[\varpi'(\mu)-\varpi'(\eta)\right]+\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{2(\eta-\mu)^{\alpha}}\left[\mathbb{J}_{\mu+}^{\alpha}\varpi(\eta)+\mathbb{J}_{\eta-}^{\alpha}\varpi(\mu)\right]\right|$$

$$\leq\frac{(\eta-\mu)^{2}}{2}\left[\left(\int_{0}^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|d\xi\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_{0}^{\frac{1}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}+\frac{3\xi}{4}-\frac{1}{2}\right|\left[\xi\left|\varpi''(\eta)\right|^{q}+(1-\xi)\left|\varpi''(\mu)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{1}{3}}^{\frac{2}{3}}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{1}{4}\right|\left[\xi\left|\varpi''(\eta)\right|^{q}+(1-\xi)\left|\varpi''(\mu)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|d\xi\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{2}{3}}^{1}\left|\frac{\xi^{\alpha+1}+(1-\xi)^{\alpha+1}}{\alpha+1}-\frac{3\xi}{4}+\frac{1}{4}\right|\left[\xi\left|\varpi''(\eta)\right|^{q}+(1-\xi)\left|\varpi''(\mu)\right|^{q}\right]d\xi\right)^{\frac{1}{q}}\right]$$

$$=\frac{(\eta-\mu)^{2}}{2}\left[(\Omega_{4}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{1}(\alpha)\left|\varpi''(\eta)\right|^{q}+(\Omega_{4}(\alpha)-\Omega_{1}(\alpha))\left|\varpi''(\mu)\right|^{q}\right)^{\frac{1}{q}}\right]\right.$$

$$+(\Omega_{5}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{2}(\alpha)\left|\varpi''(\eta)\right|^{q}+(\Omega_{5}(\alpha)-\Omega_{2}(\alpha))\left|\varpi''(\mu)\right|^{q}\right)^{\frac{1}{q}}\right]$$

$$\left.+(\Omega_{6}(\alpha))^{1-\frac{1}{q}}\left[\left(\Omega_{3}(\alpha)\left|\varpi''(\eta)\right|^{q}+(\Omega_{6}(\alpha)-\Omega_{3}(\alpha))\left|\varpi''(\mu)\right|^{q}\right)^{\frac{1}{q}}\right]\right].$$

Thus, we obtain the desired result of Theorem 2.6. □

Remark 2.7. *Let us consider $\alpha=1$ in Theorem 2.6. Then, the following Newton-type inequality holds:*

$$\left|\frac{1}{8}\left[\varpi(\mu)+3\varpi\left(\frac{2\mu+\eta}{3}\right)+3\varpi\left(\frac{\mu+2\eta}{3}\right)+\varpi(\eta)\right]-\frac{1}{\eta-\mu}\int_{\mu}^{\eta}\varpi(\xi)\,d\xi\right|$$

$$\leq\frac{(\eta-\mu)^{2}}{2}\left\{\left(\frac{19}{3^{4}\cdot 2^{6}}\right)^{1-\frac{1}{q}}\left[\left(\frac{27\left|\varpi''(\eta)\right|^{q}+125\left|\varpi''(\mu)\right|^{q}}{3^{4}\cdot 2^{9}}\right)^{\frac{1}{q}}\right.\right.$$

$$\left.\left.+\left(\frac{125\left|\varpi''(\eta)\right|^{q}+27\left|\varpi''(\mu)\right|^{q}}{3^{4}\cdot 2^{9}}\right)^{\frac{1}{q}}\right]\right\}$$

$$+\frac{1}{3^{4}\cdot 2^{2}}\left[\left(\frac{\left|\varpi''(\mu)\right|^{q}+\left|\varpi''(\eta)\right|^{q}}{2}\right)^{\frac{1}{q}}\right]\Bigg\},$$

which is given by [8, Theorem 2.2].

3. Conclusion

Some perturbed Newton-type inequalities are investigated for twice differentiable convex functions by using Riemann-Liouville fractional integrals. By using the special cases of our results, we show that our results reduce inequalities obtained in earlier work. In future works, mathematicians can try to generalize and refine our results by utilizing a different version of convex function classes or another type of fractional integral operator.

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