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POLLUTION DETECTION FOR THE SINGULAR LINEAR PARABOLIC EQUATION

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Abstract. In this work, we are concerned by the problem of identification of noisy terms which arise in singular problem as for remote sensing problems, and which are modeled by a linear singular parabolic equation. For the reason of missing some data that could be arisen when using the traditional sentinel method, the later will be changed by a new sentinel method for attaining the same purpose. Such new method is a particular least square-like method which permits one to distinguish between the missing terms and the pollution terms. In particular, a sentinel method will be given here in its more realistic setting for singular parabolic problems, where in this case, the observation and the control have their support in different open sets. The problem of finding a new sentinel is equivalent to finding singular optimality system of the least square control for the parabolic equation that we solve.

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1. Introduction

The detection of the noisy terms arise typically in singular parabolic problems as for remote sensing with active sensors or passive sensors as well. As it is well known, these problems generate heat spread, which can be modeled by a linear parabolic equation [1, 2]. The electromagnetic radiation that is reflected back from patterns of the Earth surface is measured by remote sensing tools. The measurements which consist of the evaluation of different wavelengths allow to distinguish the type of ocean or land covering; the water, the vegetation and the soil in general [3, 4]. The noisy terms for which we refer to pollution terms in this article are unknown and deterministic [5, 6]. They are found in the boundary

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of the domain for high wave numbers. The initial data for these problems are supposed unknown too, and we do not want to find them.

The main tool to detect missing terms is the classical least square method, but this method determines all the missing terms (the noisy terms and the missing initial data terms). We use here a version of sentinel method back to J.-L. Lions (1992), which permits to distinguish between all the missing terms. Here, we want to characterize the noisy terms independently of the missing initial data ones. It has been shown that, with the sentinel method, there is a gain of time calculations when interested in simulations. As we will see, the problem of finding a sentinel is equivalent to a null-controllability problem, see the general case in the book by J.-L. Lions [7], where the control and the observation have their support in the same open set. In our case, the sentinel theory is used in its general setting since we consider two different open sets for the control and the observation. This is a new issue for singular problems in general.

Optimal control of distributed parameter systems governed by a system of parabolic equations is of special importance for propagation processing problems which are generally expressed by the resolution of the heat equation [8, 9, 10, 11, 12]. The use of these equations may however leave a gap between the theoretical solutions and the experimental ones, then the use of optimal control allows to fill the gap, as it permits to optimize the distance between the two solutions, see [13, 14, 15, 16]. As an immediate application, the existence of a discriminating sentinel for a nonlinear singular parabolic equation can be discussed, as we will see later on. It should be observed that the backward problem appears under this form in the Lions sentinel theory [7]. The works by A. Omrane [17] as well as by Miloudi et al. in [18] and [19] are examples on parabolic equations for which in the later one the control with constraints problem was solved for the heat equation using a well adapted Carleman inequality, see [20].

From above perspective, we will prove in this work that the sentinel problem is equivalent to a null-controllability one. This is because that the general null-controllability problem for the heat equation is well understood, see [21, 22]. Indeed, with assuming the geometric control condition introduced by [22], one can establish an observation estimate which yields by the HUM method of Lions [21]. The geometric control condition is a microlocal making a like to bicharacteristic rays of the heat operator. Moreover, it is equivalent to exact controllability of the linear heat equation with stability that is considered according to small perturbations. From this point of view, we aim also to state a more general singular null-controllability problem in this article.

The paper is organized as follows. In Section 2, we give an application to the sentinel theory of Lions for nonlinear heat problems with incomplete data, and show the existence of a nontrivial sentinel for the heat equation, where the control and observation have their supports in two different open sets. In Section 3, we state and prove the singular null-controllability under the constraints which corresponds to the search of discriminating sentinels. The case of a discriminating sentinel is discussed in Section 4, whereas the final section presents the conclusions of this work.

2. Towards a sentinel problem

Let Ω be a bounded open subset of $\mathbb{R}^d$ with boundary Γ of class $\mathcal{C}^2$, where $d \in \mathbb{N}^*$. For $T > 0$, we set $Q = (0,T) \times \Omega$, $\Sigma = (0,T) \times \Gamma$, and we consider the following nonlinear heat problem in several dimensions:

$$\frac{\partial y}{\partial t} - \Delta y + f(y) \;=\; F \qquad \text{in } Q, \tag{1}$$

with the following initial data of incomplete information:

$$y(T) \;=\; y^0 + \tau \hat{y}^0 \qquad \text{in } \Omega, \tag{2}$$

where f being a $\mathcal{C}^1$ function, $y^0 \in L^2(\Omega)$ is a known function is and $\tau \hat{y}^0$ is unknown (see Lions [7] page 156). Without loss of generality, we may consider the above problem for all $\hat{y}^0$ such that $\left\|\hat{y}^0\right\|_{L^2} \leq 1$ and τ is small. In this regard, the noisy terms appear in a part of the domain as follows:

$$F \;=\; \xi_0 + \lambda \hat{\xi}_0 \qquad \text{in } Q, \tag{3}$$

where $\xi_0 \in L^2(Q)$ is given, and where $\lambda \hat{\xi}_0$ is not known too.

The goal here is to find a method to estimate the noise (missing) term $\lambda \hat{\xi}_0$. Actually, there are several methods can be used for attaining this purpose. The famous one is the least squares method. However with this method, the pollution and initial unknown terms $\tau \hat{y}^0$ and $\lambda \hat{\xi}_0$ are computed together, and we can not really separate them, see [7, 18]. Here, we use the sentinel method of Lions that can detect one parameter independently of the others. To have a chance to detect pollution, we observe the system in some open subset $\mathcal{O} \subset \Omega$ called observatory, during time T. We denote by y_{obs} this observation, which can be formulated by:

$$y_{obs} = m_0 + \sum_{i=1}^{N} \beta_i m_i, \tag{4}$$

where the functions $m_0, m_1, \cdots, m_N$ are known in $L^2(\mathcal{O} \times (0,T))$, but β_i are unknown real numbers in which they are assumed here "small". Actually, the terms β_i are called the interference (or noisy) terms as well. Without loss of generality, we can assume that the functions m_i are linearly independent for $1 \leq i \leq N$.

Remark 2.1. In the case of the heat equation, the observatory $\mathcal{O}$ can be chosen arbitrarily small, as well as the final time T.

We now introduce the notion of *sentinel* by following the definition in [17, 19]. In this definition, the observation and the control may have different support sets but are not disjoined. Indeed, one can observe somewhere in the domain, and can control in another part of the domain Ω. This *natural* definition leads to nontrivial controllability problems. In this regard, we let h_0 be a given function on $(0,T)\times\mathcal{O}$ such that:

$$h_0 \geq 0, \qquad \int_0^T \int_{\mathcal{O}} h_0 \, dxdt = 1. \tag{5}$$

Besides, let ω be an open and non empty subset of Ω. For a control function $u \in L^2((0,T)\times\omega)$, we introduce the following functional equation:

$$\mathcal{S}(\lambda,\tau) = \int_0^T \int_{\mathcal{O}} h_0 \, y(t,x;\lambda,\tau)\, dxdt + \int_0^T \int_{\omega} u \, y(t,x;\lambda,\tau)\, dxdt. \tag{6}$$

We shall say that S defines a sentinel (for the system (1-3) and (5)) if there exists u such that the pair (u,S) satisfies to the following two conditions:

- The sentinel S is insensitive at first-order with respect to the missing terms $\tau\hat{y}^0$, which means:

$$\frac{\partial \mathcal{S}}{\partial \tau}(0,0) = 0, \tag{7}$$

- The control u is of minimal norm in $L^2((0,T)\times\omega)$ in the sense:

$$\|u\|_{L^2((0,T)\times\omega)} \quad = \quad \inf_{v\in L^2((0,T)\times\omega)} \|v\|. \tag{8}$$

Remark 2.2. The classical point of view of Lions lies on h_o and u, having their support in the same open set of observation $\mathcal{O}=\omega$. In this case, the question of existence of a sentinel such that condition (7) holds is evident. Indeed, $h_o = -u$ is a solution, and the only question is the calculus of the optimal control (8).

The point of view considered here is a sentinel notion defined by the function h_o, an observation y_{obs} and a control u, but with h_o having its support in $\mathcal{O}$ and u of support in ω with $\omega \neq \mathcal{O}$. In this case, the existence of a sentinel is not guaranteed. However, in order to deal with the adjoint state or the controllability problem, we denote by $y_\tau = \frac{\partial y}{\partial \tau}(0,0)$, for $\lambda=\tau=0$. Then y_τ satisfies to the following system:

$$\left| \begin{array}{rcll} \Xi_{a_0} y_\tau & = & 0 & \text{in} \quad Q, \\ y_\tau & = & 0 & \text{on} \quad \Sigma, \\ y_\tau(0) & = & \hat{y}^T & \text{in} \quad \Omega, \end{array} \right. \tag{9}$$

where Ξ_{a_0} given by:

$$\Xi_{a_0} = \frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + a_0 \, I, \tag{10}$$

and $\Xi_{a_0}^*$ given by:

$$\Xi_{a_0}^* = -\frac{\partial}{\partial t} - \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2} + a_0 \, I, \tag{11}$$

which represents the d'Alembertian with potential:

$$a_0 := f'(y_o) \in L^\infty(Q), \tag{12}$$

where a_0 is a real-valued function and $y_o = y(t,x;0,0)$. It is well known that these linear problems admits each one a unique solution in the space $\mathcal{C}([0,T], H_0^1(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega))$. We immediately deduce that (7) is equivalent to the following equation:

$$\int_0^T \int_{\mathcal{O}} h_0\, y_\tau(t,x;\lambda,\tau)\, dxdt + \int_0^T \int_\omega w\, y_\tau(t,x;\lambda,\tau)\, dxdt = 0. \tag{13}$$

Proposition 2.1. *Let q be the solution to the ill-posed backward problem:*

$$\left\{ \begin{array}{rcll} \Xi_{a_0}^* q &=& h_o\chi_{\mathcal{O}} + w\chi_\omega & in \quad Q, \\ q &=& 0 & on \quad \Sigma, \\ q(0) &=& 0 & in \quad \Omega. \end{array} \right. \tag{14}$$

Then, the existence of a sentinel for (1-3), insensitive to the missing data (i.e. such that (13) hold), is equivalent to the null-controllability problem (14) together with

$$q(T) = 0 \quad in \ \Omega. \tag{15}$$

Proof. To prove this proposition, we first multiply (14) by y_τ, and then integrate the result by parts to obtain:

$$\begin{aligned} \int_Q \Xi_{a_0} q\, y_\tau\, dxdt = \int_Q q\Xi_{a_0} y_\tau\, dxdt + \int_\Omega q(T) y_\tau(T)\, dx - \int_\Omega q(0) y_\tau(0)\, dx \\ + \int_\Sigma \frac{\partial y_\tau}{\partial \nu} q\, d\sigma - \int_\Sigma \frac{\partial q}{\partial \nu} y_\tau\, d\sigma = \int_Q \Big(h_o\chi_{\mathcal{O}} + w\chi_\omega \Big) y_{\tau_0}\, dxdt. \end{aligned}$$

Consequently, we have:

$$\int_\Omega q(T) y_\tau(T)\, dx \;=\; \int_Q \Big(h_o\chi_{\mathcal{O}} + w\chi_\omega \Big) y_{\tau_0}\, dxdt.$$

But, y_{τ_0} is solution to system (9). Thus $q(T) = 0$ in Ω, which satisfies finally (15). The converse of this result is obvious. □

3. Null-controllability

In this part, we consider the singular parabolic system of the form:

$$\left\{ \begin{array}{rcll} \Xi_{a_0}^* q &=& v & \text{in} \quad Q, \\ q &=& 0 & \text{on} \quad \Sigma, \\ q(0) &=& q_0 & \text{in} \quad \Omega, \end{array} \right. \tag{16}$$

where the d'Alembertian $\Xi_{a_0}^*$ is given by (11) with potential a_0 given in (12). It is well known that given v ∈subspace∈ $L^1([0,T], L^2(\Omega))$ and $q_0 \in H_0^1(\Omega)$. In fact, problem (16) admits the following unique solution:

$$q \in \mathcal{C}([0,T], H_0^1(\Omega)) \cap \mathcal{C}^1([0,T], L^2(\Omega)).$$

Now, we state the problem of exact controllability for solutions of system (16). Let ω be an open subset of Ω. Denote by $(0,T) \times \omega$ the interior cylinder and χ_ω its characteristic function. Given $q_0 \in H_0^1(\Omega)$, the goal is to find a source

v in $L^2((0,T) \times \omega)$ such that the unique solution q of system (16) satisfies the following condition:

$$q(T) = 0 \qquad \text{in} \quad \Omega. \tag{17}$$

The inverse problem is to determine the conductivity distribution in Ω from boundary measurements.

4. The case of a discriminating sentinel

It is worth noticing that an application to the above result on controllability with constraints on the control can be analyzed. Such application is the sentinel theory of Lions for hyperbolic problems with missing data .

Definition 4.1. The sentinel S is said to be a discriminating sentinel for system (1-4) and (5), if there exists w such that the pair (w, S) satisfies the conditions (7-8), and if S is insensitive to interference terms $\beta_i m_i$, i.e.,

$$\int_0^T \int_{\mathcal{O}} h_0\, m_i\, dxdt + \int_0^T \int_\omega w\, m_i\, dxdt \quad = \quad 0, \quad 1 \le i \le N. \tag{18}$$

Let $\mathcal{K}$ be the vector subspace generated in $L^2((0,T)\times\omega)$ by the N independent functions $m_i\chi_\omega$ such that $1 \le i \le N$. It is easy to see that there exists a unique $k_0 \in \mathcal{K}$ such that:

$$\int_0^T \int_{\mathcal{O}} h_0\, m_i\, dxdt + \int_0^T \int_\omega k_0\, m_i\, dxdt = 0, \quad 1 \le i \le N.$$

It should be noted here that the vector space $\mathcal{K}$ plays the same role as in the previous section. In addition, condition (18) is equivalent to the following form:

$$w - k_0 = v \in \mathcal{K}^\perp. \tag{19}$$

To sum up, the problem consisting in obtaining the control item w such that the pair (w, S) satisfies (7) with (18) (the same as (7) with (19)). This is equivalent to find the control item v such that the pair (v, q) satisfies the following system:

$$\left\{ \begin{array}{ccccc} & v \in \mathcal{K}^\perp, & & & \\ \Xi^*_{a_0}\, q & = & h + v\chi_\omega & \text{in} & Q, \\ q & = & 0 & \text{on} & \Sigma, \\ q(0) & = & 0 & \text{in} & \Omega, \end{array} \right. \tag{20}$$

and

$$q(T) = 0 \ \text{ in } \Omega. \tag{21}$$

where $h = h_0\chi_{\mathcal{O}} + k_0\chi_\omega$. Hence, we are considering the original problem with the controllability under constraints.

4.1. Null-controllability under constraints. It should be recalled here that $\mathcal{K}$ is the finite dimensional linear subspace of $L^2((0,T)\times W)$ defining the under taken constraints. Thus, we need to consider the following hypothesis:

$(A1)$ The couple (ω, T) satisfies the geometric control condition.
$(A2)$ The only element $k \in \mathcal{K}$ satisfying $\Xi^*_{a_0} k = 0$ in Q is the trivial element $k \equiv 0$.

In view of the previous assumptions, we introduce in what follow another result.

Theorem 4.2. *Under assumptions* $(A1)$ *and* $(A2)$ *and for every* $q_T \in H^1_0(\Omega)$*, there exists a constrained control function* $v \in L^2((0,T)\times\omega)$ *such that the state solution* q *to problem*

$$\left\{\begin{array}{rcll} \Xi^*_{a_0} q &=& \chi_\omega v & in \quad Q, \\ q &=& 0 & on \quad \Sigma, \\ q(0) &=& q_0 & in \quad \Omega, \end{array}\right. \tag{22}$$

satisfies condition (17).

Proof. For the proof of this result, a suitable version of the HUM method can be simply used, as what it was exactly carried out in reference [17]. □

4.2. The discriminating sentinel. In this section, we aim to state and prove a new theoretical result that deals with the existence of a discriminating sentinel. The two assumptions $(A1)$ and $(A2)$ declared above will be very useful for attaining our goal.

Proposition 4.3. *Under assumptions* $(A1)$ *and* $(A2)$*, there exists a unique and nontrivial discriminating sentinel for problem (1-4), insensitive to the missing data* $\tau_0 \hat{y}^0$ *and* $\tau \hat{y}^1$ *as well as to the noise terms* $\beta_i\, m_i$*,* $1 \le i \le N$*.*

Proof. From Theorem 4.2, the existence of a sentinel, insensitive to the missing data (i.e. (7) and (19) hold) is guaranteed, as long as we prove that there exists $v \in \mathcal{K}^\perp$ such that condition (15) holds as well. To this aim, we observe that system (20) is equivalent to system (16) under the same constraints. Indeed, if we solve the following system:

$$\left\{\begin{array}{rcll} \Xi^*_{a_0} z &=& h & \text{in} \quad Q, \\ z &=& 0 & \text{on} \quad \Sigma, \\ z(0) &=& 0 & \text{in} \quad \Omega \end{array}\right.$$

and set $\bar{q} = q - z$, then $\bar{q}(0) = q(0)$. Aa a result, controlling the solution q or $\bar{q}$ of the system:

$$\left\{\begin{array}{rcll} & v \in \mathcal{K}^\perp, & & \\ \Xi^*_{a_0}\, \bar{q} &=& v\chi_\omega & \text{in} \quad Q, \\ \bar{q} &=& 0 & \text{on} \quad \Sigma, \\ \bar{q}(0) &=& 0 & \text{in} \quad \Omega \end{array}\right. \tag{23}$$

is the same. Moreover, system (23) with constraints on the control is null-controllable thanks to Theorem 4.2. □

5. Conclusion

In this work, a sentinel method has been provided in its more realistic setting for singular parabolic problems, where in this case, the observation and the control have their support in different open sets. It has been shown that the problem of finding a new sentinel is equivalent to finding singular optimality system of the least square control for a parabolic equation.

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