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PURE-DIRECT-PROJECTIVE OBJECTS IN GROTHENDIECK CATEGORIES

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Abstract. In this paper we study generalizations of the concept of pure-direct-projectivity from module categories to Grothendieck categories. We examine for which categories or under what conditions pure-direct-projective objects are direct-projective, quasi-projective, pure-projective, projective and flat. We investigate classes all of whose objects are pure-direct-projective. We give applications of some of the results to comodule categories.

1. Introduction

A right R-module M is said to be *direct-projective* if every submodule A of M with M/A isomorphic to a direct summand of M is a direct summand of M. Direct-projective modules were introduced by Nicholson in [12] and further studies on direct-projective modules were done by Tiwary and Bharadwaj in [18] and by Hausen in [10]. The notion of extending was generalized to purely extending by Fuchs in [8] and basic characterisations were given by Clark in [2]. Motivated by their work, the notion of pure-direct-projective modules were introduced and studied by Alizade and Toksoy in [1]. Namely, a right R-module is said to be *pure-direct-projective* if every pure submodule A of which with M/A isomorphic to a direct summand is a direct summand. We study generalizations of these notions to abelian categories and Grothendieck categories, i.e., cocomplete abelian categories with a family of generators and exact direct limits (see [17]). Namely, direct-projective objects and pure-direct-projective objects respectively. Some generalizations of direct-projective modules to abelian categories were studied by Crivei and Kör in [3] and Crivei and Keskin Tütüncü in [4]. An object M of an abelian category $\mathcal{A}$ is said to be *direct-projective* if every subobject A of M with M/A isomorphic to a direct summand of M is a direct summand. Let M and N be objects of an abelian category $\mathcal{A}$. M is called N-*projective* if given any epimorphism from N to an object L of $\mathcal{A}$, any

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homomorphism from M to L can be lifted to a homomorphism from M to N. M is said to be *quasi-projective* if it is M-projective. The following implications hold.

$$\text{projective} \Rightarrow \text{quasi-projective} \Rightarrow \text{direct-projective}$$

An object M of a Grothendieck category $\mathcal{A}$ is said to be *pure-projective* if M is relatively projective for every pure short exact sequence in $\mathcal{A}$ and it is said to be *pure-direct-projective* if every pure subobject A of M with M/A isomorphic to a direct summand of M is a direct summand. We also have the following implications.

$$\text{projective} \Rightarrow \text{pure-projective} \Rightarrow \text{pure-direct-projective}$$

Since every direct summand is a pure subobject, every direct-projective object is pure-direct-projective.

In Section 2, some definitions and a lemma which will be used in the next sections of the paper are recalled.

In Section 3, the notion of direct-projective objects are generalized to abelian categories. It is obtained that the class of direct-projective objects of an abelian category $\mathcal{A}$ with enough projectives need not be closed under factor objects and taking finite coproducts (Corollary 3.6 and Corollary 3.7). An abelian category $\mathcal{A}$ is said to have *enough projectives* if every object of $\mathcal{A}$ is a quotient object of a projective object (see [17]). It is shown that the coproduct of two direct-projective objects of an abelian category $\mathcal{A}$ with enough projectives is direct-projective if and only if every direct-projective object of $\mathcal{A}$ is projective (Proposition 3.8). Also it is obtained that a coalgebra C over a field is hereditary if and only if every subcomodule of a projective right C-comodule is direct-projective (Corollary 3.11).

In Section 4, the notion of pure-direct-projective objects are generalized to Grothendieck categories. Direct summands of pure-direct-projective objects of a Grothendieck category $\mathcal{A}$ are pure-direct-projective (Proposition 4.3). It is obtained that the class of pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under pure factors and taking finite coproducts (Corollary 4.6 and Corollary 4.7). It is proved that the coproduct of two pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ is pure-direct-projective if and only if every pure-direct-projective object of $\mathcal{A}$ is pure-projective (Corollary 4.8). It is also proved that a locally finitely presented Grothendieck category $\mathcal{A}$, whose class of pure-direct-projective objects is closed under finite coproducts, is pure-perfect if and only if every pure-injective object of $\mathcal{A}$ is pure-direct-projective (Proposition 4.9). It is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-direct-projective object of $\mathcal{A}$ is flat (Theorem 4.10). Also it is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is regular if and only if every pure-direct-projective object of $\mathcal{A}$ is direct-projective (Theorem 4.12). As a result of this, it is shown that a semiperfect coalgebra C over a field is cosemisimple if and only if every pure-projective right

C-comodule is projective if and only if every pure-direct-projective right C-comodule is direct-projective (Corollary 4.14). It is proved for a locally finitely presented Grothendieck category $\mathcal{A}$ that $\mathcal{A}$ is regular and the coproduct of two pure-direct-projective objects is pure-direct-projective if and only if every pure-direct-projective object of $\mathcal{A}$ is projective (Proposition 4.15). It is obtained for a locally finitely presented regular Grothendieck category $\mathcal{A}$ that if the coproduct of two pure-direct-projective objects is pure-direct-projective, then every pure-direct-projective object of $\mathcal{A}$ is quasi-projective (Corollary 4.16). Also it is proved that if every pure-direct-projective object of a locally finitely presented Grothendieck category $\mathcal{A}$ is quasi-projective, then $\mathcal{A}$ is regular (Proposition 4.17). Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category with enough projective objects whose class of pure-injective objects is closed under extensions. Then $\mathcal{A}$ is pure hereditary if and only if every subobject of a pure-projective object of $\mathcal{A}$ is pure-direct-projective (Proposition 4.20). It is obtained that the class of pure-direct-projective objects of a locally finitely presented Grothendieck category $\mathcal{A}$ need not be closed under subobjects (Corollary 4.22).

In Section 5, classes all of whose objects are direct-projective and classes all of whose objects are pure-direct-projective are investigated. It is proved that an abelian category $\mathcal{A}$ is spectral if and only if $\mathcal{A}$ is perfect and every object of $\mathcal{A}$ is direct-projective if and only if $\mathcal{A}$ is perfect and every factor object of a direct-projective object of $\mathcal{A}$ is direct-projective (Theorem 5.2). It is obtained that a locally finitely presented Grothendieck category $\mathcal{A}$ is semisimple if and only if $\mathcal{A}$ has enough projectives and every object of $\mathcal{A}$ is direct-projective if and only if has enough projectives the coproduct of two direct-projective objects of $\mathcal{A}$ is direct-projective (Corollary 5.3). It is proved that a coalgebra C over a field is cosemisimple if and only if C is right semiperfect and every right C-comodule is direct-projective if and only if C is right semiperfect and every factor comodule of a direct-projective right C-comodule is direct-projective (Corollary 5.5). Finally, it is shown that a locally finitely presented Grothendieck category $\mathcal{A}$ is pure-semisimple if and only if every object of $\mathcal{A}$ is pure-projective if and only if every object of $\mathcal{A}$ is pure-direct-projective if and only if every pure quotient of a pure-direct-projective object of $\mathcal{A}$ is pure-direct-projective (Theorem 5.7).

2. Preliminaries

Let $\mathcal{A}$ be an abelian category. For every morphism $f : M \longrightarrow N$ in an $\mathcal{A}$ we denote $\ker(f) : \mathrm{Ker}(f) \longrightarrow M$, $\mathrm{coker}(f) : N \longrightarrow \mathrm{Coker}(f)$, $\mathrm{im}(f) : \mathrm{Im}(f) \longrightarrow N$ and $\mathrm{coim}(f) : M \longrightarrow \mathrm{Coim}(f)$ the kernel, the cokernel, the image and the coimage of f respectively. Since $\mathcal{A}$ is abelian, $\mathrm{Coim}(f) \cong \mathrm{Im}(f)$. Recall that a morphism $f : A \longrightarrow B$ is called a *split monomorphism* (or *section*) if there is a morphism $g : B \longrightarrow A$ such that $gf = 1_A$ and a *split epimorphism* (or *retraction*) if there is a morphism $g : B \longrightarrow A$ such that $fg = 1_B$.

Definition 2.1. [15, p. 159] *Consider a class $\mathcal{E}$ of short exact sequences of an abelian category $\mathcal{A}$, such that every sequence isomorphic to a sequence in $\mathcal{E}$ also is in $\mathcal{E}$. The corresponding class of monomorphisms (epimorphisms) is denoted by $\mathcal{E}_m$ ($\mathcal{E}_e$). $\mathcal{E}$ is called a proper class if it satisfies the following conditions.*

P1. *Every split short exact sequence is in $\mathcal{E}$.*
P2. *If $\alpha, \beta \in \mathcal{E}_m$, then $\beta\alpha \in \mathcal{E}_m$ if defined.*
P3. *If $\alpha, \beta \in \mathcal{E}_e$, then $\beta\alpha \in \mathcal{E}_e$ if defined.*
P4. *If $\beta\alpha \in \mathcal{E}_m$, then $\alpha \in \mathcal{E}_m$.*
P5. *If $\beta\alpha \in \mathcal{E}_e$, then $\beta \in \mathcal{E}_e$.*

It is well known that $\mathcal{E}_m$ ($\mathcal{E}_e$) is closed under pushouts (pullbacks).

Definition 2.2. [16, p. 352] *An object M of a category $\mathcal{A}$ is said to be finitely generated if, whenever $M = \sum M_i$ for a directed family $(M_i)_I$ of subobjects of M, there is an $i \in I$ such that $M = M_i$.*

Definition 2.3. [16, p. 352] *M is called a finitely presented object if it is finitely generated and every epimorphism $f : L \longrightarrow M$, where L is finitely generated, has a finitely generated kernel.*

Definition 2.4. [16, p. 352] *$\mathcal{A}$ is said to be locally finitely presented if it has a family of finitely presented generator.*

Definition 2.5. [16, p. 353] *A short exact sequence*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

in a Grothendieck category $\mathcal{A}$ is said to be pure if every finitely presented object is relatively projective to it. In this case L is a pure subobject of M. Also an object M of $\mathcal{A}$ is said to be flat if every short exact sequence $0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$ is pure.

Lemma 2.6. [16, Lemma 6 (i)] *The class $\mathcal{P}ure$ of pure exact sequences of a Grothendieck category $\mathcal{A}$ forms a proper class.*

3. Direct-projective objects in abelian categories

We recall some generalizations of projectivity namely direct-projectivity. The concept of direct-projectivity was introduced by Nicholson in [12] as a generalization of quasi-projectivity for module categories. In this section we generalize the concept of direct-projective modules from module categories to abelian categories and we give some applications to comodule categories.

Definition 3.1. [3, Section 5, p. 810] *An object M of an abelian category $\mathcal{A}$ is said to be* direct-projective *if every subobject A of M with M/A isomorphic to a direct summand of M is a direct summand.*

Proposition 3.2, Proposition 3.3, Lemma 3.4, and Theorem 3.5 are immediate generalizations of results in [18] and [10] from module categories to abelian categories.

Proposition 3.2. *Let $\mathcal{A}$ be an abelian category. Then the following are equivalent for an object M of $\mathcal{A}$.*

(1) *Given any direct summand A of M with projection map $\pi : M \longrightarrow A$, for each epimorphism $\alpha : M \longrightarrow A$ there exists an endomorphism γ of M such that $\alpha\gamma = \pi$.*
(2) *M is direct-projective.*
(3) *Every exact sequence*

$$0 \longrightarrow K \longrightarrow N \longrightarrow L \longrightarrow 0$$

with N an epimorphic image of M and L a direct summand of M splits.

Proposition 3.3. *Direct summands of direct-projective objects of an abelian category $\mathcal{A}$ are direct-projective.*

Lemma 3.4. *If $M \oplus N$ is direct-projective, then every short exact sequence*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

in an abelian category $\mathcal{A}$ splits.

Theorem 3.5. *Let M be a projective object of an abelian category $\mathcal{A}$ and $f : M \longrightarrow N$ be an epimorphism. Then N is projective if and only if $M \oplus N$ is direct-projective.*

Corollary 3.6. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the class of direct-projective objects of $\mathcal{A}$ need not be closed under factor objects.*

Corollary 3.7. *Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the class of direct-projective objects of $\mathcal{A}$ need not be closed under taking finite coproducts.*

The following result generalizes [18, Proposition 2.6].

Proposition 3.8. *Let $\mathcal{A}$ be an abelian category having enough projective objects. The coproduct of two direct-projective objects of $\mathcal{A}$ is direct-projective if and only if every direct-projective object of $\mathcal{A}$ is projective.*

Proof. ($\Rightarrow$) Let M be a direct-projective object of $\mathcal{A}$. Since $\mathcal{A}$ has enough projectives, there exists a projective object P of $\mathcal{A}$ and an epimorphism $f : P \longrightarrow M$. Since projective objects are direct-projective, P is direct-projective. Then $P \oplus M$ is direct-projective by assumption and therefore M is projective by Theorem 3.5.
($\Leftarrow$) Clear. □

Recall that an abelian category $\mathcal{A}$ is said to be *hereditary* if and only if every subobject of a projective object is projective if and only if every quotient object of an injective object is injective. $\mathcal{A}$ is called *semihereditary* if every finitely generated subobject of a projective object is projective and *cosemihereditary* if every finitely cogenerated quotient object of an injective object is injective.

Theorem 3.9. *Assume that $\mathcal{A}$ is an abelian category with enough projective objects. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is (semi)hereditary.*
(2) *Every (finitely generated) subobject of a projective object of $\mathcal{A}$ is direct-projective.*

Proof. (1) $\Rightarrow$ (2) Clear.
(2) $\Rightarrow$ (1) Let P be a projective object and N be a subobject of P. Since $\mathcal{A}$ has enough projectives, there is an epimorphism $f : P_1 \longrightarrow N$ with P_1 projective. Now $P_1 \oplus N$ is a subobject of the projective object $P_1 \oplus P$ and therefore it is direct-projective by assumption. Then N is projective by Theorem 3.5. □

Let R be a unitary ring and $\mathrm{Mod}(R)$ be the category of right R-modules. $\mathrm{Mod}(R)$ is a locally finitely generated Grothendieck category with enough injectives and enough projectives (see [17]). $\mathrm{Mod}(R)$ is hereditary if and only if the ring R is right hereditary (see [3]). Then we have the following result for module categories.

Corollary 3.10. [21, Theorem 4] *Let R be a unitary ring. Then the following conditions are equivalent.*

(1) *R is right hereditary.*
(2) *Every submodule of a projective right R-module is direct-projective.*

Let C be a coalgebra over a field and $\mathcal{M}^C$ be the category of right C-comodules. $\mathcal{M}^C$ is a locally finitely generated Grothendieck category. $\mathcal{M}^C$ has enough projectives if C is a semiperfect coalgebra (see [11, Remarks (1) on p.1525]). The category $\mathcal{M}^C$ is hereditary if and only if C is a (left and right) hereditary coalgebra (see [11]). Then we have the following result for comodule categories.

Corollary 3.11. *Let C be a semiperfect coalgebra over a field. Then the following conditions are equivalent.*

(1) *C is hereditary.*
(2) *Every subcomodule of a projective right C-comodule is direct-projective.*

4. Pure-direct-projective objects in Grothendieck categories

Recently the concept of pure-direct-projectivity has introduced and studied by Alizade and Toksoy in [1]. In this section we generalize the concept of pure-direct-projectivity from module categories to Grothendieck categories and we give applications of some of the results to comodule categories.

Definition 4.1. *Let $\mathcal{A}$ be a Grothendieck category. An object M of $\mathcal{A}$ is said to be* pure-direct-projective *if every pure subobject A of M with M/A isomorphic to a direct summand of M is a direct summand of M.*

Proposition 4.2. [19, Proposition 3.3] *The following are equivalent for an object M of a Grothendieck category $\mathcal{A}$.*

(1) *Given a direct summand N of M with the projection $p : M \longrightarrow N$ and any epimorphism $f : M \longrightarrow N$ with $\mathrm{Ker}(f)$ pure in M there exists an endomorphism $g : M \longrightarrow M$ such that $fg = p$.*
(2) *M is pure-direct-projective.*
(3) *Any epimorphism $f : M \longrightarrow N$ with N a direct summand of M and $\mathrm{Ker}(f)$ pure in M splits.*

Proof. (1) $\Rightarrow$ (2) Let K be a pure subobject of M such that M/K is isomorphic to a direct summand N of M. Let $f : N \longrightarrow M/K$ be that isomorphism. By assumption there exists an endomorphism g of M such that $fg = p$, where $p : M \longrightarrow N$ is the projection map. Define $h : M \longrightarrow M$ by $h = gi$, $i : N \longrightarrow M$ being the inclusion map. Then $fh = f(gi) = (fg)i = i$ holds, so f splits. Thus K is a direct summand of M.
(2) $\Rightarrow$ (3) Clear.
(3) $\Rightarrow$ (1) Let N be a direct summand of M, $p : M \longrightarrow N$ the canonical projection map and $f : M \longrightarrow N$ an epimorphism with $\mathrm{Ker}(f)$ pure in M. By assumption, f splits. So there exists a morphism $h : N \longrightarrow M$ such that $fh = 1_N$. Define $g : M \longrightarrow M$ by $g = hp$, where $p : M \longrightarrow N$ is the canonical projection map. Then $fg = f(hp) = (fh)p = 1_N p = p$. □

Proposition 4.3. *Direct summands of pure-direct-projective objects of a Grothendieck category $\mathcal{A}$ are pure-direct-projective.*

Proof. Let M be a pure-direct-projective object of $\mathcal{A}$ and N be a direct summand of M and $\pi' : M \longrightarrow N$ be the projection map. Let K be a pure subobject of N and $\pi : N \longrightarrow K$ be the projection map. Let $f : N \longrightarrow K$ be an epimorphism with $\mathrm{Ker}(f)$ pure in N and $f' : M \longrightarrow N$ be an epimorphism. Then $ff' : M \longrightarrow K$ is also an epimorphism and since $\mathrm{Ker}(f)$ is a pure subobject of N and N is a pure subobject of M, $\mathrm{Ker}(f)$ is a pure subobject of M by [16, Lemma 6 (i)]. Since M is pure-direct-projective, there is an endomorphism $g : M \longrightarrow M$ such that $ff'g = \pi\pi'$. Let $i : K \longrightarrow N$ and $i' : N \longrightarrow M$ be inclusion maps. Put $h = f'gi'$. Then $fh = ff'gi' = \pi\pi' i' = \pi 1_N = \pi$. Thus N is pure-direct-projective by Proposition 4.2. □

Lemma 4.4. *If $M \oplus N$ is pure-direct-projective, then every pure exact sequence of the form*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

in a Grothendieck category $\mathcal{A}$ splits.

Proof. Let

$$0 \longrightarrow K \longrightarrow M \xrightarrow{g} N \longrightarrow 0$$

be a pure exact sequence in $\mathcal{A}$. Suppose that $M \oplus N$ is pure-direct-projective in $\mathcal{A}$. Let $p_1 : M \oplus N \longrightarrow M$ and $p_2 : M \oplus N \longrightarrow N$ be canonical projections. Since $M \oplus N$ is pure-direct-projective, there exists an endomorphism h of $M \oplus N$ such that $gp_1h = p_2$ by Proposition 4.2. Define $f : N \longrightarrow M$ by $f = p_1hi_2$ where $i : N \longrightarrow M \oplus N$ is the inclusion map. Then $gf = g(p_1hi_2) = p_2i_2 = 1_N$. Thus the sequence splits. □

Theorem 4.5. *Let $\mathcal{A}$ be a Grothendieck category and*

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

be a pure exact sequence with M pure-projective. Then $M \oplus N$ is pure-direct-projective if and only if N is pure-projective.

Proof. ($\Rightarrow$) Suppose that $M \oplus N$ is pure-direct-projective. Then the sequence

$$0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$$

splits by Lemma 4.4. Then N is pure-projective by [7, Proposition 4.1 (1)].
($\Leftarrow$) $M \oplus N$ is pure-projective by [7, Proposition 4.1 (1)] and therefore it is pure-direct-projective. □

Recall that every locally finitely presented Grothendieck category has enough pure-projective objects (see [16, Lemma 6(ii)]).

Corollary 4.6. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. The class of pure-direct-projective objects of $\mathcal{A}$ need not be closed under pure factors.*

Corollary 4.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. The class of pure-direct-projective objects of $\mathcal{A}$ need not be closed under taking finite coproducts.*

Corollary 4.8. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Every pure-direct-projective object of $\mathcal{A}$ is pure-projective if and only if the coproduct of two pure-direct-projective objects of $\mathcal{A}$ is pure-direct-projective.*

Proof. ($\Rightarrow$) Clear by [7, Proposition 4.1].
($\Leftarrow$) Let M be a pure-direct-projective object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projectives by [16, Lemma 6(ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. Since P is pure-projective, P is pure-direct-projective and therefore $P \oplus M$ is pure-direct-projective by assumption . So M is pure-projective by Theorem 4.5. □

Recall that a pure epimorphism $f : M \longrightarrow N$ in $\mathcal{A}$ is *purely minimal* if a morphism $g : X \longrightarrow M$ is a pure epimorphism whenever fg is a pure epimorphism. A locally finitely presented Grothendieck category $\mathcal{A}$ is said to be *pure-perfect* if every object M in $\mathcal{A}$ has a pure-projective cover, i.e. there exists a purely minimal epimorphism from a pure-projective object P to M (see [13]). An object M of a Grothendieck category $\mathcal{A}$ is *pure-injective* if it is relatively injective for every pure exact sequence in $\mathcal{A}$.

Proposition 4.9. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category whose class of pure-direct-projective objects is closed under finite coproducts. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is pure-perfect.*
(2) *Every pure-injective object of $\mathcal{A}$ is pure-direct-projective.*

Proof. (1) ⇒ (2) Suppose that $\mathcal{A}$ is pure-perfect and I is a pure-injective object of $\mathcal{A}$. Then I is pure-projective by [13, Theorem 6.3] and therefore I is pure-direct-projective.
(2) ⇒ (1) Let I be a pure-injective object of $\mathcal{A}$. Since there are enough pure-projective objects in $\mathcal{A}$ by [16, Lemma 6 (ii)], there is a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow I \longrightarrow 0$$

with P pure-projective. Now $P \oplus I$ is pure-direct-projective by the statement. Then I is pure-projective by Theorem 4.5 and therefore $\mathcal{A}$ is pure-perfect by [13, Theorem 6.3]. □

Recall that a Grothendieck category $\mathcal{A}$ is said to be *regular* if every object M of $\mathcal{A}$ is regular in the sense that every short exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

is pure in $\mathcal{A}$ (see [20]).

Theorem 4.10. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is regular.*
(2) *Every pure-direct-projective object is flat.*

Proof. (1) ⇒ (2) Clear by [16, Theorem 4].
(2) ⇒ (1) Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projective objects by [16, Lemma 6 (ii)], we have a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. So P is flat by assumption. Then M is flat by [5, Proposition 2.2. (c) (i)]. □

Lemma 4.11. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the followings are equivalent.*

(1) *$\mathcal{A}$ is regular.*

(2) *Every pure-projective object is projective.*
(3) *Every pure-projective object is flat.*

Proof. (1) ⇒ (2) Since $\mathcal{A}$ is regular, every short exact sequence in $\mathcal{A}$ is pure by [16, Theorem 4].
(2) ⇒ (3) Since every projective object is flat by [16, Lemma 7 (i)], it is clear.
(3) ⇒ (1) Let M be an object. Since $\mathcal{A}$ has enough pure-projective objects by [16, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. Now P is flat by assumption and therefore M is flat by [5, Proposition 2.2 (c) (ii)]. Thus $\mathcal{A}$ is regular by [16, Theorem 4]. □

Theorem 4.12. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projective objects. Then the following conditions are equivalent.*
(1) *$\mathcal{A}$ is regular.*
(2) *Every pure-direct-projective object of $\mathcal{A}$ is direct-projective.*

Proof. (1) ⇒ (2) Clear since every short exact sequence is pure exact in a regular category by [16, Theorem 4].
(2) ⇒ (1) Let M be a pure-projective object in $\mathcal{A}$. Since $\mathcal{A}$ has enough projectives, there is a projective object P and an epimorphism $f : P \longrightarrow M$. Since P is projective, it is pure-projective and so $M \oplus P$ is pure-projective by [7, Proposition 4.1 (1)]. Therefore $M \oplus P$ is direct-projective by assumption. Thus M is projective by Theorem 3.5 and so $\mathcal{A}$ is regular by Lemma 4.11. □

Now we have the following corollary of Theorem 4.12 for module categories.

Corollary 4.13. [1, Proposition 2.10] *The following conditions are equivalent for a unitary ring R.*
(1) *R is a von Neumann regular ring.*
(2) *Every pure-projective right R-module is projective.*
(3) *Every pure-direct-projective right R-module is direct-projective.*

Also we have the following corollary of Theorem 4.12 for comodule categories.

Corollary 4.14. *Let C be a semiperfect coalgebra over a field. Then the following statements are equivalent.*
(1) *C is cosemisimple.*
(2) *Every pure-projective right C-comodule is projective.*
(3) *Every pure-direct-projective right C-comodule is direct-projective.*

Proof. C has enough projectives if and only if C is right semiperfect (see [6, Theorem 3.2.3]). Since every cosemisimple coalgebra is right semiperfect, C has enough projectives. C is cosemisimple if and only if every right C-comodule is injective if and only if every right C-comodule is projective by [6, Theorem 3.1.5]. If a coalgebra C over a field is cosemisimple, then the category of right

C-comodules $\mathcal{M}^C$ is regular. Conversely, if $\mathcal{M}^C$ is regular, then every right C-comodule K is FP-injective, that is every short exact sequence of the form $0 \longrightarrow K \longrightarrow M \longrightarrow N \longrightarrow 0$ is pure (see [16]). The category of right C-comodules $\mathcal{M}^C$ coincides with the category $\sigma[{}^*_C\, C]$ of submodules of C-generated left C^*-modules (see [16, Section 2.5]). Since $\mathcal{M}^C$ is locally noetherian, every FP-injective right C-comodule is injective by [20, 35.7]. Therefore every right C-comodule is injective. Hence C is cosemisimple by [16, Theorem 3.1.5]. □

Proposition 4.15. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is regular and the coproduct of two pure-direct-projective objects is pure-direct-projective.*
(2) *Every pure-direct-projective object of $\mathcal{A}$ is projective.*

Proof. (1) ⇒ (2) Let M be a pure-direct-projective object of $\mathcal{A}$. Since every locally finitely presented Grothendieck category $\mathcal{A}$ has enough pure-projective objects by [16, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. Now $M \oplus P$ is pure-direct-projective by assumption. Then M is pure-projective by Theorem 3.5 and therefore M is projective by Theorem 4.11.
(2) ⇒ (1) Let M be a pure-direct-projective object of $\mathcal{A}$. Then M is projective by assumption and therefore it is flat by [16, Lemma 7 (i)]. So $\mathcal{A}$ is regular by Theorem 4.10. The rest of the proof is clear. □

Corollary 4.16. *Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category. If the coproduct of two pure-direct-projective objects is pure-direct-projective, then every pure-direct-projective object of $\mathcal{A}$ is quasi-projective.*

Proposition 4.17. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category with enough projective objects. If every pure-direct-projective object is quasi-projective, then $\mathcal{A}$ is regular.*

Proof. Let M be a finitely presented object of $\mathcal{A}$. So M is pure-projective. Since there are enough projective objects in $\mathcal{A}$, there is an epimorphism $f : P \longrightarrow M$ with P projective. Now $P \oplus M$ is pure-projective and therefore it is pure-direct-projective. So $P \oplus M$ is quasi-projective by assumption. Since every quasi-projective object is direct-projective, $P \oplus M$ is direct-projective. Then M is projective by Theorem 3.5. Hence $\mathcal{A}$ is regular by [15, Theorem 4]. □

Definition 4.18. *A Grothendieck category $\mathcal{A}$ is said to be* pure hereditary *if every quotient of an injective object of $\mathcal{A}$ is pure-injective.*

Recall that a class $\mathcal{C}$ of objects of a category is said to be *closed under extensions* if $A, M/A \in \mathcal{C}$ implies that $M \in \mathcal{C}$. In this case M is an extension of A and M/A. We have the following result which generalizes [9, Proposition 2.14].

Proposition 4.19. *Suppose that $\mathcal{A}$ is a Grothendieck category with enough projective objects and the class of pure-injective objects in $\mathcal{A}$ is closed under extensions. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is pure hereditary.*
(2) *Every pure subobject of any projective object is projective.*
(3) *Every flat object is of projective-dimension at most 1.*

Proof. (1) $\Rightarrow$ (2) Let M be a projective object of $\mathcal{A}$ and P be a pure subobject of M. Let $\beta : I \longrightarrow L$ be an epimorphism with I injective and let $f : P \longrightarrow L$ be a morphism from P to L. Since $\mathcal{A}$ is pure hereditary, L is pure-injective. So there exists a morphism $g : M \longrightarrow L$ such that $gh = f$. Since M is projective, there exists a morphism $\alpha : M \longrightarrow I$ such $\beta\alpha = g$. Put $\gamma = \alpha h : P \longrightarrow I$. Then gives $\beta\gamma = \beta(\alpha h) = gh = f$. Hence P is projective.
(2) $\Rightarrow$ (3) Let M be a flat object. Since $\mathcal{A}$ has enough projectives, there exists an epimorphism $f : P \longrightarrow M$ with P projective. Then we have the short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

which is pure since M is flat. So K is pure in P and therefore K is projective by assumption. Thus projective dimension of M is at most 1.
(3) $\Rightarrow$ (1) Let I be an injective object of $\mathcal{A}$ and N be a subobject of I. Then we have a short exact sequence

$$0 \longrightarrow N \longrightarrow I \longrightarrow I/N \longrightarrow 0.$$

Let M be a flat object of $\mathcal{A}$. Since projective dimension of M is at most 1 by assumption, $Ext^1_{\mathcal{A}}(M, I/N) = 0$. So I/N is a cotorsion object of $\mathcal{A}$. Therefore I/N is pure-injective by [21, Theorem 3.5.1] whose proof works in locally finitely presented Grothendieck categories. So $\mathcal{A}$ is pure-hereditary. □

Corollary 4.20. *Let $\mathcal{A}$ be a locally finitely presented regular Grothendieck category with enough projective objects. If the class of pure-injective objects is closed under extensions then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is pure hereditary.*
(2) *Every subobject of a projective object of $\mathcal{A}$ is pure-direct-projective.*
(3) *Every subobject of a pure-projective object of $\mathcal{A}$ is pure-direct-projective.*

Proof. Clear by Theorem 3.9 and Proposition 4.19. □

We have the following result for module categories.

Corollary 4.21. [1, Corollary 2.7] *Let R be a von Neumann regular ring. Then the following conditions are equivalent.*

(1) *R is pure hereditary.*
(2) *Every submodule of a projective right R-module is pure-direct-projective.*

(3) *Every submodule of a pure-projective right R-module is pure-direct-projective.*

Also we have the following result.

Corollary 4.22. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the class of pure-direct-projective objects of $\mathcal{A}$ need not be closed under subobjects.*

5. Classes all of whose objects are (pure-)direct-projective

In this section we investigate classes of (Grothendieck) abelian categories all of whose objects are (pure-)direct-projective. Recall that a morphism $f : P \longrightarrow M$ is said to be *projective cover* of an object M of an abelian category $\mathcal{A}$ if P is projective and $\mathrm{Ker}(f) \ll P$, i.e. for any subobject X of M, $\mathrm{Ker}(f) + X = M$ implies $X = M$. An abelian category $\mathcal{A}$ is called a *perfect* category if every object of $\mathcal{A}$ has a projective cover. If every subobject L of an object M of an abelian category $\mathcal{A}$ contains a direct summand K of M such that $L/K \ll M/K$, then M is said to be *lifting* (see [4]).

Theorem 5.1. [4, Theorem 3.5] *Let $\mathcal{A}$ be an abelian category. Then $\mathcal{A}$ is perfect if and only if it has enough projectives and every projective object of $\mathcal{A}$ is lifting.*

Recall that an abelian category $\mathcal{A}$ is called a *spectral* category if every short exact sequence in $\mathcal{A}$ splits (see [17, Definition p.129]).

Theorem 5.2. *Let $\mathcal{A}$ be an abelian category. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is spectral.*
(2) *$\mathcal{A}$ is perfect and every object of $\mathcal{A}$ is direct-projective.*
(3) *$\mathcal{A}$ is perfect and every factor object of a direct-projective object is direct-projective.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (2) Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough projective objects by Theorem 5.1, there exists an epimorphism $f : P \longrightarrow M$ with P projective. Therefore M is direct-projective being a quotient object of a direct-projective object P.
(2) $\Rightarrow$ (1) Let M be an object of $\mathcal{A}$. Since there are enough projective objects in $\mathcal{A}$ by Theorem 5.1, there is a short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P projective. Since every object of $\mathcal{A}$ is direct-projective, $P \oplus M$ is direct-projective. Then the sequence splits by Corollary 3.4. Therefore $\mathcal{A}$ is spectral. □

Recall that a Grothendieck category $\mathcal{A}$ is called *semisimple* if every object of $\mathcal{A}$ is semisimple, that is, a coproduct of simple objects. A locally finitely generated Grothendieck category is semisimple if and only if its spectral (see [17, Proposition 6.7, Chapter V]). Therefore we have the following result which generalizes [21, Theorem 9].

Corollary 5.3. *Let $\mathcal{A}$ be a locally finitely generated Grothendieck category. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is semisimple.*
(2) *$\mathcal{A}$ has enough projectives and every object of $\mathcal{A}$ is direct-projective.*
(3) *$\mathcal{A}$ has enough projectives and the coproduct of two direct-projective objects is direct-projective.*

Proof.
(1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear.
(3) $\Rightarrow$ (1) Let S be a simple object in $\mathcal{A}$. Since $\mathcal{A}$ has enough projective objects, there exists an epimorphism $f : P \longrightarrow S$ with P projective. S is clearly quasi-projective and therefore direct-projective. So $P \oplus S$ is direct-projective by assumption. Thus S is projective by Theorem 3.5. Then $\mathcal{A}$ is semisimple by [20, 20.7] whose proof works for locally finitely generated Grothendieck categories. □

Remark 5.4. *Let C be a coalgebra over a field. Then the category $\mathcal{M}^C$ of right C-comodules is spectral if and only if $\mathcal{M}^C$ is semisimple if and only if C is cosemisimple.*

Now we have the following result of Theorem 5.2 for comodule categories.

Corollary 5.5. *Let C be a coalgebra over a field. Then the following conditions are equivalent.*

(1) *C is cosemisimple.*
(2) *C is right semiperfect and every right C-comodule is direct-projective.*
(3) *C is right semiperfect and every factor comodule of a direct-projective right C-comodule is direct-projective.*

A Grothendieck category $\mathcal{A}$ is said to be *pure-semisimple* if it is locally finitely presented and each of its objects is pure-projective ([14]). A locally finitely presented Grothendieck category $\mathcal{A}$ is pure-semisimple if it has pure global dimension zero, which means that each of its objects is a direct summand of a coproduct of finitely presented objects ([13]). $\mathcal{A}$ is pure-semisimple if and only if it satisfies the pure noetherian property a coproduct of any family of pure-injective objects in $\mathcal{A}$ is pure-injective (see [14, Theorem 1.9]).

Lemma 5.6. *Let $\mathcal{A}$ be a finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is pure-semisimple.*
(2) *Every pure exact sequence in $\mathcal{A}$ splits.*

Proof. (1) $\Rightarrow$ (2) By definition every object in $\mathcal{A}$ is pure-projective. Since any pure exact sequence ending with pure-projective object splits by [20, 33.6] whose proof works for locally finitely presented Grothendieck categories, every pure-exact sequence in $\mathcal{A}$ splits.
(2) $\Rightarrow$ (1) Suppose that every pure exact sequence in $\mathcal{A}$ splits. Let M be an object of $\mathcal{A}$. We want to show that M is pure-projective. Since every locally finitely presented Grothendieck category has enough pure-projective objects by [16, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

with P pure-projective. This sequence splits by assumption and therefore M is a direct summand of the pure-projective object P. Then M is pure-projective by [7, Proposition 4.1]. This means that $\mathcal{A}$ is pure-semisimple. □

Theorem 5.7. *Let $\mathcal{A}$ be a locally finitely presented Grothendieck category. Then the following conditions are equivalent.*

(1) *$\mathcal{A}$ is pure-semisimple.*
(2) *Every object of $\mathcal{A}$ is pure-projective.*
(3) *Every object of $\mathcal{A}$ is pure-direct-projective.*
(4) *Every pure quotient of a pure-direct-projective object is pure-direct-projective.*

Proof. (1) $\Rightarrow$ (2) Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projective objects by [16, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

with N pure-projective. Since $\mathcal{A}$ is pure-semisimple,

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

splits by Lemma 5.6. So M is pure-projective [7, Proposition 4.1].
(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear.
(4) $\Rightarrow$ (3) Since $\mathcal{A}$ has enough pure-projective objects by [16, Lemma 6 (ii)], every object N of $\mathcal{A}$ is a pure quotient of a pure-projective object of $\mathcal{A}$.
(3) $\Rightarrow$ (1) Let M be an object of $\mathcal{A}$. Since $\mathcal{A}$ has enough pure-projective objects by [16, Lemma 6 (ii)], there exists a pure exact sequence

$$0 \longrightarrow K \longrightarrow N \longrightarrow M \longrightarrow 0$$

with N pure-projective. Since $N \oplus M$ is pure-direct-projective by assumption, M is pure-projective by Theorem 3.5. □

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