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SOME NEW RESULTS ON POWER CORDIAL LABELING†

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Abstract. A power cordial labeling of a graph $G = (V(G), E(G))$ is a bijection $f : V(G) \to \{1, 2, ..., |V(G)|\}$ such that an edge $e = uv$ is assigned the label 1 if $f(u) = (f(v))^n$ or $f(v) = (f(u))^n$, For some $n \in \mathbb{N} \cup \{0\}$ and the label 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. In this paper, we investigate power cordial labeling for helm graph, flower graph, gear graph, fan graph and jewel graph as well as larger graphs obtained from star and bistar using graph operations.

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Key words and phrases : Graph labeling, graph operation, cordial labeling, power cordial labeling, power cordial graph.

1. Introduction

We begin with simple, finite and undirected graph $G = (V(G), E(G))$. For all standard terminologies and notations we refer to Gross and Yellen [8]. We provide a brief summary and definitions that are relevant to the current investigations.

Definition 1.1. A *graph labeling* is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (an edge labeling).

In 1967, Rosa [2] introduced β - valuation of a graph. Golamb [15] subsequently called such labeling as a graceful labeling.

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Harmonious labeling has been introduced by Graham and Sloane [10] in 1980 during their study on modular versions of additive bases problem stemming from error correcting codes.

The famous Ringle-Kotzing [5] conjecture, 'All trees are graceful' generates great interest for the study of graph labeling. Many researchers have given varies graph labeling schemes.

For a dynamic survey of various graph labelings along with bibliographic references we refer to Gallian [7].

In 1987, Cahit [6] introduced cordial labeling as a weaker version of graceful labeling and harmonious labeling. Variants of cordial labeling like H-cordial labeling, A-cordial labeling, prime cordial labeling, product cordial labeling were also introduced.

Motiveted by this, Barasara and Thakkar [3] have introduced power cordial labeling of graph as a variant of cordial labeling and defined it as follows:

Definition 1.2. For a graph $G = (V(G), E(G))$, the vertex labeling function is defined as a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ and induced edge labeling function $f^* : E(G) \to \{0, 1\}$ is given by

$$f^*(e = uv) = \begin{cases} 1, & \text{if } f(u) = (f(v))^n \text{ or } f(v) = (f(u))^n, \text{ for some } n \in \mathbb{N} \cup \{0\}; \\ 0, & \text{otherwise.} \end{cases}$$

The number of edges labeled with 0 and 1 is denoted by $e_f(0)$ and $e_f(1)$ respectively. f is called *power cordial labeling* of graph G if $|e_f(0) - e_f(1)| \leq 1$. The graph that admits a power cordial labeling is called a *power cordial graph*.

Barasara and Thakkar [3] have discussed power cordial labeling for path, cycle, complete graph, wheel, tadpole, $K_{2,n}$, $K_{3,n}$, star and bistar.

Labeled graph have applications in many different field such as coding theory, communication networks, determination of optimal circuit layouts etc. A study on wide verity of applications of labeled graph have been exploited by Bloom and Golomb [4].

There are three types of problems that can be considered in this area.

(1) Construct new families of power cordial graphs by finding suitable labeling.
(2) How power cordiality is affected under various graph operations.
(3) Given a graph theoretic property P, characterize the class of graphs with property P that are power cordial.

This paper is focused on problems of first two types.

In this paper, we investigate power cordial labeling for helm graph, flower graph, gear graph, fan graph and jewel graph as well as larger graphs obtained from star and bistar using graph operations.

2. Power cordial labeling for some standard graphs

Definition 2.1. Let G_1 and G_2 be two graphs with no vertex in common, We define the *join* of G_1 and G_2 denoted by G_1+G_2 to be the graph with vertex set $V(G_1+G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1+G_2) = E(G_1) \cup E(G_2) \cup J$, where $J = \{x_1x_2 : x_1 \in V(G_1), x_2 \in V(G_2)\}$.

Definition 2.2. The *wheel* W_n is defined to be the join $K_1 + C_n$, the vertex corresponding to K_1 is known as an apex vertex and vertices corresponding to C_n are known as rim vertices.

Definition 2.3. The *helm* H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.

Theorem 2.4. *The helm H_n is a power cordial graph for $n \leq 9$ and $n = 13$ and not a power cordial graph for $n = 10, 11, 12$ and $n \geq 14$.*

Proof. Let v be the apex vertex and v_i, for $1 \leq i \leq n$ be the rim vertices of wheel W_n. To obtain helm H_n, join vertices u_i to the vertices v_i, for $1 \leq i \leq n$ of wheel W_n. Then $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$.

We define a bijection $f : V(H_n) \to \{1, 2, \ldots, 2n+1\}$ by following five cases.

Case 1: For $n = 3, 4$.

$$\begin{aligned}
&f(v) = 1, \\
&f(v_i) = 2i; && \text{for } 1 \leq i \leq n, \\
&f(u_i) = 2i+1; && \text{for } 1 \leq i \leq n.
\end{aligned}$$

In view of above defined labeling pattern, we have $e_f(0) = 5$ and $e_f(1) = 4$ for $n = 3$ and $e_f(0) = 6$ and $e_f(1) = 6$ for $n = 4$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 2: For $n = 5, 6$ and 7.

$$\begin{aligned}
&f(v) = 1, \\
&f(v_i) = i+1; && \text{for } 1 \leq i \leq 2, \\
&f(u_i) = (i+1)^2; && \text{for } 1 \leq i \leq 2, \\
&f(v_n) = 2^3.
\end{aligned}$$

Now assign the labels to remaining vertices arbitrarily.

In view of above defined labeling pattern, we have $e_f(0) = 7$ and $e_f(1) = 8$ for $n = 5$, $e_f(0) = 9$ and $e_f(1) = 9$ for $n = 6$ and $e_f(0) = 11$ and $e_f(1) = 10$ for $n = 7$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 3: For $n = 8, 9$.

$$\begin{aligned}
&f(v) = 1, \\
&f(v_i) = 2^i; && \text{for } 1 \leq i \leq 2, \\
&f(v_{i+2}) = 3^i; && \text{for } 1 \leq i \leq 2, \\
&f(u_i) = 2^{i+2}; && \text{for } 1 \leq i \leq 2.
\end{aligned}$$

Now assign the labels to remaining vertices arbitrarily.

In view of above defined labeling pattern, we have $e_f(0) = 12$ and $e_f(1) = 12$ for $n = 8$ and $e_f(0) = 14$ and $e_f(1) = 13$ for $n = 9$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 4: For $n = 13$.

$$\begin{aligned}
&f(v) = 1, && \\
&f(v_i) = 2^i; && \text{for } 1 \leq i \leq 2, \\
&f(v_{i+2}) = 3^i; && \text{for } 1 \leq i \leq 2, \\
&f(v_{i+4}) = 5^i; && \text{for } 1 \leq i \leq 2, \\
&f(u_i) = 2^{i+2}; && \text{for } 1 \leq i \leq 2, \\
&f(u_3) = 2n + 1. &&
\end{aligned}$$

Now assign the labels to remaining vertices arbitrarily.

In view of above defined labeling pattern, we have $e_f(0) = 20$ and $e_f(1) = 19$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Case 5: For $n = 10, 11, 12$ and $n \geq 14$.

In order to satisfy the edge condition for power cordial labeling of graph it is essential to label at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges with label 0 as well as at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges with label 1 out of total $3n$ edges. All the possible assignment of vertex labels will give rise at most $\left\lfloor \frac{3n}{2} \right\rfloor - 1$ edges with label 1 and at least $\left\lfloor \frac{3n}{2} \right\rfloor + 1$ edges with label 0 out of total $3n$ edges. Therefore, $|e_f(0) - e_f(1)| \geq 2$. Thus, the helm H_n is not a power cordial graph for $n = 10, 11, 12$ and $n \geq 14$.

Hence, the helm H_n is a power cordial graph for $n \leq 9$ and $n = 13$ and not a power cordial graph for $n = 10, 11, 12$ and $n \geq 14$. $\square$

Example 2.5. The helm H_8 and its power cordial labeling is shown in Fig 1.

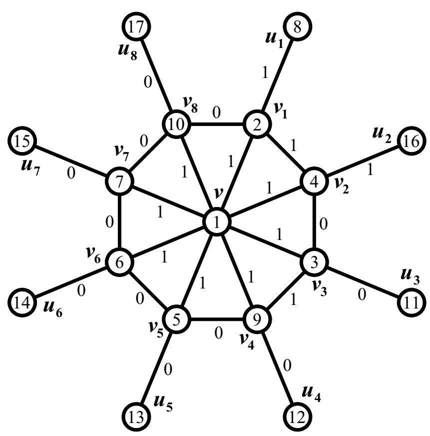

Fig 1: The helm H_8 and its power cordial labeling.

Definition 2.6. The *flower* Fl_n is the graph obtained from a helm H_n by joining each pendant vertex to the apex of the helm.

Theorem 2.7. *The flower Fl_n is a power cordial graph.*

Proof. Let v be the apex vertex and v_i, for $1 \leq i \leq n$ be the rim vertices of wheel W_n. To obtain helm H_n, join vertices u_i to the vertices v_i, for $1 \leq i \leq n$ of wheel W_n. To obtain flower Fl_n from helm H_n, join the apex vertex v to the vertices u_i, for $1 \leq i \leq n$. Then $|V(Fl_n)| = 2n + 1$ and $|E(Fl_n)| = 4n$.

We define a bijection $f : V(Fl_n) \to \{1, 2, \ldots, 2n + 1\}$ as follows:

$$\begin{aligned}
&f(v) = 1, \\
&f(v_1) = 2, \\
&f(v_i) = 2i + 1; \quad \text{for } 2 \leq i \leq n, \\
&f(u_1) = 3, \\
&f(u_i) = 2i; \qquad\quad \text{for } 2 \leq i \leq n.
\end{aligned}$$

In view of above defined labeling pattern, we have $e_f(0) = 2n$ and $e_f(1) = 2n$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 0$.

Hence, the flower Fl_n is a power cordial graph. □

Example 2.8. The flower Fl_6 and its power cordial labeling is shown in Fig 2.

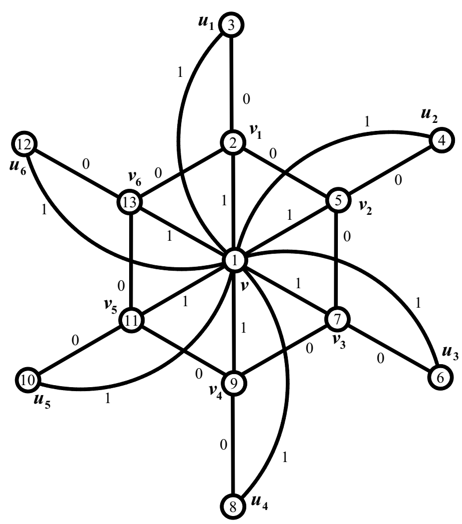

Fig 2: The flower Fl_6 and its power cordial labeling.

Definition 2.9. Let $e = uv$ be an edge of the graph G and w is not a vertex of G. The edge e is called *subdivided* when it is replaced by edges $e' = uw$ and $e'' = wv$.

Definition 2.10. The *gear graph* G_n is the graph obtained from the wheel W_n by subdividing each of its rim edge.

Theorem 2.11. *The gear graph G_n is a power cordial graph for $n \leq 9$ and $n = 13$ and not a power cordial graph for $n = 10, 11, 12$ and $n \geq 14$.*

Proof. Let v be the apex vertex and v_i, for $1 \leq i \leq n$ be the rim vertices of wheel W_n. To obtain gear graph G_n from wheel, subdivide each rim edge e_i of wheel by a new vertex u_i, for $1 \leq i \leq n$ respectively. Then $|V(G_n)| = 2n + 1$ and $|E(G_n)| = 3n$.

We define a bijection $f : V(G_n) \to \{1, 2, \ldots, 2n + 1\}$ by following six cases.

Case 1: For $n = 3$.

$$\begin{aligned}
&f(v) = 1,\\
&f(v_i) = i + 1; \quad \text{for } 1 \leq i \leq 2,\\
&f(v_3) = 7;\\
&f(u_i) = i + 3; \quad \text{for } 1 \leq i \leq 3.
\end{aligned}$$

In view of above defined labeling pattern, we have $e_f(0) = 5$ and $e_f(1) = 4$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Case 2: For $n = 4, 5$.

$$\begin{aligned}
&f(v) = 1,\\
&f(v_i) = i + 1; && \text{for } 1 \leq i \leq 2,\\
&f(v_i) = i + 2; && \text{for } 3 \leq i \leq n,\\
&f(u_i) = (i + 1)^2; && \text{for } 1 \leq i \leq 2.
\end{aligned}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0) = 6$ and $e_f(1) = 6$ for $n = 4$ and $e_f(0) = 8$ and $e_f(1) = 7$ for $n = 5$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 3: For $n = 6, 7$.

$$\begin{aligned}
&f(v) = 1,\\
&f(v_i) = i + 1; && \text{for } 1 \leq i \leq 2,\\
&f(v_i) = i + 7; && \text{for } 3 \leq i \leq n,\\
&f(u_i) = (i + 1)^2; && \text{for } 1 \leq i \leq 2,\\
&f(u_n) = 2^3.
\end{aligned}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0) = 9$ and $e_f(1) = 9$ for $n = 6$ and $e_f(0) = 11$ and $e_f(1) = 10$ for $n = 7$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 4: For $n = 8, 9$.

$$\begin{aligned}
&f(v) = 1,\\
&f(v_1) = 2,\\
&f(v_2) = 2^4,\\
&f(v_3) = 3,\\
&f(v_i) = i + 6; && \text{for } 4 \leq i \leq n,\\
&f(u_i) = (i + 1)^2; && \text{for } 1 \leq i \leq 2,\\
&f(u_n) = 2^3.
\end{aligned}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0) = 12$ and $e_f(1) = 12$ for $n = 8$ and $e_f(0) = 14$ and $e_f(1) = 13$ for $n = 9$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 5: For $n = 13$.

$$\begin{aligned}
&f(v) = 1,\\
&f(v_1) = 2,\\
&f(v_2) = 2^4,\\
&f(v_3) = 3,\\
&f(v_4) = 5,\\
&f(v_i) = i + 5; && \text{for } 5 \leq i \leq 10,\\
&f(v_i) = i + 6; && \text{for } 11 \leq i \leq n,\\
&f(u_i) = (i + 1)^2; && \text{for } 1 \leq i \leq 2,\\
&f(u_3) = 3^3,\\
&f(u_4) = 5^2,\\
&f(u_n) = 2^3.
\end{aligned}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0) = 20$ and $e_f(1) = 19$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Case 6: For $n = 10, 11, 12$ and $n \geq 14$.

In order to satisfy the edge condition for power cordial labeling of graph it is essential to label at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges with label 0 as well as at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges with label 1 out of total $3n$ edges. All the possible assignment of vertex labels will give rise at most $\left\lfloor \frac{3n}{2} \right\rfloor - 1$ edges with label 1 and at least $\left\lfloor \frac{3n}{2} \right\rfloor + 1$ edges with label 0 out of total $3n$ edges. Therefore, $|e_f(0) - e_f(1)| \geq 2$. Thus, the gear graph G_n is not a power cordial graph for $n = 10, 11, 12$ and $n \geq 14$.

Hence, the gear graph G_n is a power cordial graph for $n \leq 9$ and $n = 13$ and not a power cordial graph for $n = 10, 11, 12$ and $n \geq 14$. □

Example 2.12. The gear graph G_7 and its power cordial labeling is shown in Fig 3.

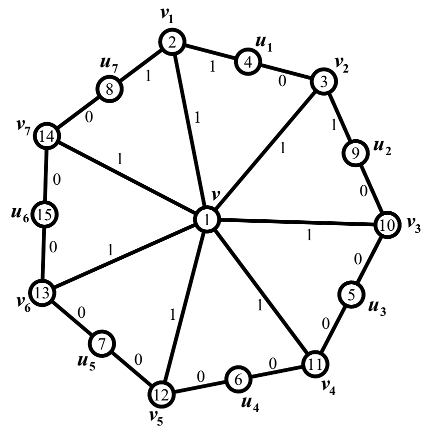

Fig 3: The gear graph G_7 and its power cordial labeling.

Definition 2.13. The *fan* F_n is defined to be the join $P_n + K_1$, the vertex corresponding to K_1 is known as an apex vertex.

Theorem 2.14. *The fan* F_n *is a power cordial graph.*

Proof. Let v_i, for $1 \leq i \leq n$ be the vertices of path P_n and v be the vertex of complete graph K_1. To obtain fan $F_n = P_n + K_1$, join vertex v to the vertices v_i, for $1 \leq i \leq n$. Then $|V(F_n)| = n + 1$ and $|E(F_n)| = 2n - 1$.

We define a bijection $f : V(F_n) \to \{1, 2, \ldots, n + 1\}$ as follows:

$$f(v) = 1,$$
$$f(v_i) = i + 1; \quad \text{for } 1 \leq i \leq n.$$

In view of above defined labeling pattern, we have $e_f(0) = n-1$ and $e_f(1) = n$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Hence, the fan F_n is a power cordial graph. □

Example 2.15. The fan F_5 and its power cordial labeling is shown in Fig 4.

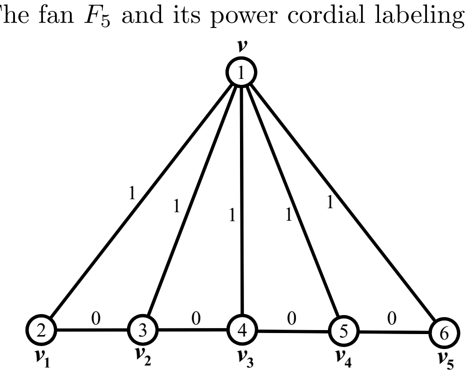

Fig 4: The fan F_5 and its power cordial labeling.

Definition 2.16. The *jewel graph* J_n is the graph with vertex set $\{u, x, v, y, u_i : 1 \leq i \leq n\}$ and edge set $\{ux, vx, uy, vy, xy, uu_i, vu_i : 1 \leq i \leq n\}$.

Theorem 2.17. *The jewel graph J_n is a power cordial graph.*

Proof. Let J_n be the jewel graph with vertex set $V(J_n) = \{u, x, v, y, u_i : 1 \leq i \leq n\}$ and edge set $E(J_n) = \{ux, vx, uy, vy, xy, uu_i, vu_i : 1 \leq i \leq n\}$. Then $|V(J_n)| = n + 4$ and $|E(J_n)| = 2n + 5$.

We define a bijection $f : V(J_n) \to \{1, 2, \ldots, n+4\}$ as follows:

$$\begin{aligned} &f(u) = 1, \\ &f(x) = 2, \\ &f(y) = 3, \\ &f(v) = p; \quad \text{where } p = \text{largest prime number} \leq n + 4, \end{aligned}$$

Now assign the labels to remaining vertices arbitrarily.

In view of above defined labeling pattern, we have $e_f(0) = n + 3$ and $e_f(1) = n + 2$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Hence, the jewel graph J_n is a power cordial graph. □

Example 2.18. The jewel graph J_4 and its power cordial labeling is shown in Fig 5.

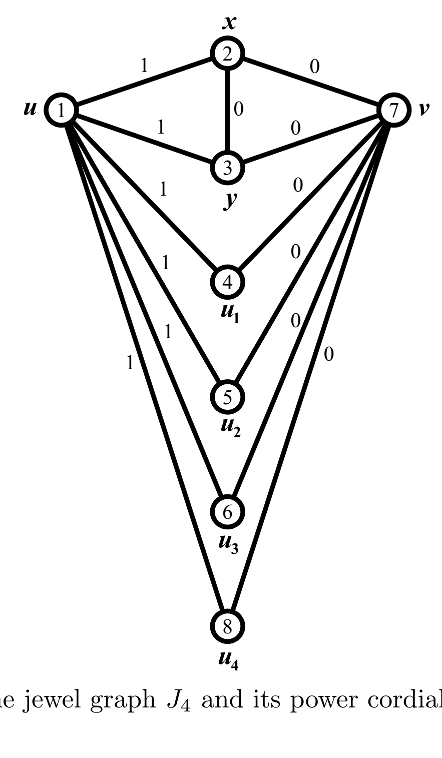

Fig 5: The jewel graph J_4 and its power cordial labeling.

3. Power cordial labeling for graphs obtained using graph operations

Definition 3.1. A complete bipartite graph $K_{1,n}$ is called *star*. A vertex corresponding to partition having one vertex is known as an apex vertex.

Definition 3.2. *Subdivision* of the graph G is obtained from G by subdividing each of its edge. It is denoted by $S(G)$.

Theorem 3.3. *$S(K_{1,n})$ is a power cordial graph.*

Proof. Let $K_{1,n}$ be the star with apex vertex v and other vertices $u_1, u_2, ..., u_n$. Subdivide each edge vu_i by vertex v_i for $i = 1, 2, ..., n$ to obtain graph $S(K_{1,n})$. Then $|V(S(K_{1,n}))| = 2n + 1$ and $|E(S(K_{1,n}))| = 2n$.

We define a bijection $f : V(S(K_{1,n})) \to \{1, 2, \ldots, 2n + 1\}$ as follows:

$$
\begin{aligned}
&f(v) = 1, \\
&f(v_i) = 2i; && \text{for } 1 \leq i \leq n, \\
&f(u_i) = 2i + 1; && \text{for } 1 \leq i \leq n.
\end{aligned}
$$

In view of above defined labeling pattern, we have $e_f(0) = n$ and $e_f(1) = n$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 0$.

Hence, $S(K_{1,n})$ is a power cordial graph. □

Example 3.4. The graph $S(K_{1,8})$ and its power cordial labeling is shown in Fig 6.

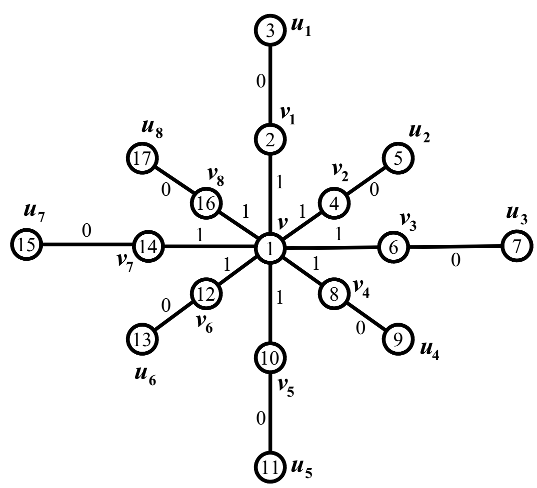

Fig 6: The graph $S(K_{1,8})$ and its power cordial labeling.

Theorem 3.5. *$2K_{1,n}$ is a power cordial graph.*

Proof. Let u be the apex vertex and $u_1, u_2, ..., u_n$ be the other vertices of first star $K_{1,n}$ and v be the apex vertex and $v_1, v_2, ..., v_n$ be the other vertices of second star $K_{1,n}$. Then $|V(2K_{1,n})| = 2n + 2$ and $|E(2K_{1,n})| = 2n$.

We define a bijection $f : V(2K_{1,n}) \to \{1, 2, \ldots, 2n + 2\}$ as follows:

$$f(u) = 1,$$
$$f(u_i) = 2i + 2; \quad \text{for } 1 \leq i \leq n,$$
$$f(v) = 2,$$
$$f(v_i) = 2i + 1; \quad \text{for } 1 \leq i \leq n.$$

In view of above defined labeling pattern, we have $e_f(0) = n$ and $e_f(1) = n$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 0$.

Hence, $2K_{1,n}$ is a power cordial graph. □

Example 3.6. The graph $2K_{1,6}$ and its power cordial labeling is shown in Fig 7.

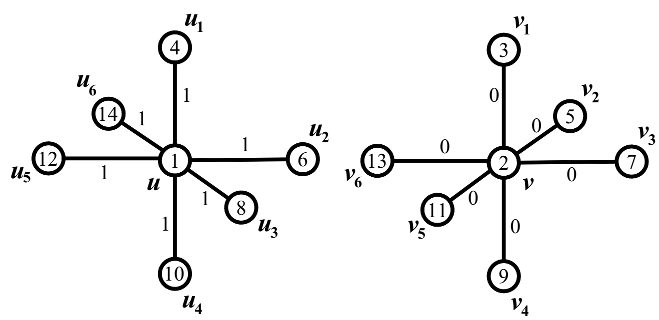

Fig 7: The graph $2K_{1,6}$ and its power cordial labeling.

Definition 3.7. For a graph G, the *splitting graph* of G, is obtained from G by adding a new vertex u for each vertex v of G so that u is adjacent to every vertex that is adjacent to v. It is denoted by $S'(G)$.

Theorem 3.8. *$S'(K_{1,n})$ is a power cordial graph for $n \leq 5$ and $n = 7$ and not a power cordial graph for $n = 6$ and $n \geq 8$.*

Proof. Let $v_1, v_2, ..., v_n$ be the pendant vertices and v be the apex vertex of $K_{1,n}$ and $u, u_1, u_2, ..., u_n$ are added vertices corresponding to $v, v_1, v_2, ..., v_n$ respectively to obtain $S'(K_{1,n})$. Then $|V(S'(K_{1,n}))| = 2n+2$ and $|E(S'(K_{1,n}))| = 3n$.

We define a bijection $f : V(S'(K_{1,n})) \to \{1, 2, \ldots, 2n + 2\}$ by following two cases.

Case 1: For $n \leq 5$ and $n = 7$.

$$\begin{aligned} &f(u) = 1, \\ &f(u_i) = 2i + 2; \quad \text{for } 1 \leq i \leq n, \\ &f(v) = 2, \\ &f(v_i) = 2i + 1; \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

In view of above defined labeling pattern, we have $e_f(0) = 1$ and $e_f(1) = 2$ for $n = 1$, $e_f(0) = 3$ and $e_f(1) = 3$ for $n = 2$, $e_f(0) = 4$ and $e_f(1) = 5$ for $n = 3$, $e_f(0) = 6$ and $e_f(1) = 6$ for $n = 4$, $e_f(0) = 8$ and $e_f(1) = 7$ for $n = 5$ and $e_f(0) = 11$ and $e_f(1) = 10$ for $n = 7$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| \leq 1$.

Case 2: For $n = 6$ and $n \geq 8$.

In order to satisfy the edge condition for power cordial labeling of graph it is essential to label at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges with label 0 as well as at least $\left\lfloor \frac{3n}{2} \right\rfloor$ edges with label 1 out of total $3n$ edges. All the possible assignment of vertex labels will give rise at most $\left\lfloor \frac{3n}{2} \right\rfloor - 1$ edges with label 1 and at least $\left\lfloor \frac{3n}{2} \right\rfloor + 1$ edges with label 0 out of total $3n$ edges. Therefore, $|e_f(0) - e_f(1)| \geq 2$. Thus, $S'(K_{1,n})$ is not a power cordial graph for $n = 6$ and $n \geq 8$.

Hence, $S'(K_{1,n})$ is a power cordial graph for $n \leq 5$ and $n = 7$ and not a power cordial graph for $n = 6$ and $n \geq 8$. □

Example 3.9. The graph $S'(K_{1,4})$ and its power cordial labeling is shown in Fig 8.

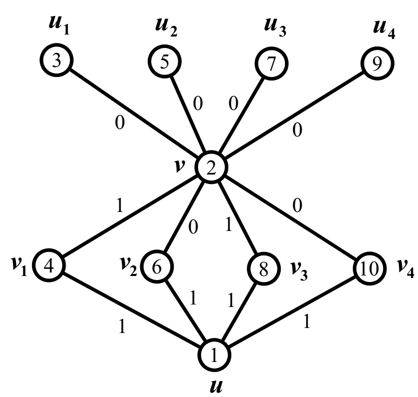

Fig 8: The graph $S'(K_{1,4})$ and its power cordial labeling.

Definition 3.10. The graph $G = \langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$ is obtained by joining apex vertices of two stars $K_{1,n}$ to a new vertex x.

Theorem 3.11. *The graph* $G = \langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$ *is a power cordial graph.*

Proof. Let u be the apex vertex and $u_1, u_2, ..., u_n$ be the other vertices of first star $K_{1,n}$ and v be the apex vertex and $v_1, v_2, ..., v_n$ be the other vertices of second star $K_{1,n}$ and $G = \langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$ be the graph obtained by joining apex vertices of two stars to a new vertex x. Then $|V(G)| = 2n+3$ and $|E(G)| = 2n+2$.

We define a bijection $f : V(G) \to \{1, 2, \ldots, 2n+3\}$ as follows:

$$\begin{aligned}
&f(x) = 2n+3,\\
&f(u) = 1,\\
&f(u_i) = 2i+2; \quad \text{for } 1 \le i \le n,\\
&f(v) = 2,\\
&f(v_i) = 2i+1; \quad \text{for } 1 \le i \le n.
\end{aligned}$$

In view of above defined labeling pattern, we have $e_f(0) = n+1$ and $e_f(1) = n+1$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 0$.

Hence, the graph $G = \langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$ is a power cordial graph. □

Example 3.12. The graph $G = \langle K_{1,7}^{(1)}, K_{1,7}^{(2)} \rangle$ and its power cordial labeling is shown in Fig 9.

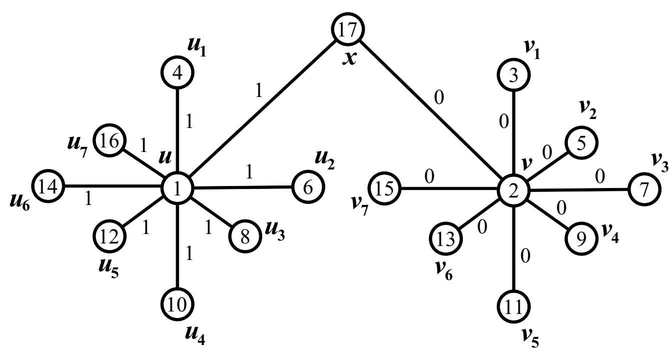

Fig 9: The graph $G = \langle K_{1,7}^{(1)}, K_{1,7}^{(2)} \rangle$ and its power cordial labeling.

Definition 3.13. The *bistar* $B_{n,n}$ is graph obtained by joining the apex vertices of two copies of stars $K_{1,n}$ by an edge.

Theorem 3.14. $S'(B_{n,n})$ *is a power cordial graph for* $n \le 8$ *and not a power cordial graph for* $n > 8$.

Proof. Let $B_{n,n}$ be the bistar with vertex set $V(B_{n,n}) = \{u, v, u_i, v_i : 1 \le i \le n\}$. In order to obtain $S'(B_{n,n})$, the splitting graph of a bistar $B_{n,n}$, add vertices u', v', u'_i, v'_i for $1 \le i \le n$ corresponding to vertices u, v, u_i, v_i for $1 \le i \le n$. Then $|V(S'(B_{n,n}))| = 4n+4$ and $|E(S'(B_{n,n}))| = 6n+3$.

We define a bijection $f : V(S'(B_{n,n})) \to \{1, 2, \ldots, 4n+4\}$ by following five cases.

Case 1: For $n = 1$.

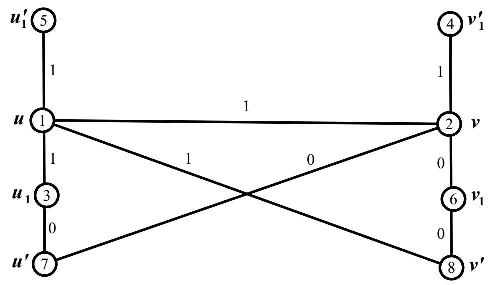

Fig 10: The graph $S'(B_{1,1})$ and its power cordial labeling.

Case 2: For $n = 2, 3$.

$$\begin{aligned}
&f(u) = 1, \\
&f(v) = 2, \\
&f(v'_i) = 2^{i+1}; \quad \text{for } 1 \leq i \leq n, \\
&f(u_i) = 3i; \quad \text{for } 1 \leq i \leq n, \\
&f(u'_i) = 5i; \quad \text{for } 1 \leq i \leq n, \\
&f(v') = 7.
\end{aligned}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0) = 7$ and $e_f(1) = 8$ for $n = 2$ and $e_f(0) = 10$ and $e_f(1) = 11$ for $n = 3$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Case 3: For $n = 4, 5$.

$$\begin{aligned}
&f(u) = 1, \\
&f(v) = 2, \\
&f(v'_i) = 2^{i+1}; \quad \text{for } 1 \leq i \leq 3, \\
&f(v') = 3, \\
&f(v') = 3, \\
&f(v_1) = 3^2, \\
&f(u_i) = 9 + i; \quad \text{for } 1 \leq i \leq n, \\
&f(u'_i) = 16 + i; \quad \text{for } 1 \leq i \leq n.
\end{aligned}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0) = 13$ and $e_f(1) = 14$ for $n = 4$ and $e_f(0) = 17$ and $e_f(1) = 16$ for $n = 5$. Thus, f satisfies the condition $|e_f(0) - e_f(1)| = 1$.

Case 4: For $6 \leq n \leq 8$.

$$\begin{array}{ll} f(u)=1, & \\ f(v)=2, & \\ f(v_i')=2^{i+1}; & \text{for } 1\le i\le \left\lceil \frac{n}{2}\right\rceil, \\ f(v')=3, & \\ f(v_i)=3^{i+1}; & \text{for } 1\le i\le 2, \\ f(u')=5, & \\ f(u_1)=5^2, & \\ f(u_i)=2i+7; & \text{for } 2\le i\le n, \\ f(u_i')=2(2i+1); & \text{for } 1\le i\le n. \end{array}$$

Now label the remaining vertices in such a way that they do not generate edge label 1.

In view of above defined labeling pattern, we have $e_f(0)=19$ and $e_f(1)=20$ for $n=6$, $e_f(0)=22$ and $e_f(1)=23$ for $n=7$ and $e_f(0)=26$ and $e_f(1)=25$ for $n=8$. Thus, f satisfies the condition $|e_f(0)-e_f(1)|=1$.

Case 5: For $n>8$.

In order to satisfy the edge condition for power cordial labeling of graph it is essential to label at least $\left\lfloor \frac{6n+3}{2}\right\rfloor$ edges with label 0 as well as at least $\left\lfloor \frac{6n+3}{2}\right\rfloor$ edges with label 1 out of total $6n+3$ edges. All the possible assignment of vertex labels will give rise at most $\left\lfloor \frac{6n+3}{2}\right\rfloor-1$ edges with label 1 and at least $\left\lfloor \frac{6n+3}{2}\right\rfloor+1$ edges with label 0 out of total $6n+3$ edges. Therefore, $|e_f(0)-e_f(1)|\ge 2$. Thus, $S'(B_{n,n})$ is not a power cordial graph for $n>8$.

Hence, $S'(B_{n,n})$ is a power cordial graph for $n\le 8$ and not a power cordial graph for $n>8$. □

Example 3.15. The graph $S'(B_{4,4})$ and its power cordial labeling is shown in Fig 11.

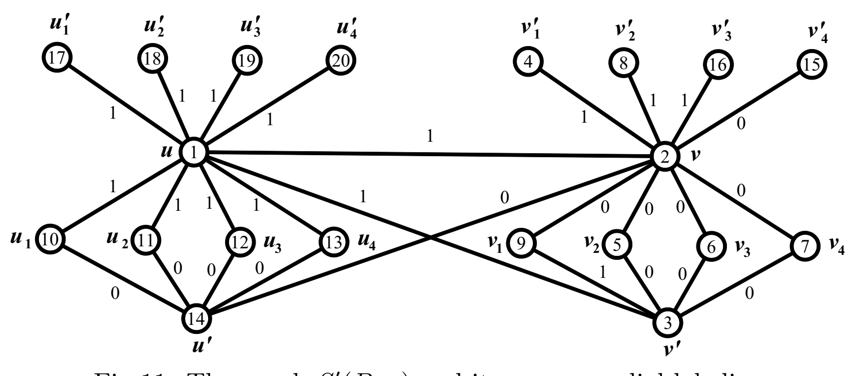

Fig 11: The graph $S'(B_{4,4})$ and its power cordial labeling.

4. Concluding remarks

Every power cordial graph is divisor cordial but every divisor cordial graph need not be power cordial. Hence, it is interesting to study power cordial labeling for divisor cordial graphs. Varatharajan *et al.* [11, 12] have proved that $S(K_{1,n})$, $2K_{1,n}$ and $G = \langle K_{1,n}^{(1)}, K_{1,n}^{(2)} \rangle$ are divisor cordial graphs while Vaidya and Shah [13, 14] have discussed divisor cordial labeling for $S'(K_{1,n})$, $S'(B_{n,n})$, helm H_n, flower Fl_n and gear graph G_n. Divisor cordial labeling for fan F_n is studied by Murugan and Devakiruba [1]. Raj and Manoharan [9] have obtained divisor cordial labeling for jewel graph J_n. In this paper, we have investigated power cordial labeling for the same graphs.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

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References

1. A.N. Murugan and G. Devakiruba, *Cycle Related Divisor Cordial Graphs*, Int. J. Math. Trends Tech. **12** (2014), 34-43.
2. A. Rosa, *On certain valuations of the vertices of a graph*, In Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris 1967, 349-355.
3. C.M. Barasara and Y.B. Thakkar, *Power Cordial Graphs*, Communicated.
4. G.S. Bloom and S.W. Golomb, *Applications of numbered undirected graphs*, Proceedings of IEEE **65** (1977), 562-570.
5. G.S. Bloom, *A choronology of Ringle-Kotzig conjecture and the counting quest to call all trees graceful*, in: Topics in Graph Theory (edited by F Harary), New York Academy of Sciences **328** (1979), 32–51.
6. I. Cahit, *Cordial Graphs: A weaker version of graceful and harmonious Graphs*, Ars Combinatoria **23** (1987), 201-207.
7. J.A. Gallian, *A Dynamic Survey of Graph Labeling*, The Electronics Journal of Combinatorics **24** (2021), #DS6.
8. J. Gross and J. Yellen, *Graph Theory and its applications*, CRC Press, 1999.
9. P.L.R. Raj and R.L.J. Manoharan, *Some results on divisor cordial labeling of graphs*, Int. J. Innov. Sci., Eng., Tech. **1** (2014), 226-231.
10. R.L. Graham and N.J.A. Sloane, *On additive bases and harmonious graphs*, SIAM J. Alg. Discrete Methods **1** (1980), 382-404.
11. R. Varatharajan, S. Navanaeethakrishnan and K. Nagarajan, *Divisor Cordial Graphs*, Int. J. of Mathematics and Combinatorics **4** (2011), 15-25.
12. R. Varatharajan, S. Navanaeethakrishnan and K. Nagarajan, *Special Classes of Divisor Cordial Graphs*, Int. Mathematical Forum **7** (2012), 1737-1749.
13. S.K. Vaidya and N.H. Shah, *Some star and bistar related divisor cordial graphs*, Annals Pure Appl. Math. **3** (2013), 67-77.

14. S.K. Vaidya and N.H. Shah, *Further results on divisor cordial labeling*, Annals Pure Appl. Math. **4** (2013), 150-159.
15. S.W. Golomb, *How to number a graph*, in Graph Theory and Computing, R. C. Read, ed., Academic Press, New York, 1972, 23-37.

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