Honam Mathematical J. **45** (2023), No. 2, pp. 198–214
https://doi.org/10.5831/HMJ.2023.45.2.198

QUATERNIONS AND HOMOTHETIC MOTIONS IN EUCLIDEAN AND LORENTZIAN SPACES

Gülsüm YÜCA* and Yusuf YAYLI

Abstract. In the present paper, we investigate homothetic motions determined by quaternions, which is a general form of our previous paper [20]. We introduce a transition between homothetic motions in 3D and 4D Euclidean and Lorentzian spaces. In other words, we give a new method that works as a handy tool for obtaining Lorentzian homothetic motions from Euclidean homothetic motions. Moreover, some remarkable properties of homothetic motions, which are given in former studies on this subject, are also examined by dual transformations. Then, we present applications and visualize them with 3D-plots. Finally, we investigate homothetic motions in dual spaces because of the importance in many fields related to kinematics.

1. Introduction

Kinematics is the branch of mechanics that deals with the description of motion and plays a central role in a great variety of fields. Motion is the phenomenon of constant displacement of a rigid body relative to a certain reference point. Displacement of a rigid body is used to describe the motion of systems in engineering fields such as robotics and in other areas related. Homothetic motions of a rigid body in n-dimensional Euclidean space are generated by the homothetic transformations. In [6], the n-dimensional homothetic motion of a body in Euclidean space is generated by the transformation

$$Y = h.AX + C, \tag{1}$$

where $h = h.I_n$ is a scalar matrix, $A \in \mathrm{SO}(n)$ and $C \in \mathbb{R}^n_1$. Homothetic motions are studied by several authors [4], [14] and [15].

In this research, we examine homothetic motions provided by quaternions. W. R. Hamilton discovered quaternions in 1843 and gave the fundamental formula with the symbols, i, j, k;

$$i^2 = j^2 = k^2 = ijk = -1.$$

Received March 25, 2022. Revised August 15, 2022. Accepted February 2, 2023.
2020 Mathematics Subject Classification. 15B33, 53A17, 53B30.
Key words and phrases. Homothetic motion, Hamilton operator, quaternion, Lorentzian space, dual space, kinematics.
*Corresponding author

The set of quaternions denoted by $\mathbb{H}$ can be represented as

$$\mathbb{H} = \{q = x_0 + x_1 i + x_2 j + x_3 k | x_0, x_1, x_2, x_3 \in \mathbb{R}\}.$$

Quaternions have long been used in many fields including kinematics, robotics, computer graphics, aerospace, quantum physics, and other areas related. Quaternions also have attracted attention as tools for rotations in recent years. The groups SO(3) and SO(4) of rotations in 3D and 4D spaces will be used for dual transformations during this study. The transition between Euclidean and Lorentzian rotational motion matrices is given with the help of dual transformations between $\mathrm{SO}(n)\backslash\{a_{nn} = 0\}$ and $\mathrm{SO}(n-1,1)$ in [5]. In the light of this study, we investigate dual transformations in dual spaces by examining invariant axes in both spaces, see [18]. Then, kinematics applications of dual transformations are given in [17]. Additionally, we carry this research into Galilean spaces in [19]. Afterward, we studied quaternions and dual transformations in [20]. Lastly, homothetic motions are defined by using dual transformations in [21]

The objective of this paper is to obtain homothetic motions determined by quaternions, which is a general form of our previous paper [20]. In other words, if we take $h \equiv 1$ in Eq. 1, then we have the results in [20]. We introduce a transition between homothetic motions in 3D and 4D Euclidean and Lorentzian spaces. Furthermore, some notable properties of homothetic motions, which are given in former studies on this subject, are also examined by dual transformations. Then, we give examples with figures. Finally, we investigate homothetic motions in dual spaces because of the importance in kinematics and other areas related.

2. Preliminaries

This section includes three subsections to provide a background for Lorentzian space, quaternions, and dual transformations. Since the concepts to be presented in the subsections will be given with their dual notions, it would be appropriate to give the definitions that belong to dual space beforehand.

Definition 2.1. If a and a^* are real numbers and $\epsilon^2 = 0$, the combination $\widehat{a} = a + \epsilon a^*$ is called a *dual number*, where ϵ is the dual unit.

Definition 2.2. The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$. The set $\mathbb{D}^3 = \{\overrightarrow{\widehat{a}} = (\widehat{a}_1, \widehat{a}_2, \widehat{a}_3) | \widehat{a}_i \in \mathbb{D}, 1 \leq i \leq 3\}$ is called a $\mathbb{D}$-*module* or *dual space.*

Definition 2.3. The elements of $\mathbb{D}^3$ are called *dual vectors.* A dual vector $\overrightarrow{\widehat{a}}$ can be written $\overrightarrow{\widehat{a}} = \overrightarrow{a} + \epsilon \overrightarrow{a}^*$, where $\overrightarrow{a}$ and $\overrightarrow{a}^*$ are real vectors in $\mathbb{R}^3$.

Definition 2.4. The norm of a dual vector $\overrightarrow{\widehat{a}}$ is defined by $|\overrightarrow{\widehat{a}}| = |\overrightarrow{a}| + \epsilon \dfrac{\langle \overrightarrow{a}, \overrightarrow{a}^* \rangle}{|\overrightarrow{a}|^2}$.

See [16].

2.1. Background on Lorentzian Space

In current years, studies on Lorentzian space have taken an important place in the field of mathematical research. It can be seen as a part of differential geometry (see [12], [13]) as well as robotics, computer graphics, most areas of physics, and kinematics (see [16], [9], [10]). We mention some fundamental definitions and properties in Lorentzian space that we use in this paper.

Definition 2.5. The Lorentzian metric $\langle , \rangle$ defined by

$$\langle u, v\rangle = u_1v_1 + u_2v_2 + ... + u_{n-1}v_{n-1} - u_nv_n \tag{2}$$

in E_1^n will be used in this study.

It is pointed out that $\langle , \rangle$ is a non-degenerate metric of index 1. It can also be written in the form:

$$\langle u, v\rangle = u^T \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{bmatrix} v = u^T G v. \tag{3}$$

Definition 2.6. A vector $v \in E_1^n$ is called spacelike if $\langle v, v\rangle > 0$ or $v = 0$, timelike if $\langle v, v\rangle < 0$, lightlike if $\langle v, v\rangle = 0$ and $v \neq 0$.

Definition 2.7. An $n \times n$ matrix S is called semi symmetric if $S^T = GSG$ or $S = GS^TG$, semi skew-symmetric if $S^T = -GSG$ or $S = -GS^TG$, semi-orthogonal if $S^T = GS^{-1}G$ or $S^{-1} = GS^TG$, where G is the sign matrix of Lorentzian space.

Definition 2.8. The Lorentzian inner product of dual vectors $\overrightarrow{\widehat{a}}$ and $\overrightarrow{\widehat{b}}$ is defined by

$$\left\langle \overrightarrow{\widehat{a}}, \overrightarrow{\widehat{b}} \right\rangle = \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle + \epsilon(\left\langle \overrightarrow{a}, \overrightarrow{b}^* \right\rangle + \left\langle \overrightarrow{a}^*, \overrightarrow{b} \right\rangle)$$

with $\overrightarrow{\widehat{a}} = \overrightarrow{a} + \epsilon \overrightarrow{a}^*$ and $\overrightarrow{\widehat{b}} = \overrightarrow{b} + \epsilon \overrightarrow{b}^*$. A dual vector $\overrightarrow{\widehat{a}}$ is called *timelike* if $\left\langle \overrightarrow{\widehat{a}}, \overrightarrow{\widehat{a}} \right\rangle < 0$, *spacelike* if $\left\langle \overrightarrow{\widehat{a}}, \overrightarrow{\widehat{a}} \right\rangle > 0$ and *lightlike* (or *null*) if $\left\langle \overrightarrow{\widehat{a}}, \overrightarrow{\widehat{a}} \right\rangle = 0$, where $\langle , \rangle$ is Lorentzian inner product. We call the dual space $\mathbb{D}^3$ together with this Lorentzian inner product as *dual Lorentzian space* and indicate it by $\mathbb{D}_1^3$.

2.2. Background on Quaternions

We now give some concepts of real and dual quaternions to provide a background for the main results of this study. A real quaternion q is an expression of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and x_3 are real numbers, and i, j, k are real quaternionic units which satisfy the non-commutative multiplication rules

$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = -ji = k,\ jk = -kj = i,\ ki = -ik = j.$$

A real quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is pieced into two parts with scalar piece $S_q = x_0$ and vectorial piece $\overrightarrow{V_q} = x_1 i + x_2 j + x_3 k$. We also write $q = S_q + \overrightarrow{V_q}$. The quaternionic conjugate of $q = S_q + \overrightarrow{V_q}$ is defined as $\bar{q} = S_q - \overrightarrow{V_q}$. The norm of a real quaternion $q = x_0 + x_1 i + x_2 j + x_3 k$ is $|q| = q\bar{q} = \bar{q}q = x_0^2 + x_1^2 + x_2^2 + x_3^2 \geq 0$, where $q \in \mathbb{R}$. If $|q| = 1$ then q is called unit real quaternion. For more details about concepts and properties of real quaternions see [2], [7], [8] and [11].

The set of dual quaternions is denoted by $\mathbb{H}_D$ can be represented as

$$\mathbb{H}_D = \{\widehat{q} = \widehat{x_0} + \widehat{x_1} i + \widehat{x_2} j + \widehat{x_3} k | \widehat{x_0}, \widehat{x_1}, \widehat{x_2}, \widehat{x_3} \in \mathbb{D}\}.$$

The ring of dual quaternions is defined as the four-dimensional vector space over dual numbers $\mathbb{D}$ having a basis {1,i,j,k} with the same multiplication property of the basis elements in real quaternions. A dual quaternion $\widehat{q} = \widehat{x_0} + \widehat{x_1} i + \widehat{x_2} j + \widehat{x_3} k$ can be written as $\widehat{q} = q + \epsilon q^*$, where q and q^* are real and pure dual quaternion components, respectively. We may consider a dual quaternion as $\widehat{q} = x_0 + x_1 i + x_2 j + x_3 k + \epsilon(x_0^* + x_1^* i + x_2^* j + x_3^* k)$. For more details about concepts and properties of dual quaternions see [1], [3].

2.3. Dual Transformations

The dual transformation between $\mathrm{SO}(n) \backslash \{a_{nn} = 0\}$ and $\mathrm{SO}(n-1,1)$ which is defined below will be used for obtaining semi-orthogonal matrices from orthogonal matrices.

Definition 2.9. Dual transformation between $\mathrm{SO}(n) \backslash \{a_{nn} = 0\}$ and $\mathrm{SO}(n-1,1)$ is defined in [5]. The two sets to use when applying dual transformation are as follows

$$\mathrm{SO}(n) = \{A \in GL(n,\mathbb{R}) | A^T A = AA^T = I_n, \det A = 1\},$$
$$\mathrm{SO}(n-1,1) = \{A \in GL(n,\mathbb{R}) | A^T G A = AGA^T = G, \det A = 1\},$$

where $G = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & -1 \end{array}\right]$ and I_n is $n \times n$ identity matrix.

Let $A \in \mathrm{SO}(n)$. Then it can be written in the block form as

$$A = \left[\begin{array}{c|c} B & C \\ \hline D & a_{nn} \end{array}\right],$$

where $a_{nn} \neq 0$. Here, B is an $(n-1) \times (n-1)$ square matrix, C is a column matrix and D is a row matrix. Since $a_{nn} \neq 0$, we can use the following two sets given by

$$\mathfrak{S}_1 = \{A \in \mathrm{SO}(n) | a_{nn} \neq 0\},$$

$$\mathfrak{S}_2 = \{A \in \mathrm{SO}(n-1,1) | a_{nn} \neq 0\}.$$

Thus, the dual transformation can be defined as

$$f : \mathfrak{S}_1 \to \mathfrak{S}_2$$

$$f : A \mapsto f(A) = \frac{1}{a_{nn}} \left[\begin{array}{c|c} a_{nn}(B^{-1})^T & C \\ \hline -D & 1 \end{array} \right], \tag{4}$$

where T denotes transposition.

We now give the definition of dual transformation in dual spaces. We will use it for obtaining dual semi-orthogonal matrices from dual orthogonal matrices.

In order to define f dual transformation, we need the following sets given by

$$\mathrm{S}\widehat{O}(n) = \{\widehat{A} \in GL(n,\mathbb{D}) | \widehat{A}^T \widehat{A} = \widehat{A}\widehat{A}^T = I_n, \det \widehat{A} = 1\},$$

$$\mathrm{S}\widehat{O}(n-1,1) = \{\widehat{A} \in GL(n,\mathbb{D}) | \widehat{A}^T G \widehat{A} = \widehat{A} G \widehat{A}^T = G, \det \widehat{A} = 1\},$$

where $G = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & -1 \end{array} \right]$ and I_n is $n \times n$ identity matrix.

We write the dual matrix $\widehat{A} \in \mathrm{S}\widehat{O}(n)$ in the block form as

$$\widehat{A} = \left[\begin{array}{c|c} \widehat{B} & \widehat{C} \\ \hline \widehat{D} & \widehat{a}_{nn} \end{array} \right],$$

where $\widehat{a}_{nn} \neq 0$. Since $\widehat{a}_{nn} \neq 0$, then two sets can be written as

$$\widehat{\mathfrak{S}_1} = \{\widehat{A} \in \mathrm{S}\widehat{O}(n) | \widehat{a}_{nn} \neq 0\},$$

$$\widehat{\mathfrak{S}_2} = \{\widehat{A} \in \mathrm{S}\widehat{O}(n-1,1) | \widehat{a}_{nn} \neq 0\}.$$

Then, f dual transformation can be defined as below

$$f : \widehat{\mathfrak{S}_1} \to \widehat{\mathfrak{S}_2}$$

$$f : \widehat{A} \mapsto f(\widehat{A}) = \frac{1}{\widehat{a}_{nn}} \left[\begin{array}{c|c} \widehat{a}_{nn}(\widehat{B}^{-1})^T & \widehat{C} \\ \hline -\widehat{D} & 1 \end{array} \right]. \tag{5}$$

See [18].

3. Homothetic Motions With Dual Transformations in E^3 and E_1^3

In this section, we obtain a Lorentzian homothetic motion from a Euclidean homothetic motion with the help of quaternions and dual transformations in 3D-spaces. First, we give the one-parameter homothetic motion along a curve in 3D Euclidean space which is investigated in [14]. In 3D Euclidean space, one-parameter homothetic motions of a rigid body are generated by the transformation $Y = h.AX + C$ with the matrix representation as

$$\begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} h.A & C \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X \\ 1 \end{bmatrix}, \tag{6}$$

where h is a homothetic scalar and $A \in \mathrm{SO}(3)$. The matrix $B = h.A$ is called a homothetic matrix and Y, X and $C \in \mathbb{R}^3_1$. The homothetic scalar h and the elements of A and C are continuously differentiable functions of a real parameter t. Y and X correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space R_o and the fixed space R, respectively. At the initial time $t = t_o$, we consider the coordinate systems of R_o and R are coincident. To avoid the case of affine transformation we assume that $h(t) \neq cons.$ and to avoid the case of a pure translation or a pure rotation, we assume that $\dfrac{d}{dt}(hA) \neq 0$, $\dfrac{d}{dt}(C) \neq 0$.

Let us consider the quaternion curve $\alpha : I \subset \mathbb{R} \to E^4$ defined by $\alpha(t) = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t))$, for every $t \in I$. We suppose that the curve $\alpha(t)$ is a differentiable regular curve of order r which does not pass through the origin.

Let us write the matrix B as follows:

(7)
$$B = \begin{bmatrix} \alpha_0(t)^2 + \alpha_1(t)^2 - \alpha_2(t)^2 - \alpha_3(t)^2 & -2\alpha_0(t)\alpha_3(t) + 2\alpha_1(t)\alpha_2(t) & 2\alpha_0(t)\alpha_2(t) + 2\alpha_1(t)\alpha_3(t) \\ 2\alpha_1(t)\alpha_2(t) + 2\alpha_3(t)\alpha_0(t) & \alpha_0(t)^2 - \alpha_1(t)^2 + \alpha_2(t)^2 - \alpha_3(t)^2 & 2\alpha_2(t)\alpha_3(t) - 2\alpha_1(t)\alpha_0(t) \\ 2\alpha_1(t)\alpha_3(t) - 2\alpha_2(t)\alpha_0(t) & 2\alpha_1(t)\alpha_0(t) + 2\alpha_2(t)\alpha_3(t) & \alpha_0(t)^2 - \alpha_1(t)^2 - \alpha_2(t)^2 + \alpha_3(t)^2 \end{bmatrix}.$$

For the matrix B, we have $BB^T = B^T B = h^2 I_3$ and $detB = h^3$,
where
$$h : I \subset \mathbb{R} \to \mathbb{R},$$
$$t \mapsto h(t) = \alpha_0(t)^2 + \alpha_1(t)^2 + \alpha_2(t)^2 + \alpha_3(t)^2.$$
We can represent the matrix B as

$$B = h. \begin{bmatrix} \frac{b_{11}}{h} & \frac{b_{12}}{h} & \frac{b_{13}}{h} \\ \frac{b_{21}}{h} & \frac{b_{22}}{h} & \frac{b_{23}}{h} \\ \frac{b_{31}}{h} & \frac{b_{32}}{h} & \frac{b_{33}}{h} \end{bmatrix} = h.A. \tag{8}$$

Here, $A \in \mathrm{SO}(3)$, B is a homothetic matrix and Eq.6 determines a homothetic motion. See [14].

We give the following theorem presents the transition between Euclidean and Lorentzian homothetic motion matrices. After that, we can provide examples and theorems of homothetic motions in Lorentzian space.

Theorem 3.1. Let $\bar{B} \in E^3$ determines a homothetic motion of a rigid body given by

$$\bar{B} = h.A + C, \tag{9}$$

where $h = h.I_3$ is a scalar matrix, $A \in \mathrm{SO}(3)$ and $C \in \mathbb{R}^3_1$. f_h defines a dual transformation between Euclidean and Lorentzian homothetic motion matrices in 3D-spaces.

$$f_h : E^3 \to E_1^3$$

$$\bar{B} \mapsto f_h(\bar{B}) = h.f(A) + C, \tag{10}$$

where f is the dual transformation given in Eq.4, $f(A) \in \mathrm{SO}(2,1)$. Eq.10 determines a homothetic motion of a rigid body in 3D Lorentzian space.

We have denoted $h.A$ as a homothetic matrix $B \in E^3$ in Eq. 8. Then, we now denote $h.f(A)$ as B_L since it is a homothetic matrix in E_1^3.

Proof. We observe that

$$\begin{aligned} f_h^2(\bar{B}) &= f_h(f_h(\bar{B})) \\ &= f_h(\bar{B}_L), \quad f^2 = id. \\ &= \bar{B} \\ &\Longrightarrow f_h^2 = id. \end{aligned}$$

Thus, f_h is a dual transformation. □

Example 3.2. Let us consider a curve $\alpha : I \subset \mathbb{R} \to E^4$ given by $\alpha(t) = (\frac{1}{\sqrt{2}}\cos(t), \frac{1}{\sqrt{2}}\sin(t), 1, \frac{t}{\sqrt{2}})$, for every $t \in I$. $\alpha(t)$ is a differentiable regular curve of order r. Since $\alpha(t)$ does not pass through the origin, we can write the matrix B as follows

$$B = \begin{bmatrix} -\frac{t^2}{2} - \frac{1}{2} & -t\cos(t) + \sqrt{2}\sin(t) & \sqrt{2}\cos(t) + t\sin(t) \\ t\cos(t) + \sqrt{2}\sin(t) & \frac{\cos^2(t) - \sin^2(t)}{2} - \frac{t^2}{2} + 1 & \sqrt{2}t - \sin(t)\cos(t) \\ t\sin(t) - \sqrt{2}\cos(t) & \sin(t)\cos(t) + \sqrt{2}t & \frac{\cos^2(t) - \sin^2(t)}{2} + \frac{t^2}{2} - 1 \end{bmatrix}$$

$$= (\frac{3+t^2}{2}) \begin{bmatrix} \frac{-t^2-1}{t^2+3} & \frac{2\sqrt{2}\sin(t) - 2t\cos(t)}{t^2+3} & \frac{2\sqrt{2}\cos(t) + 2t\sin(t)}{t^2+3} \\ \frac{2\sqrt{2}\sin(t) + 2t\cos(t)}{t^2+3} & \frac{2\cos^2(t) - t^2 + 1}{t^2+3} & \frac{2\sqrt{2}t - 2\sin(t)\cos(t)}{t^2+3} \\ \frac{2t\sin(t) - 2\sqrt{2}\cos(t)}{t^2+3} & \frac{2\sin(t)\cos(t) + 2\sqrt{2}t}{t^2+3} & \frac{2\cos^2(t) + t^2 - 3}{t^2+3} \end{bmatrix}$$

$$= (\frac{3+t^2}{2})A,$$

where $h(t) = \frac{3+t^2}{2}$, $A \in \mathrm{SO}(3)$. Therefore, B is a homothetic matrix and it determines a homothetic motion in E^3.

Now, we obtain B_L by applying the f_h dual transformation in Eq. 10 to the matrix B.

$$B_L = (\frac{3+t^2}{2}) \begin{bmatrix} \frac{2\cos^2(t) - t^2 + 1}{2\cos^2(t) + t^2 - 3} & \frac{-2\sqrt{2}\sin(t) - 2t\cos(t)}{2\cos^2(t) + t^2 - 3} & \frac{2\sqrt{2}\cos(t) + 2t\sin(t)}{2\cos^2(t) + t^2 - 3} \\ \frac{-2\sqrt{2}\sin(t) + 2t\cos(t)}{2\cos^2(t) + t^2 - 3} & \frac{-t^2-1}{2\cos^2(t) + t^2 - 3} & \frac{2\sqrt{2}t - 2\sin(t)\cos(t)}{2\cos^2(t) + t^2 - 3} \\ \frac{-2t\sin(t) + 2\sqrt{2}\cos(t)}{2\cos^2(t) + t^2 - 3} & \frac{-2\sin(t)\cos(t) - 2\sqrt{2}t}{2\cos^2(t) + t^2 - 3} & \frac{t^2+3}{2\cos^2(t) + t^2 - 3} \end{bmatrix}$$

$$= (\frac{3+t^2}{2})f(A),$$

where $h(t) = (\frac{3+t^2}{2})$, $f(A) \in SO(2,1)$. Therefore, B_L is a homothetic matrix and it determines a homothetic motion in E_1^3.

We continue with obtaining surfaces from homothetic matrices by multiplying with the curve $\gamma(s) = (\cos(s), \sin(s), s^3)$. Initially, we get a surface from Euclidean homothetic matrix by multiplying the curve $\gamma(s)$. The elements of the matrix $B.\gamma(s)$ can be represented by the surface $S_1 = ((-t^2-1)/2)\cos(s) + (-\cos(t)t + \sqrt{2}\sin(t))\sin(s) + (\sqrt{2}\cos(t) + \sin(t)t)s^3, ((2\cos(t)^2 - t^2+1)/2)\sin(s) + (\cos(t)t + \sqrt{2}\sin(t))\cos(s) + (\sqrt{2}t - \cos(t)\sin(t))s^3, (\sin(t)\cos(t) + \sqrt{2}t)\sin(s) + (\sin(t)t - \sqrt{2}\cos(t))\cos(s) + ((2\cos(t)^2 + t^2 - 3)/2)s^3$. See Fig. 1

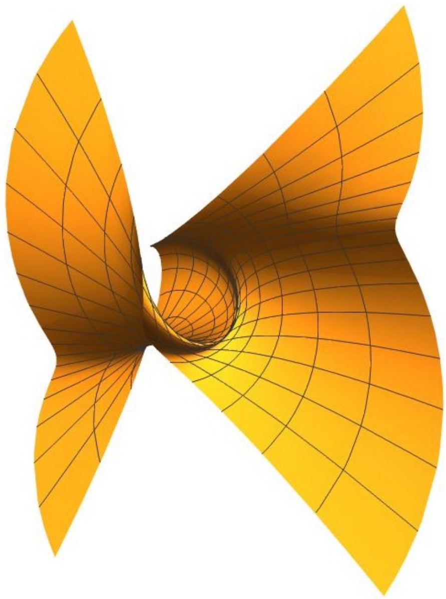

FIGURE 1. The surface $S_1 \in \mathbb{R}^3$

Subsequently, we obtain the surface from Lorentzian homothetic matrix B_L by multiplying with the same curve $\gamma(s) = (\cos s, \sin s, s^3)$. The elements of the matrix $B_L.\gamma(s)$ can be represented by the surface $S_2 = \delta(t,s) = 2\sin(s)(-\cos(t)t - \sqrt{2}\sin(t)/(\cos(t)^2 - \sin(t)^2 + t^2 - 2) + \cos(s)(\cos(t)^2 - \sin(t)^2 - t^2+2)/(\cos(t)^2 - \sin(t)^2 + t^2 - 2) + 2s^3(\sin(t)t + \sqrt{2}\cos(t))/(\cos(t)^2 - \sin(t)^2 + t^2 - 2), \sin(s)(\cos(t)^2 + \sin(t)^2 - t^2 - 2)/(\cos(t)^2 - \sin(t)^2 + t^2 - 2) + (2\cos(s)(\cos(t)t - \sqrt{2}\sin(t)))/(\cos(t)^2 - \sin(t)^2 + t^2 - 2) + s^3(2\sqrt{2}t - \cos(t)\sin(t))/(\cos(t)^2 -$

$\sin(t)^2 + t^2 - 2), 2\sin(s)(-2\sin(t)\cos(t) - 2\sqrt{2}t)/(\cos(t)^2 - \sin(t)^2 + t^2 - 2) + 2\cos(s)(-\sin(t)t + \sqrt{2}\cos(t))/(\cos(t)^2 - \sin(t)^2 + t^2 - 2) + 2s^3/(\cos(t)^2 - \sin(t)^2 + t^2 - 2))$. See Fig. 2.

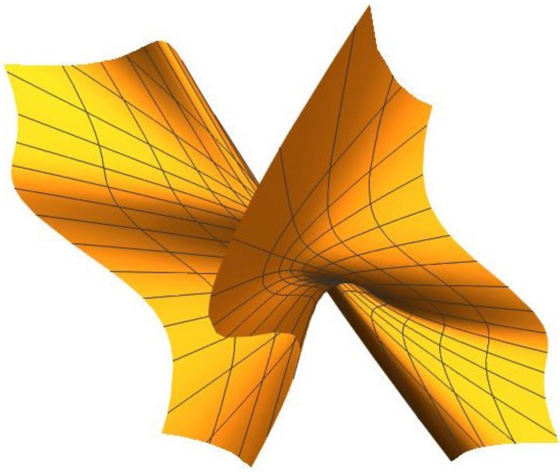

FIGURE 2. The surface $S_2 \in \mathbb{R}^3_1$

We obtained surfaces S_1 and S_2. Drawing these figures allowed us to visualize the applications. We have defined dual transformations in Euclidean and Lorentzian spaces with the help of homothetic motions. Through these visuals, we have the opportunity to compare the surfaces drawn with the help of these motions in both spaces. If we examine the figures in the examples carefully, we can imagine the figures obtained in Lorentzian space as the more opened form of that obtained in Euclidean space. In our previous studies (see [20], [18] and [17]), we compared the images we obtained in Euclidean space and Lorentz space. We have also captured similar views and considered it appropriate to include them in this study. Although the spaces are different, we think it is interesting that the resulting figures resemble each other.

4. Homothetic Motions With Dual Transformations in E^4 and E^4_1

In this section, we investigate homothetic motions produced by Hamilton operators with the help of dual transformations. First, we use the orthogonal matrix representation of a quaternion in 4D space. Afterward, we obtain a Lorentzian homothetic motion from a Euclidean homothetic motion.

Let us consider the quaternion curve $\alpha : I \subset \mathbb{R} \to E^4$ defined by $\alpha(t) = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t))$, for every $t \in I$. We suppose that the curve $\alpha(t)$ is a differentiable regular curve of order r. The operator Q, called the Hamilton operator corresponding to $\alpha(t)$ is determined by the following matrix:

$$H_Q = \begin{bmatrix} \alpha_0(t) & -\alpha_1(t) & -\alpha_2(t) & -\alpha_3(t) \\ \alpha_1(t) & \alpha_0(t) & -\alpha_3(t) & \alpha_2(t) \\ \alpha_2(t) & \alpha_3(t) & \alpha_0(t) & -\alpha_1(t) \\ \alpha_3(t) & -\alpha_2(t) & \alpha_1(t) & \alpha_0(t) \end{bmatrix},$$

where $\alpha_0(t) \neq 0$. Let $\alpha(t)$ be a unit velocity curve. Since $\alpha(t)$ does not pass through the origin, we can write the matrix H_Q as follows

$$(11) \qquad H_Q = h. \begin{bmatrix} \frac{\alpha_0(t)}{h} & \frac{-\alpha_1(t)}{h} & -\frac{\alpha_2(t)}{h} & \frac{-\alpha_3(t)}{h} \\ \frac{\alpha_1(t)}{h} & \frac{\alpha_0(t)}{h} & \frac{-\alpha_3(t)}{h} & \frac{\alpha_2(t)}{h} \\ \frac{\alpha_2(t)}{h} & \frac{\alpha_3(t)}{h} & \frac{\alpha_0(t)}{h} & \frac{-\alpha_1(t)}{h} \\ \frac{\alpha_3(t)}{h} & \frac{-\alpha_2(t)}{h} & \frac{\alpha_1(t)}{h} & \frac{\alpha_0(t)}{h} \end{bmatrix} = h.Q,$$

where

$$h : I \subset \mathbb{R} \to \mathbb{R},$$
$$t \mapsto h(t) = \| \alpha(t) \| = \sqrt{(\alpha_0(t)^2 + \alpha_1(t)^2 + \alpha_2(t)^2 + \alpha_3(t)^2)}.$$

Here, $Q \in SO(4)$.

Hamiltonian motion is generated by the transformation

$$(12) \qquad Y = h.QX + C$$

with the matrix representation as

$$(13) \qquad \begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} h.Q & C \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} X \\ 1 \end{bmatrix},$$

where h is a homothetic scalar and $Q \in SO(4)$. The matrix $H_Q = h.Q$ is called a homothetic matrix and Y, X and $C \in \mathbb{R}^4_1$. Y and X correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space R_o and the fixed space R, respectively. Hamiltonian motion defined by Eq. 13 in 4D Euclidean space is a homothetic motion, it was shown in [6].

Theorem 4.1. The derivation operator $\dot{H}_Q$ of the Hamilton operator $H_Q = hQ$ is a real orthogonal matrix.

Proof. $\dot{H}_Q.\dot{H}_Q{}^T = \dot{H}_Q{}^T.\dot{H}_Q = I_4$ and $det\dot{H}_Q = 1$. □

See [15].

Theorem 4.2. In E^4, the Hamilton motion that is represented by the matrix H_Q is a regular motion and it is independent of h.

Proof. This motion is regular as $det\dot{H}_Q = 1$ and the value of $det\dot{H}_Q$ is independent of h. □

See [15].

We now give the following theorem that presents the transition between Euclidean and Lorentzian homothetic motions in 4D-spaces.

Theorem 4.3. Let $H_Q \in E^4$ determines a homothetic motion given by

$$H_Q = h.Q + C, \tag{14}$$

where $h = h.I_4$ is a scalar matrix, $Q \in SO(4)$ and $C \in \mathbb{R}_1^4$. f_h defines a dual transformation,

$$f_h : E^4 \to E_1^4$$

$$H_Q \mapsto f_h(H_Q) = H_{Q_L} = h.f(Q) + C, \tag{15}$$

where $f(Q) = Q_L \in SO(3,1)$. We can write the matrix H_{Q_L} by using f_h dual transformation as follows:

$$H_{Q_L} = h.\begin{bmatrix} \frac{\alpha_0(t)^2+\alpha_3(t)^2}{h.\alpha_0(t)} & \frac{-\alpha_0(t)\alpha_1(t)-\alpha_2(t)\alpha_3(t)}{h.\alpha_0(t)} & \frac{-\alpha_0(t)\alpha_2(t)+\alpha_1(t)\alpha_3(t)}{h.\alpha_0(t)} & \frac{-\alpha_3(t)}{h.\alpha_0(t)} \\ \frac{\alpha_0(t)\alpha_1(t)-\alpha_2(t)\alpha_3(t)}{h.\alpha_0(t)} & \frac{\alpha_0(t)^2+\alpha_2(t)^2}{h.\alpha_0(t)} & \frac{-\alpha_0(t)\alpha_3(t)-\alpha_1(t)\alpha_2(t)}{h.\alpha_0(t)} & \frac{\alpha_2(t)}{h.\alpha_0(t)} \\ \frac{\alpha_0(t)\alpha_2(t)+\alpha_1(t)\alpha_3(t)}{h.\alpha_0(t)} & \frac{\alpha_0(t)\alpha_3(t)-\alpha_1(t)\alpha_2(t)}{h.\alpha_0(t)} & \frac{\alpha_0(t)^2+\alpha_1(t)^2}{h.\alpha_0(t)} & \frac{-\alpha_1(t)}{h.\alpha_0(t)} \\ \frac{-\alpha_3(t)}{h.\alpha_0(t)} & \frac{\alpha_2(t)}{h.\alpha_0(t)} & \frac{-\alpha_1(t)}{h.\alpha_0(t)} & \frac{1}{h.\alpha_0(t)} \end{bmatrix}.$$

The matrix H_{Q_L} represents the homothetic motion in 4-dimensional Lorentzian space.

Proof. We show that

$$\begin{aligned} f_h^2(H_Q) &= f_h(f_h(H_Q)) \\ &= f_h(H_{Q_L}), \quad f^2 = id. \\ &= H_Q \\ &\implies f_h^2 = id. \end{aligned}$$

Hence, f_h is a dual transformation. □

We now give an example of one-parameter homothetic motions produced by Hamilton operators with the help of dual transformations.

Example 4.4. Let us consider a curve $\alpha : I \subset \mathbb{R} \to E^4$ given by $\alpha(t) = (\cos(\frac{t}{\sqrt{2}}), \sin(\frac{t}{\sqrt{2}}), \cos(\frac{t}{\sqrt{2}}), \sin(\frac{t}{\sqrt{2}}))$, for every $t \in I$. $\alpha(t)$ is a differentiable regular curve of order r. Since $\alpha(t)$ does not pass through the origin, we can write the matrix $H_Q(t)$ as follows:

$$\begin{aligned} H_Q = \sqrt{2}.&\begin{bmatrix} \frac{\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{-\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{-\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{-\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} \\ \frac{\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{-\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} \\ \frac{\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{-\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} \\ \frac{\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{-\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{\sin(\frac{t}{\sqrt{2}})}{\sqrt{2}} & \frac{\cos(\frac{t}{\sqrt{2}})}{\sqrt{2}} \end{bmatrix}, \\ &= \sqrt{2}.Q, \end{aligned}$$

where $h = \sqrt{2}$, $Q \in SO(4)$. Therefore, H_Q is a homothetic matrix and it determines a homothetic motion in E^4.

We now obtain the Lorentzian matrix H_{Q_L} by using f_h as below

$$H_{Q_L} = \sqrt{2}. \begin{bmatrix} \dfrac{1}{\sqrt{2}\cos(\frac{t}{\sqrt{2}})} & -\sqrt{2}\sin(\frac{t}{\sqrt{2}}) & \dfrac{-\cos^2(\frac{t}{\sqrt{2}}) + \sin^2(\frac{t}{\sqrt{2}})}{\sqrt{2}\cos(\frac{t}{\sqrt{2}})} & -\tan(\frac{t}{\sqrt{2}}) \\ 0 & \sqrt{2}\cos(\frac{t}{\sqrt{2}}) & -\sqrt{2}\sin(\frac{t}{\sqrt{2}}) & -1 \\ \dfrac{1}{\sqrt{2}\cos(\frac{t}{\sqrt{2}})} & 0 & \dfrac{1}{\sqrt{2}\cos(\frac{t}{\sqrt{2}})} & -\tan(\frac{t}{\sqrt{2}}) \\ -\tan(\frac{t}{\sqrt{2}}) & -1 & -\tan(\frac{t}{\sqrt{2}}) & \dfrac{\sqrt{2}}{\cos(\frac{t}{\sqrt{2}})} \end{bmatrix},$$

where $h = \sqrt{2}$, $f(Q) \in SO(3,1)$. Therefore, H_{Q_L} is a homothetic matrix and it determines a homothetic motion in E_1^4.

5. Dual Homothetic Motions With Dual Transformations

In this section, firstly, we acquire a dual homothetic motion in $\mathbb{D}_1^3$ from a dual homothetic motion in $\mathbb{D}^3$ with the help of dual quaternions and dual transformations. Secondly, we investigate dual homothetic motions produced by Hamilton operators with the help of dual transformations between $\mathbb{D}^4$ and $\mathbb{D}_1^4$.

In $\mathbb{D}^3$, we consider the dual motional space $\tilde{R}_o$ and the dual fixed space $\tilde{R}$, respectively. This dual motion can be expressed as follows:

$$\begin{bmatrix} \widehat{Y} \\ 1 \end{bmatrix} = \begin{bmatrix} \widehat{h}.\widehat{A} & \widehat{C} \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} \widehat{X} \\ 1 \end{bmatrix}, \tag{16}$$

or equivalently

$$\widehat{Y} = \widehat{h}.\widehat{A}\widehat{X} + \widehat{C}, \tag{17}$$

where dual position vectors of any point respectively in $\tilde{R}_o$ and $\tilde{R}$ are represented by $\widehat{Y}$ and $\widehat{X}$, and $\widehat{C}$ represents any dual translation vector.
In $\mathbb{D}^3$, the one-parameter dual homothetic motion of a body is generated by the transformation given in Eq. 17, where $\widehat{h}$ is called the homothetic scalar, which is a dual scalar matrix, $\widehat{A} \in S\widehat{O}(3)$ dual orthogonal matrix, $\widehat{X}$, and $\widehat{C}$ are 3×1 dual matrices, and $\widehat{A}$, $\widehat{C}$ and $\widehat{h}$ are differentiable functions of C^r class of a parameter t.

In order not to encounter the case of affine transformation we suppose that $\widehat{h} = h(t) + \epsilon h^*(t) \neq cons.$, $\quad h(t) \neq 0$, and to avoid the cases of pure rotation and pure translation we also suppose that $\dfrac{d}{dt}(\widehat{h}\widehat{A}) \neq 0$, $\dfrac{d}{dt}(\widehat{C}) \neq 0$.

Let us consider the following parametrized dual curve:

$\widehat{\alpha} : I \subset \mathbb{R} \to \mathbb{D}^4$ defined by $\widehat{\alpha}(t) = (\widehat{\alpha}_0(t), \widehat{\alpha}_1(t), \widehat{\alpha}_2(t), \widehat{\alpha}_3(t))$, for every $t \in I$. $\widehat{\alpha}(t) = \overrightarrow{\alpha}(t) + \epsilon \overrightarrow{\alpha}^*(t)$, where $\overrightarrow{\alpha}(t) = (\alpha_0(t), \alpha_1(t), \alpha_2(t), \alpha_3(t))$, $\overrightarrow{\alpha}^*(t) = (\alpha_0^*(t), \alpha_1^*(t), \alpha_2^*(t), \alpha_3^*(t))$ are curves in $\mathbb{R}^4$. We suppose that the curve $\widehat{\alpha}(t)$ is a differentiable regular curve of order r which does not pass through the origin.

Let us write the dual matrix $\widehat{B}$ as follows:

(18)
$$\widehat{B} = \begin{bmatrix} \widehat{\alpha_0}(t)^2 + \widehat{\alpha_1}(t)^2 - \widehat{\alpha_2}(t)^2 - \widehat{\alpha_3}(t)^2 & -2\widehat{\alpha_0}(t)\widehat{\alpha_3}(t) + 2\widehat{\alpha_1}(t)\widehat{\alpha_2}(t) & 2\widehat{\alpha_0}(t)\widehat{\alpha_2}(t) + 2\widehat{\alpha_1}(t)\widehat{\alpha_3}(t) \\ 2\widehat{\alpha_1}(t)\widehat{\alpha_2}(t) + 2\widehat{\alpha_3}(t)\widehat{\alpha_0}(t) & \widehat{\alpha_0}(t)^2 - \widehat{\alpha_1}(t)^2 + \widehat{\alpha_2}(t)^2 - \widehat{\alpha_3}(t)^2 & 2\widehat{\alpha_2}(t)\widehat{\alpha_3}(t) - 2\widehat{\alpha_1}(t)\widehat{\alpha_0}(t) \\ 2\widehat{\alpha_1}(t)\widehat{\alpha_3}(t) - 2\widehat{\alpha_2}(t)\widehat{\alpha_0}(t) & 2\widehat{\alpha_1}(t)\widehat{\alpha_0}(t) + 2\widehat{\alpha_2}(t)\widehat{\alpha_3}(t) & \widehat{\alpha_0}(t)^2 - \widehat{\alpha_1}(t)^2 - \widehat{\alpha_2}(t)^2 + \widehat{\alpha_3}(t)^2 \end{bmatrix}.$$

We can represent the matrix $\widehat{B}$ as

$$\widehat{B} = \widehat{h}.\widehat{A}, \tag{19}$$

where $\widehat{A} \in S\widehat{O}(3)$, $\widehat{B}$ is a homothetic matrix and Eq.16 determines a homothetic motion. Here,

$$\widehat{h} : I \subset \mathbb{R} \to \mathbb{D},$$
$$t \mapsto \widehat{h}(t) = \parallel \widehat{\alpha}(t) \parallel = (\widehat{\alpha}_0(t)^2 + \widehat{\alpha}_1(t)^2 + \widehat{\alpha}_2(t)^2 + \widehat{\alpha}_3(t)^2).$$

Theorem 5.1. Let $\bar{\bar{\widehat{B}}} \in \mathbb{D}^3$ determines a dual homothetic motion of a rigid body given by

$$\bar{\bar{\widehat{B}}} = \widehat{h}.\widehat{A} + \widehat{C}, \tag{20}$$

where $\widehat{h}$ is a homothetic scalar, $\widehat{A} \in S\widehat{O}(3)$ and $\widehat{C}$ is a 3×1 dual matrix. f_h defines a dual transformation between dual Euclidean and dual Lorentzian homothetic motion matrices in dual 3D-spaces.

$$f_h : \mathbb{D}^3 \to \mathbb{D}_1^3$$

$$\bar{\bar{\widehat{B}}} \mapsto f_h(\bar{\bar{\widehat{B}}}) = \widehat{h}.f(\widehat{A}) + \widehat{C}, \tag{21}$$

where f is the dual transformation given in Eq.5, $f(\widehat{A}) \in S\widehat{O}(2,1)$. Eq.21 determines a dual homothetic motion of a rigid body in dual 3D Lorentzian space.

We have denoted $\widehat{h}.\widehat{A}$ as a dual homothetic matrix $\widehat{B} \in \mathbb{D}^3$ in Eq. 18. Then, we now denote $\widehat{h}.f(\widehat{A})$ as $\widehat{B}_L$ since it is a dual homothetic matrix in $\mathbb{D}_1^3$.

Proof. We observe that

$$\begin{aligned} f_h^2(\bar{\bar{\widehat{B}}}) &= f_h(f_h(\bar{\bar{\widehat{B}}})) \\ &= f_h(\bar{\bar{\widehat{B}}}_L), \quad f^2 = id. \\ &= \bar{\bar{\widehat{B}}} \\ &\Longrightarrow f_h^2 = id. \end{aligned}$$

Thus, f_h is a dual transformation. □

We now investigate dual homothetic motions produced by Hamilton operators with the help of dual transformations. First, we use the dual orthogonal matrix representation of a dual quaternion in dual 4D space. Afterward, we obtain a dual Lorentzian homothetic motion from a dual Euclidean homothetic motion.

Let us consider the dual quaternion curve $\widehat{\alpha} : I \subset \mathbb{R} \to \mathbb{D}^4$ defined by $\widehat{\alpha}(t) = (\widehat{\alpha}_0(t), \widehat{\alpha}_1(t), \widehat{\alpha}_2(t), \widehat{\alpha}_3(t))$, for every $t \in I$. We suppose that the dual curve $\widehat{\alpha}(t)$ is a differentiable regular dual curve of order r. The operator $\widehat{Q}$, called the Hamilton operator corresponding to $\widehat{\alpha}(t)$ is determined by the following dual matrix:

$$\widehat{H}_Q = \begin{bmatrix} \widehat{\alpha}_0(t) & -\widehat{\alpha}_1(t) & -\widehat{\alpha}_2(t) & -\widehat{\alpha}_3(t) \\ \widehat{\alpha}_1(t) & \widehat{\alpha}_0(t) & -\widehat{\alpha}_3(t) & \widehat{\alpha}_2(t) \\ \widehat{\alpha}_2(t) & \widehat{\alpha}_3(t) & \widehat{\alpha}_0(t) & -\widehat{\alpha}_1(t) \\ \widehat{\alpha}_3(t) & -\widehat{\alpha}_2(t) & \widehat{\alpha}_1(t) & \widehat{\alpha}_0(t) \end{bmatrix},$$

where $\widehat{\alpha}_0(t) \neq 0$.

Let $\widehat{\alpha}(t)$ be a unit velocity dual curve. Since $\widehat{\alpha}(t)$ does not pass through the origin, we can write the dual matrix $\widehat{H}_Q$ as follows

$$\widehat{H}_Q = \widehat{h}. \begin{bmatrix} \frac{\widehat{\alpha}_0(t)}{\widehat{h}} & \frac{-\widehat{\alpha}_1(t)}{\widehat{h}} & -\frac{\widehat{\alpha}_2(t)}{\widehat{h}} & \frac{-\widehat{\alpha}_3(t)}{\widehat{h}} \\ \frac{\widehat{\alpha}_1(t)}{h} & \frac{\widehat{\alpha}_0(t)}{\widehat{h}} & \frac{-\widehat{\alpha}_3(t)}{\widehat{h}} & \frac{\widehat{\alpha}_2(t)}{\widehat{h}} \\ \frac{\widehat{\alpha}_2(t)}{\widehat{h}} & \frac{\widehat{\alpha}_3(t)}{\widehat{h}} & \frac{\widehat{\alpha}_0(t)}{\widehat{h}} & \frac{-\widehat{\alpha}_1(t)}{\widehat{h}} \\ \frac{\widehat{\alpha}_3(t)}{\widehat{h}} & \frac{-\widehat{\alpha}_2(t)}{\widehat{h}} & \frac{\widehat{\alpha}_1(t)}{\widehat{h}} & \frac{\widehat{\alpha}_0(t)}{\widehat{h}} \end{bmatrix} = \widehat{h}.\widehat{Q}, \tag{22}$$

where

$$\widehat{h} : I \subset \mathbb{R} \to \mathbb{D},$$
$$t \mapsto \widehat{h}(t) = \parallel \widehat{\alpha}(t) \parallel = \sqrt{(\widehat{\alpha}_0(t)^2 + \widehat{\alpha}_1(t)^2 + \widehat{\alpha}_2(t)^2 + \widehat{\alpha}_3(t)^2)}.$$

Here, $\widehat{Q} \in S\widehat{O}(4)$.

Hamiltonian dual motion is generated by the transformation

$$\widehat{Y} = \widehat{h}.\widehat{Q}\widehat{X} + \widehat{C} \tag{23}$$

with the dual matrix representation as

$$\begin{bmatrix} \widehat{Y} \\ 1 \end{bmatrix} = \begin{bmatrix} \widehat{h}.\widehat{Q} & \widehat{C} \\ 0 & 1 \end{bmatrix} . \begin{bmatrix} \widehat{X} \\ 1 \end{bmatrix}, \tag{24}$$

where $\widehat{h}$ is a homothetic scalar and $\widehat{Q} \in S\widehat{O}(4)$. The matrix $\widehat{H}_Q = \widehat{h}.\widehat{Q}$ is called a dual homothetic matrix and $\widehat{Y}$, $\widehat{X}$ and $\widehat{C}$ is a 4×1 dual matrix. $\widehat{Y}$ and $\widehat{X}$ correspond to the position vectors of the same point with respect to the rectangular coordinate systems of the moving space $\tilde{R}_o$ and the fixed space $\tilde{R}$, respectively.

Theorem 5.2. The derivation operator $\dot{\widehat{H}}_Q$ of dual homothetic matrix $\widehat{H}_Q = \widehat{h}\widehat{Q}$ is a dual orthogonal matrix.

Proof. $\dot{\widehat{H}}_Q.\dot{\widehat{H}}_Q^{\,T} = \dot{\widehat{H}}_Q^{\,T}.\dot{\widehat{H}}_Q = I_4$ and $det\dot{\widehat{H}}_Q = 1$. □

Theorem 5.3. In $\mathbb{D}^4$, the Hamiltonian dual motion that is represented by the dual matrix $\widehat{H}_Q$ is a regular motion and it is independent of $\widehat{h}$.

Proof. This dual motion is regular as $det\dot{\widehat{H}}_Q = 1$ and the value of $det\dot{\widehat{H}}_Q$ is independent of $\widehat{h}$. □

We now give the following theorem that presents the transition between dual Euclidean and dual Lorentzian homothetic motions in dual 4D-spaces.

Theorem 5.4. Let $\widehat{H}_Q \in \mathbb{D}^4$ determines a homothetic motion given by

$$\widehat{H}_Q = \widehat{h}.\widehat{Q} + \widehat{C}, \tag{25}$$

where $\widehat{h}$ is a homothetic scalar, $\widehat{Q} \in S\widehat{O}(4)$ and $\widehat{C}$ is a 4×1 dual matrix. f_h defines a dual transformation,

$$f_h : \mathbb{D}^4 \to \mathbb{D}^4_1$$

$$\widehat{H}_Q \mapsto f_h(\widehat{H}_Q) = \widehat{H}_{Q_L} = h.f(\widehat{Q}) + \widehat{C}, \tag{26}$$

where $f(\widehat{Q}) = \widehat{Q}_L \in S\widehat{O}(3,1)$. We can write the dual matrix $\widehat{H}_{Q_L}$ by using f_h dual transformation as follows:

$$\widehat{H}_{Q_L} = \widehat{h}. \begin{bmatrix} \frac{\widehat{\alpha}_0(t)^2 + \widehat{\alpha}_3(t)^2}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{-\widehat{\alpha}_0(t)\widehat{\alpha}_1(t) - \widehat{\alpha}_2(t)\widehat{\alpha}_3(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{-\widehat{\alpha}_0(t)\widehat{\alpha}_2(t) + \widehat{\alpha}_1(t)\widehat{\alpha}_3(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{-\widehat{\alpha}_3(t)}{\widehat{h}.\widehat{\alpha}_0(t)} \\ \frac{\widehat{\alpha}_0(t)\widehat{\alpha}_1(t) - \widehat{\alpha}_2(t)\widehat{\alpha}_3(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{\widehat{\alpha}_0(t)^2 + \widehat{\alpha}_2(t)^2}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{-\widehat{\alpha}_0(t)\widehat{\alpha}_3(t) - \widehat{\alpha}_1(t)\widehat{\alpha}_2(t)}{\widehat{h\alpha}_0(t)} & \frac{\widehat{\alpha}_2(t)}{\widehat{h}.\widehat{\alpha}_0(t)} \\ \frac{\widehat{\alpha}_0(t)\widehat{\alpha}_2(t) + \widehat{\alpha}_1(t)\widehat{\alpha}_3(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{\widehat{\alpha}_0(t)\widehat{\alpha}_3(t) - \widehat{\alpha}_1(t)\widehat{\alpha}_2(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{\widehat{\alpha}_0(t)^2 + \widehat{\alpha}_1(t)^2}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{-\widehat{\alpha}_1(t)}{\widehat{h}.\widehat{\alpha}_0(t)} \\ \frac{-\widehat{\alpha}_3(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{\widehat{\alpha}_2(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{-\widehat{\alpha}_1(t)}{\widehat{h}.\widehat{\alpha}_0(t)} & \frac{1}{\widehat{h}.\widehat{\alpha}_0(t)} \end{bmatrix}.$$

The dual matrix $\widehat{H}_{Q_L}$ represents the dual homothetic motion in dual 4-dimensional Lorentzian space.

Proof. We show that

$$\begin{aligned} f_h^2(\widehat{H}_Q) &= f_h(f_h(\widehat{H}_Q)) \\ &= f_h(\widehat{H}_{Q_L}), \quad f^2 = id. \\ &= \widehat{H}_Q \\ &\implies f_h^2 = id. \end{aligned}$$

Hence, f_h is a dual transformation. □

References

[1] O. P. Agrawal, *Hamilton operators and dual-number-quaternions in spatial kinematics*, Mech. and Mach. Theory **22** (1987), 569–575.

[2] S. Aslan, M. Bekar, and Y. Yaylı, *Ruled surfaces constructed by quaternions*, Journal of Geometry and Physics **161** (2021), 104048.

[3] M. Bekar and Y. Yaylı, *Kinematics of dual quaternion involution matrices*, SDU Journal of Science (E-Journal) **11** (2016), 121–132.

[4] E. D. Çetin and Y. Yaylı, *Homothetic motions and surfaces with H2-K=0 in Lorentz 3-Space*, International Journal of Geometry **10** (2021), 11–20.

[5] R. Dohi, Y. Maeda, M. Mori, and H. Yoshida, *A dual transformation between* $SO(n+1)\backslash\{(00-component)=0\}$ *and* $SO(n,1)$ *and its geometric applications*, Linear Algebra and its Applications **432** (2010), 770–776.

[6] H. H. Hacısalihoglu, *On The Rolling of one curve or surface upon another*, Mathematical Proceeding of the R. Irish Academy **71** (1971), 13–17.

[7] R. W. Hamilton, *On quaternions*, Proceedings of the Royal Irish Academy **3** (1947), 1–16.

[8] A. J. Hanson, *Visualizing Quaternions*, Elsevier, 2006.

[9] R. López, *Differential geometry of curves and surfaces in Lorentz-Minkowski space*, International Electronic Journal of Geometry **7** (2014), 44–107.

[10] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, 103, Academic Press, Inc., New York, 1983.

[11] J. P. Ward, *Quaternions and Cayley numbers*, Springer, Netherlands, 1997.

[12] A. Yavuz and M. Erdoğdu, *A different approach by system of differential equations for the characterization position vector of spacelike curves*, Punjab University Journal of Mathematics **53** (2021), 231–245.

[13] A. Yavuz and Y. Yaylı, *Ruled surfaces with constant slope ruling according to Darboux frame in Minkowski space*, International Journal of Analysis and Applications **18** (2020), 900–919.

[14] Y. Yaylı, *Hamilton Motions and Lie Groups*, Phd. Thesis, Gazi University Graduate School of Natural and Applied Sciences, 1988.

[15] Y. Yaylı, *Homothetic motions at* E^4. Mech. Mach. Theory **27** (1992), 303–305.

[16] Y. Yaylı, A. Çalışkan, and H. H. Uğurlu, *The E. Study maps of circles on dual hyperbolic and Lorentzian unit spheres* H_0^2 *and* S_1^2, Math. Proc. R. Ir. Acad. **102A** (2002), 37–47.

[17] G. Yüca, *Kinematics applications of dual transformations*, Journal of Geometry and Physics **163** (2021), 104139.

[18] G. Yüca and Y. Yaylı, *A dual transformation between* $\widehat{SO}(3)$ *and* $\widehat{SO}(2,1)$ *and its geometric applications*, Proc. Natl. Acad. Sci., India, Sect. A. Phys. Sci. **88** (2018), 267–273.

[19] G. Yüca and Y. Yaylı, *Dual transformations in Galilean Spaces*, International Electronic Journal of Geometry **13** (2020), 52–61.

[20] G. Yüca and Y. Yaylı, *Dual transformations and quaternions*, Mathematical Methods in the Applied Sciences **44** (2021), 10957–10971.

[21] G. Yüca and Y. Yaylı, *Homothetic motions and dual transformations*, Erciyes University Journal of Institue Of Science and Technology **37** (2021), 194–205.

Gülsüm YÜCA
Department of Mathematics, Aksaray University,
Aksaray 68100, Turkey.

E-mail: gulsumbicer@gmail.com

Yusuf YAYLI
Department of Mathematics, Ankara University,
Ankara 06100, Turkey.
E-mail: yayli@science.ankara.edu.tr