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SOME IDENTITIES RELATED TO THE EULER NUMBERS AND POLYNOMIALS

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Abstract. In this note, we give a proof of the p-adic analogue of mild generalization of classical zeta functions by modifying Osipov's method. In addition, we obtain some identities for the p-adic integration, from which, some classical formulas for Euler numbers and polynomials have been deduced.

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1. Introduction and definitions

In this paper p will denote an odd rational prime number, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of fractions of $\mathbb{Z}_p$, and $\mathbb{C}_p$ the p-adic completion of the algebraic closure $\overline{\mathbb{Q}}_p$. Let v_p be the p-adic valuation of $\mathbb{C}_p$ normalized so that $|p|_p = p^{-v_p(p)} = p^{-1}$. Let $\mathbb{Z}_p^\times$ be the multiplicative group of all p-adic units. Divisibility and congruences are always understood within the ring of p-adic integers, i.e., $a \mid b$ iff $v_p(a) \le v_p(b)$.

Let d be a fixed positive integer. We set $X_d = \varprojlim_n (\mathbb{Z}/dp^n\mathbb{Z})$, the map from $\mathbb{Z}/dp^l\mathbb{Z}$ to $\mathbb{Z}/dp^n\mathbb{Z}$ for $l \ge n$ is a reduction mod dp^n. For $d = 1$, $X_1 = \mathbb{Z}_p$.

Let ℓ be a fixed positive integer. $\mathfrak{a}, \mathfrak{d}$ and $\mathfrak{x}$ will denote $(a_1, \ldots, a_\ell)$,$(d_1, \ldots, d_\ell)$ and $(x_1, \ldots, x_\ell)$, respectively. For $\mathfrak{x} = (x_1, \ldots, x_\ell)$ and $\mathfrak{m} = (m_1, \ldots, m_\ell)$ we denote

$$\mathfrak{x}^{\mathfrak{m}} = x_1^{m_1} \cdots x_\ell^{m_\ell}, \quad |\mathfrak{x}| = x_1 + \cdots + x_\ell.$$

We set

$$X_{\mathfrak{d}} = \prod_{\substack{d_i \in \mathbb{N} \\ i=1,\ldots,\ell}} X_{d_i} \tag{1.1}$$

with the product topology, so $X_{\mathfrak{d}}$ is compact since X_{d_i} is compact for $i = 1, \ldots, \ell$.

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Let $\mathfrak{d}$ be the the point in $\mathbb{N}^\ell$. We denote the polydisc as

$$\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell=\{\mathfrak{x}\in\mathbb{Q}_p^\ell \mid x_i\equiv a_i \pmod{d_ip^{h_in_i}},\ i=1,\dots,\ell\},$$

where $\mathfrak{a}\in\mathbb{Q}_p^\ell$ and $\mathfrak{h},\mathfrak{n}\in\mathbb{N}^\ell$. The polydisc $\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell$ is the product of discs $a_i+d_ip^{h_in_i}\mathbb{Z}_p$ for $i=1,\dots,k$, that is, $\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell=\prod_{i=1}^\ell(a_i+d_ip^{h_in_i}\mathbb{Z}_p)$.

Definition 1.1 ([6, 11])**.** Let $\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell$ be a polydisc with $\mathfrak{a}\in\mathbb{Q}_p^\ell$ and $\mathfrak{d}\in\mathbb{N}^\ell$. Let μ_i be a distribution on $\mathbb{Z}_p$ for $i=1,\dots,\ell$. We define a formal direct product of distributions $\mu(\mathfrak{a})$ by $\mu(\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell)=\prod_{i=1}^\ell\mu_i(a_i+d_ip^{h_in_i}\mathbb{Z}_p)$.

We use the symbols $0\le\mathfrak{a}<\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}$ to denote $0\le a_i<d_ip^{h_in_i}$ for each i. From now on, when we write $\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell$ we will assume that $0\le\mathfrak{a}<\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}$.

Definition 1.2 ([6, p. 322])**.** Let $f:X_\mathfrak{d}\to\mathbb{C}_p$ be any continuous function, and it can be written as a uniform limit of locally constant function f_i. For $\mathfrak{n}=(n_1,\dots,n_\ell)\in\mathbb{N}^\ell$, we will assume that $\mathfrak{n}\to\infty$ when $n_1\to\infty,\dots,n_\ell\to\infty$. Define

$$\begin{aligned}\int_{X_\mathfrak{d}}f(\mathfrak{x})d\mu(\mathfrak{x})&=\lim_{\mathfrak{n}\to\infty}\sum_{0\le\mathfrak{a}<\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}}f(\mathfrak{a})\mu(\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell)\\&=\lim_{\substack{n_1\to\infty\\\cdots\\n_\ell\to\infty}}\sum_{a_1=0}^{d_1p^{h_1n_1}-1}\cdots\sum_{a_\ell=0}^{d_\ell p^{h_\ell n_\ell}-1}f(a_1,\dots,a_\ell)\mu(a_1,\dots,a_\ell)\end{aligned}$$

(cf. [11], [14], [19] for a slightly different formulation).

Also, for any compact open subset O of $X_\mathfrak{d}$, the integral of f on O is defined by

$$\int_O f(\mathfrak{x})d\mu(\mathfrak{x})=\int_{X_\mathfrak{d}}f(\mathfrak{x})\cdot(\text{characteristic function of } O)\,d\mu(\mathfrak{x})$$

(cf. [14, Chapter II]).

Definition 1.3. For $i=1,\dots,\ell$, let ε_i be roots of unity with order relatively prime with p, and let $\varepsilon_i\ne1$ for each i. Set $\tilde\varepsilon=(\varepsilon_1,\dots,\varepsilon_\ell)$. The higher order Euler polynomials with parameter $\tilde\varepsilon$, $H_m^{(\ell)}(x,\tilde\varepsilon)$, are defined by

$$g_{\tilde\varepsilon}^{(\ell)}(t)e^{xt}=\sum_{m=0}^\infty H_m^{(\ell)}(x,\tilde\varepsilon)\frac{t^m}{m!}\tag{1.2}$$

with the function

$$g_{\tilde\varepsilon}^{(\ell)}(t)=\prod_{i=1}^\ell\frac{1-\varepsilon_i}{1-\varepsilon_ie^t}.\tag{1.3}$$

The values at $x=0$ in (1.2), $H_m^{(\ell)}(0,\tilde\varepsilon)$, are called the higher order Euler numbers with parameter $\tilde\varepsilon$; when $\ell=1$, the polynomials or numbers are called ordinary. When $\ell=1$, $H_m(x,\varepsilon)$ and $H_m(\varepsilon)$ are denoted by $H_m^{(1)}(x,\varepsilon)$ and $H_m^{(1)}(0,\varepsilon)$, respectively. If $\ell=1$ in (1.2), it is well known that the explicit representations for the ordinary Euler polynomials, complementing those given in [18].

In Definition 1.3, when $\varepsilon = (\varepsilon, \ldots, \varepsilon)$ with a slight abuse of notation, the higher order Euler polynomials with parameter ε, $H_m^{(\ell)}(x, \varepsilon)$, are defined by

$$g_\varepsilon^{(\ell)}(t)e^{xt} = \sum_{m=0}^{\infty} H_m^{(\ell)}(x, \varepsilon)\frac{t^m}{m!}. \tag{1.4}$$

Definition 1.4. Let x be a positive real number and $\varepsilon^r = 1, \varepsilon \neq 1$. A mild generalization of classical Riemann zeta function $\zeta(s)$ might be

$$\zeta_\ell(s, x, \varepsilon) = \sum_{0 \leq \mathfrak{n} < \infty} \varepsilon^{|\mathfrak{n}|}(x + |\mathfrak{n}|)^{-s}, \tag{1.5}$$

which was defined by Barnes for $\mathrm{Re}(s) > \ell$ and has a meromorphic continuation to all $s \in \mathbb{C}$ except for simple poles at $s = j$ $(1 \leq j \leq \ell)$ (for details see [2], [17], [19]). We'll want to do is to get rid of the terms $1/(x + |\mathfrak{n}|)^s$ with $|\mathfrak{n}|$ divisible by p. Now define

$$\tilde{\zeta}_\ell(s, x + \ell, \varepsilon) = \sum_{\substack{1 \leq \mathfrak{n} < \infty \\ (p, |\mathfrak{n}|) = 1}} \varepsilon^{|\mathfrak{n}|}(x + |\mathfrak{n}|)^{-s}. \tag{1.6}$$

From (1.6), we have

$$\tilde{\zeta}_\ell(s, x + \ell, \varepsilon) = \zeta_\ell(s, x + \ell, \varepsilon) - p^{-s}\zeta_\ell\left(s, \frac{x}{p} + \ell, \varepsilon\right). \tag{1.7}$$

Hence

$$\tilde{\zeta}_\ell(-m, x + \ell, \varepsilon) = \zeta_\ell(-m, x + \ell, \varepsilon) - p^{-s}\zeta_\ell\left(-m, \frac{x}{p} + \ell, \varepsilon\right) \tag{1.8}$$

for $m \geq 0$. We note that $\zeta_\ell(s, x, \varepsilon)$ is also defined for $\varepsilon = 1$. Thus we set $\zeta_\ell(s, x) = \zeta_\ell(s, x, 1)$. The zeta function $\zeta_\ell(s, x, \varepsilon)$ is expressed as an integral,

$$\Gamma(s)\zeta_\ell(s, x, \varepsilon) = \int_0^\infty \frac{e^{-xt}t^{s-1}}{(1 - \varepsilon e^{-t})^\ell}dt,$$

where $\mathrm{Re}(s) > \ell$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$. The above expression gives us the analytic continuation of $\zeta_\ell(s, x, \varepsilon)$ to the whole complex plane.

2. Some properties of the higher order Euler numbers and polynomials

For a fixed $\tilde{\varepsilon}$ we adopt the following notations: $f(t) = a_0t^0 + a_1t + \cdots + a_mt^m + \cdots$, then

$$f(H^{(\ell)}(x, \tilde{\varepsilon})) = a_0H_0^{(\ell)}(x, \tilde{\varepsilon}) + a_1H_1^{(\ell)}(x, \tilde{\varepsilon}) + \cdots + a_mH_m^{(\ell)}(x, \tilde{\varepsilon}) + \cdots. \tag{2.1}$$

Thus, by (1.2) and (2.1), we have

$$g_{\tilde{\varepsilon}}^{(\ell)}(t) = e^{H^{(\ell)}(\tilde{\varepsilon})t}, \quad g_{\tilde{\varepsilon}}^{(\ell)}(t)e^{xt} = e^{H^{(\ell)}(x, \tilde{\varepsilon})t} = e^{(H^{(\ell)}(\tilde{\varepsilon}) + x)t}. \tag{2.2}$$

So

$$H_m^{(\ell)}(x,\tilde{\varepsilon}) = (H^{(\ell)}(\tilde{\varepsilon})+x)^m = \sum_{i=0}^{m}\binom{m}{i}H_i^{(\ell)}(\tilde{\varepsilon})x^{m-i}, \quad m \geq 0. \tag{2.3}$$

Theorem 2.1. *If m is a nonnegative integer, then*

$$H_0^{(\ell)}(\tilde{\varepsilon}) = 1, \quad H_{m+1}^{(\ell)}(\tilde{\varepsilon}) = \sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{i}\binom{i}{j}H_j^{(\ell)}(\tilde{\varepsilon})\sum_{n=1}^{\ell}\frac{\varepsilon_n H_{i-j}(\varepsilon_n)}{1-\varepsilon_n}.$$

Proof. Using (1.3) and (2.1), the derivative $\frac{\mathrm{d}}{\mathrm{d}t}(g_{\tilde{\varepsilon}}^{(\ell)}(t))$ of $g_{\tilde{\varepsilon}}^{(\ell)}(t)$ is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(g_{\tilde{\varepsilon}}^{(\ell)}(t)) &= \prod_{i=1}^{\ell}(1-\varepsilon_i)\frac{\mathrm{d}}{\mathrm{d}t}[(1-\varepsilon_1 e^t)^{-1}\cdots(1-\varepsilon_\ell e^t)^{-1}] \\
&= \prod_{i=1}^{\ell}(1-\varepsilon_i)[\varepsilon_1 e^t(1-\varepsilon_1 e^t)^{-2}\cdots(1-\varepsilon_\ell e^t)^{-1} \\
&\qquad + \cdots + \varepsilon_\ell e^t(1-\varepsilon_1 e^t)^{-1}\cdots(1-\varepsilon_\ell e^t)^{-2}] \\
&= \prod_{i=1}^{\ell}\frac{1-\varepsilon_i}{1-\varepsilon_i e^t}\left[\frac{\varepsilon_1}{1-\varepsilon_1 e^t}+\cdots+\frac{\varepsilon_\ell}{1-\varepsilon_\ell e^t}\right]e^t \\
&= g_{\tilde{\varepsilon}}^{(\ell)}(t)\sum_{n=1}^{\ell}e^{(H(\varepsilon_n)+1)t}\frac{\varepsilon_n}{1-\varepsilon_n} = \sum_{n=1}^{\ell}e^{(H^{(\ell)}(\tilde{\varepsilon})+H(\varepsilon_n)+1)t}\frac{\varepsilon_n}{1-\varepsilon_n}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
H_{m+1}^{(\ell)}(\tilde{\varepsilon}) &= \sum_{n=1}^{\ell}(H^{(\ell)}(\tilde{\varepsilon})+H(\varepsilon_n)+1)^m\frac{\varepsilon_n}{1-\varepsilon_n} \\
&= \sum_{n=1}^{\ell}\sum_{i=0}^{m}\binom{m}{i}(H^{(\ell)}(\tilde{\varepsilon})+H(\varepsilon_n))^i\frac{\varepsilon_n}{1-\varepsilon_n}.
\end{aligned}$$

Since

$$(H^{(\ell)}(\tilde{\varepsilon})+H(\varepsilon_n))^i = \sum_{j=0}^{i}\binom{i}{j}H_j^{(\ell)}(\tilde{\varepsilon})H_{i-j}(\varepsilon_n),$$

so we have

$$H_0^{(\ell)}(\tilde{\varepsilon}) = 1, \quad H_{m+1}^{(\ell)}(\tilde{\varepsilon}) = \sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{i}\binom{i}{j}H_j^{(\ell)}(\tilde{\varepsilon})\sum_{n=1}^{\ell}\frac{\varepsilon_n H_{i-j}(\varepsilon_n)}{1-\varepsilon_n} \tag{2.4}$$

for $m \geq 0$. This completes the proof. □

From (2.4) (for $\varepsilon = (\varepsilon, \cdots, \varepsilon)$) it follows that

$$H_0^{(\ell)}(\varepsilon) = 1, \quad H_{m+1}^{(\ell)}(\varepsilon) = \frac{\ell\varepsilon}{1-\varepsilon}\sum_{i=0}^{m}\binom{m}{i}\sum_{j=0}^{i}\binom{i}{j}H_j^{(\ell)}(\varepsilon)H_{i-j}(\varepsilon). \tag{2.5}$$

For example, we have following some values of $H_m^{(\ell)}(\varepsilon)$:

$$H_1^{(\ell)}(\varepsilon) = \frac{-\ell\varepsilon}{\varepsilon - 1}, H_2^{(\ell)}(\varepsilon) = \frac{\ell\varepsilon + \ell^2\varepsilon^2}{(\varepsilon-1)^2}, H_3^{(\ell)}(\varepsilon) = \frac{-\ell\varepsilon - \ell\varepsilon^2 - 3\ell^2\varepsilon^2 - \ell^3\varepsilon^3}{(\varepsilon-1)^3}.$$

Recall that the higher order Bernoulli polynomials, $B_m^{(\ell)}(x)$, are defined by

$$\left(\frac{t}{e^t - 1}\right)^\ell e^{xt} = \sum_{m=0}^{\infty} B_m^{(\ell)}(x)\frac{t^m}{m!} = e^{B^{(\ell)}(x)t}, \tag{2.6}$$

and $B_m^{(\ell)}(0) = B_m^{(\ell)}$, the higher order Bernoulli numbers. In particular, $B_m = B_m^{(1)}(0)$ is the ordinary Bernoulli numbers. Using (2.1) and (2.6), it is easily seen that for any x

$$e^{xt}e^{B^{(\ell)}t} = e^{(x+B^{(\ell)})t} = e^{B^{(\ell)}(x)t}.$$

So

$$B_m^{(\ell)}(x) = (x + B^{(\ell)})^m = \sum_{l=0}^{m}\binom{m}{l}B_l^{(\ell)}x^{m-l}.$$

Theorem 2.2. *If* $\varepsilon^r = 1$ *and* $\varepsilon \neq 1$, *then*

$$H_m^{(\ell)}(x,\varepsilon) = \frac{1}{r^\ell}\frac{(\varepsilon-1)^\ell}{(m+\ell)\cdots(m+1)}\sum_{0\leq\mathfrak{a}<r}\varepsilon^{|\mathfrak{a}|}(x+|\mathfrak{a}|+rB^{(\ell)})^{m+\ell}, \quad m \geq 0.$$

Proof. Let $\varepsilon^r = 1, \varepsilon \neq 1$. Note that

$$\sum_{0\leq\mathfrak{a}<r}\varepsilon^{|\mathfrak{a}|}(x+|\mathfrak{a}|+rB^{(\ell)})^m = 0 \quad \text{for } m = 0,\ldots,\ell-1, \tag{2.7}$$

by using $\sum_{a=0}^{r-1}\varepsilon^a = 0$. Using (2.1), (1.4) and (2.6), we see that

$$\begin{aligned} g_\varepsilon^{(\ell)}(t)e^{xt} &= \frac{(\varepsilon-1)^\ell}{r^\ell}\sum_{0\leq\mathfrak{a}<r}\varepsilon^{|\mathfrak{a}|}e^{(x+|\mathfrak{a}|)t}e^{rB^{(\ell)}t}\frac{1}{t^\ell} \\ &= \sum_{m=0}^{\infty}\left(\frac{(\varepsilon-1)^\ell}{r^\ell}\sum_{0\leq\mathfrak{a}<r}\varepsilon^{|\mathfrak{a}|}\frac{(x+|\mathfrak{a}|+rB^{(\ell)})^{m+\ell}}{(m+\ell)\cdots(m+1)}\right)\frac{t^m}{m!}. \end{aligned} \tag{2.8}$$

Comparing the coefficients of the terms $t^m/m!$ in (1.4) and (2.8), we obtain the required result. □

Example 2.3. 1. For $\ell = 1$ and $m = 0$, Theorem 2.2 gives us $H_0(x,\varepsilon) = \frac{\varepsilon-1}{r}\sum_{a=0}^{r-1}\varepsilon^a(x+a+rB_1)$. This means that

$$\frac{r}{\varepsilon-1} = \sum_{a=0}^{r-1}\varepsilon^a a$$

for $\varepsilon^r = 1, \varepsilon \neq 1$, since $H_0(x,\varepsilon) = 1$ and $B_1 = -\frac{1}{2}$.

2. Put $m=1$ and $\ell=2$ in (2.7). Note that $\sum_{a=0}^{r-1}\varepsilon^a=0$ and $\sum_{a=0}^{r-1}a\varepsilon^a\neq 0$. Thus we have

$$\begin{aligned}\sum_{\mathfrak{a}=0}^{r-1}\varepsilon^{|\mathfrak{a}|}(x+|\mathfrak{a}|+rB^{(2)})^1 &= \sum_{a_1=0}^{r-1}\sum_{a_2=0}^{r-1}\varepsilon^{a_1+a_2}(x+a_1+a_2+rB^{(2)}) \\ &= \sum_{a_1=0}^{r-1}\varepsilon_1^{a_1}a_1\sum_{a_2=0}^{r-1}\varepsilon_1^{a_2}+\sum_{a_1=0}^{r-1}\varepsilon_1^{a_1}\sum_{a_2=0}^{r-1}\varepsilon_1^{a_2}a_2 \\ &= 0.\end{aligned}$$

3. p-adic Integral representations

Lemma 3.1 ([18])**.** *Let $\varepsilon^r=1, \varepsilon\neq 1$ and $(r,p)=1$. Then there exists h such that $r \mid (p^h-1)$, and*

$$H_0(\varepsilon)=1, \quad \lim_{n\to\infty}\sum_{a=0}^{dp^{hn}-1}a^m\varepsilon^a=\frac{1-\varepsilon^d}{1-\varepsilon}H_m(\varepsilon), \quad m\geq 1.$$

For $d=1$ in Lemma 3.1 we have [18, Eq. (19)]

$$\lim_{n\to\infty}\sum_{a=0}^{p^{hn}-1}a^m\varepsilon^a=H_m(\varepsilon), \quad m\geq 1.$$

Lemma 3.2 ([14])**.** *If $f: X_{\mathfrak{d}}\to\mathbb{C}_p$ is a continuous function such that $|f(\mathfrak{x})|_p\leq A$ for all $\mathfrak{x}\in X_{\mathfrak{d}}$, and if $|\mu(O)|_p\leq B$ for all compact-open $O\subset X_{\mathfrak{d}}$, then $|\int f\mu|_p\leq AB$.*

For each i, take a r_i-th root of unity ε_i with $\varepsilon_i\neq 1$, and set $\tilde{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_\ell)$. Let r_i be prime to p for all i. By Lemma 3.1, there exists h_i such that $r_i \mid (p^{h_i}-1)$, e.g., one can take $h_i=\varphi(r_i)$, where φ is the Euler function. For h_i satisfying $r_i \mid (p^{h_i}-1)$ we denote $\mathfrak{h}=(h_1,\ldots,h_\ell)$. Let's define the measure $\mu_{\tilde{\varepsilon}}$ on $\mathbb{Z}_p^\ell$ by (cf. [13], [18, Eq. (16)])

$$\mu_{\tilde{\varepsilon}}(\mathfrak{a})=\mu_{\tilde{\varepsilon}}(\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell)=\tilde{\varepsilon}^{\mathfrak{a}}, \tag{3.1}$$

where $0\leq\mathfrak{a}<\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}$ and $\tilde{\varepsilon}^{\mathfrak{a}}=\varepsilon_1^{a_1}\cdots\varepsilon_\ell^{a_\ell}$. The measure $\mu_{\tilde{\varepsilon}}$ is $\mathbb{Q}_p(\tilde{\varepsilon})$-valued.

Theorem 3.3. *Let $\mathcal{O}$ be an open set in $\mathbb{C}_p^\ell$ with $\mathfrak{a}+\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}\mathbb{Z}_p^\ell\subset X_{\mathfrak{d}}\subset\mathcal{O}$. $\mathcal{B}$ is a Banach space over $\mathbb{C}_p$ and $f:\mathcal{O}\to\mathcal{B}$ is locally holomorphic. Define*

$$S_{\mathfrak{d}}(\mathfrak{n},\mathfrak{h})=\sum_{0\leq\mathfrak{a}<\mathfrak{d}p^{\mathfrak{h}\mathfrak{n}}}f(\mathfrak{a})\mu_{\tilde{\varepsilon}}(\mathfrak{a}),$$

where we are summing over all $\mathfrak{a}=(a_1,\ldots,a_\ell)$ as in Definition 1.2. We will assume that $\mathfrak{n}\to\infty$ when $n_1\to\infty,\ldots,n_\ell\to\infty$. Then

(1) *$L=\lim_{\mathfrak{n}\to\infty}S_{\mathfrak{d}}(\mathfrak{n},\mathfrak{h})$ exists;*
(2) *L is dependent of the $\mathfrak{d}$ used;*
(3) *L may be calculated by iteration of the limit in any order.*

Proof. Let f be holomorphic on $\mathcal{O}$ with $\mathbb{Z}_p^\ell \subset \mathcal{O}$. Then we can write $f(\mathfrak{x}) = \sum_{0\le\mathfrak{m}<\infty} a_{\mathfrak{m}}\mathfrak{x}^{\mathfrak{m}}$, where the right side represents a power series in k variables with $\mathfrak{m}$ running through the k-tuples of nonnegative integers. Thus $a_{\mathfrak{m}} \to 0$. From (1.2), (3.1) and Lemma 3.1 we have

$$\begin{aligned}\lim_{\mathfrak{n}\to\infty} S_{\mathfrak{d}}(\mathfrak{n},\mathfrak{h}) &= \lim_{\mathfrak{n}\to\infty} \sum_{0\le\mathfrak{a}<\mathfrak{d}p^{\mathfrak{h}\,\mathfrak{n}}} \left(\sum_{0\le\mathfrak{m}<\infty} \mathfrak{a}^{\mathfrak{m}}\right)\tilde{\varepsilon}^{\mathfrak{a}} \\ &= \prod_{i=1}^{\ell} \frac{1-\varepsilon_i^{d_i}}{1-\varepsilon_i} \sum_{0\le\mathfrak{m}<\infty} a_{\mathfrak{m}} \widehat{H}_{\mathfrak{m}}(\tilde{\varepsilon}),\end{aligned} \tag{3.2}$$

where $\widehat{H}_{\mathfrak{m}}(\tilde{\varepsilon}) = H_{m_1}(\varepsilon_1)\cdots H_{m_\ell}(\varepsilon_\ell)$. Note that $|1-\varepsilon_i|_p = 1$ if $(p,r_i)\ne 1, \varepsilon_i \ne 1$ and $\varepsilon_i^{r_i} = 1$ (see [10, p.38, Lemma 2.10]). From Lemma 3.2, we obtain that $|H_{m_i}(\varepsilon_i)|_p \le 1$ for $i = 1,\ldots,\ell$, whence

$$|\widehat{H}_{\mathfrak{m}}(\tilde{\varepsilon})|_p = \prod_{i=1}^{\ell} |H_{m_i}(\varepsilon_i)|_p \le 1.$$

Thus $\sum_{0\le\mathfrak{m}<\infty} a_{\mathfrak{m}}\widehat{H}_{\mathfrak{m}}(\tilde{\varepsilon})$ converges. We can now conclude that $L = \lim_{\mathfrak{n}\to\infty} S_{\mathfrak{d}}(\mathfrak{n},\mathfrak{h})$ exists, but it is in fact dependent of $\mathfrak{d}$, as is shown in (3.2). Part (3) follows immediately from (3.2). □

Namely, we're in business as long as $|\mathfrak{x}| \not\equiv 0 \pmod p$. Let $\mathbb{Z}_p^\times$ be the group of p-adic units. To make all of our $|\mathfrak{x}|$'s in the domain of integration to have this property, we must take $\{\mathfrak{x}\in\mathbb{Z}_p^\ell \mid |\mathfrak{x}| \in \mathbb{Z}_p^\times\}$ and $\{\mathfrak{x}\in\mathbb{Z}_p^\ell \mid |\mathfrak{x}| \in p\mathbb{Z}_p\}$. It is easy to see that

$$\int_{\substack{\mathbb{Z}_p^\ell \\ |\mathfrak{x}|\in\mathbb{Z}_p^\times}} (\alpha+|\mathfrak{x}|)^m d\mu_{\tilde{\varepsilon}}(\mathfrak{x}) = \int_{\mathbb{Z}_p^\ell} (\alpha+|\mathfrak{x}|)^m d\mu_{\tilde{\varepsilon}}(\mathfrak{x}) - \int_{\substack{\mathbb{Z}_p^\ell \\ |\mathfrak{x}|\in p\mathbb{Z}_p}} (\alpha+|\mathfrak{x}|)^m d\mu_{\tilde{\varepsilon}}(\mathfrak{x}) \tag{3.3}$$

(cf. [19, p.34]). Thus, we claim that the expression

$$\int_{\substack{\mathbb{Z}_p^\ell \\ |\mathfrak{x}|\in\mathbb{Z}_p^\times}} (\alpha+|\mathfrak{x}|)^m d\mu_{\tilde{\varepsilon}}(\mathfrak{x}) \tag{3.4}$$

can be interpolated.

Theorem 3.4 ([20, Theorem 3.6])**.** *Let α be a positive integer with $(p,\alpha)\ne 1$. Let $\mathcal{X}_p = \mathbb{Z}_p \times (\mathbb{Z}/(p-1)\mathbb{Z})$. The function*

$$-m \longmapsto H_m^{(\ell)}(\alpha,\tilde{\varepsilon}) - p^m h_{p,\tilde{\varepsilon}}^{(\ell)} \sum_{\substack{0\le\mathfrak{a}<p \\ (p,|\mathfrak{a}|)\ne 1}} \tilde{\varepsilon}^{\mathfrak{a}} H_m^{(\ell)}\left(\frac{\alpha+|\mathfrak{a}|}{p}, \tilde{\varepsilon}^p\right)$$

admits a continuation from the sense subset $\{0,-1,\ldots\}\subset\mathbb{Z}_p$ *to a continuous function* $\zeta_{p,\ell}(s,\alpha,\tilde{\varepsilon}):\mathcal{X}_p\to\mathbb{Q}_p(\tilde{\varepsilon})$ *and*

$$\zeta_{p,\ell}(s,\alpha,\tilde{\varepsilon})=\int_{\substack{\mathbb{Z}_p^\ell\\ |\mathfrak{x}|\in\mathbb{Z}_p^\times}}(\alpha+|\mathfrak{x}|)^{-s}d\mu_{\tilde{\varepsilon}}(\mathfrak{x}).$$

By (1.5), we have

$$\begin{aligned}\zeta_\ell(-m,\alpha,\varepsilon)&=\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m e^{\alpha t}\sum_{0\le\mathfrak{n}<\infty}(\varepsilon e^t)^{|\mathfrak{n}|}\Big|_{t=0}\\&=\frac{1}{(1-\varepsilon)^\ell}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m g_\varepsilon^{(\ell)}(t)e^{\alpha t}\Big|_{t=0}\end{aligned}$$

for a nonnegative integer m. By (1.4), we obtain

$$\zeta_\ell(-m,\alpha,\varepsilon)=(1-\varepsilon)^{-\ell}H_m^{(\ell)}(\alpha,\varepsilon). \tag{3.5}$$

Further by (2.6), it is known (cf. e.g., [12], [17], [19]) that

$$\zeta_\ell(-m,\alpha)=(-1)^m((m+\ell)\cdots(m+1))^{-1}B_{m+\ell}^{(\ell)}(\alpha) \tag{3.6}$$

for $m\ge 0$. From (1.5) and (1.6), it is also easy to see that

$$\tilde{\zeta}_\ell(s,\alpha,\varepsilon)=\zeta_\ell(s,\alpha,\varepsilon)-p^{-s}\sum_{\substack{0\le\mathfrak{a}<p\\(p,\alpha+|\mathfrak{a}|)\ne1}}\varepsilon^{\mathfrak{a}}\zeta_\ell\left(s,\frac{\alpha+|\mathfrak{a}|}{p},\varepsilon^p\right). \tag{3.7}$$

It follows from (3.5) and (3.7) that

$$\begin{aligned}&H_m^{(\ell)}(\alpha,\varepsilon)-p^m h_{p,\varepsilon}^{(\ell)}\sum_{\substack{0\le\mathfrak{a}<p\\(p,\alpha+|\mathfrak{a}|)\ne1}}\varepsilon^{\mathfrak{a}}H_m^{(\ell)}\left(\frac{\alpha+|\mathfrak{a}|}{p},\varepsilon^p\right)\\&\quad=(1-\varepsilon)^\ell\left(\zeta_\ell(-m,\alpha,\varepsilon)-p^m\sum_{\substack{0\le\mathfrak{a}<p\\(p,\alpha+|\mathfrak{a}|)\ne1}}\varepsilon^{\mathfrak{a}}\zeta_\ell\left(-m,\frac{\alpha+|\mathfrak{a}|}{p},\varepsilon^p\right)\right)\\&\quad=(1-\varepsilon)^\ell\tilde{\zeta}_\ell(-m,\alpha,\varepsilon),\end{aligned} \tag{3.8}$$

where $h_{p,\varepsilon}^{(\ell)}=((1-\varepsilon)/(1-\varepsilon^p))^\ell$.

The statement follows from Theorem 3.4 and (3.7).

Theorem 3.5. *The function* $-m\to(1-\varepsilon)^\ell\tilde{\zeta}_\ell(-m,\alpha,\varepsilon)$ *admits a continuation from the dense subset* $\{0,-1,-2,\ldots\}\subset\mathcal{X}_p$ *to a continuous* $\mathbb{Q}_p(\varepsilon)$*-valued function*

$$\zeta_{p,\ell}(s,\alpha,\varepsilon)=\int_{\substack{\mathbb{Z}_p^\ell\\ |\mathfrak{x}|\in\mathbb{Z}_p^\times}}(\alpha+|\mathfrak{x}|)^{-s}d\mu_\varepsilon(\mathfrak{x}),$$

defined on $\mathcal{X}_p$.

Let $\varepsilon^r = 1, \varepsilon \neq 1$ and let $r > 1, (r,p) = 1$, and $r \mid (p^h - 1)$. Then by (1.5), we have

$$\varepsilon^\alpha \sum_{\substack{\varepsilon^r=1\\ \varepsilon\neq 1}} \tilde{\zeta}_\ell(s,\alpha,\varepsilon) = \begin{cases} (1-p^{-s})\sum_{0\leq \mathfrak{a}<\infty} \frac{1}{(\alpha+|\mathfrak{n}|)^s}(r-1), & r \mid (\alpha+|\mathfrak{n}|) \\ -(1-p^{-s})\sum_{0\leq \mathfrak{a}<\infty} \frac{1}{(\alpha+|\mathfrak{n}|)^s}, & \text{otherwise} \end{cases}$$

$$= (1-p^{-s})(r^{1-s}-1)\zeta_\ell(s,\alpha)$$

for $\mathrm{Re}(s) > k$. From the uniqueness of the analytic representation it follows that

$$\sum_{\substack{\varepsilon^r=1\\ \varepsilon\neq 1}} \tilde{\zeta}_\ell(s,\alpha,\varepsilon) = \varepsilon^{-\alpha}(1-p^{-s})(r^{1-s}-1)\zeta_\ell(s,\alpha). \tag{3.9}$$

which is valid for all $s \in \mathbb{C}$. Setting $s = -1, -2, \ldots$ in (3.9) and making use of (3.6) and (3.8), we obtain an identity which connects the higher order Euler numbers $H_m^{(\ell)}(\alpha,\varepsilon)$ with the higher order Bernoulli numbers $B_{m+k}^{(\ell)}(\alpha)$:

$$\begin{aligned} &\sum_{\substack{\varepsilon^r=1\\ \varepsilon\neq 1}} \frac{\varepsilon^\alpha}{(1-\varepsilon)^\ell}\left(H_m^{(\ell)}(\alpha,\varepsilon) - p^m h_{p,\varepsilon}^{(\ell)} \sum_{\substack{0\leq \mathfrak{a}<p\\ (p,\alpha+|\mathfrak{a}|)\neq 1}} \varepsilon^{\mathfrak{a}} H_m^{(\ell)}\left(\frac{\alpha+|\mathfrak{a}|}{p},\varepsilon^p\right)\right) \\ &= (-1)^\ell((m+\ell)\cdots(m+1))^{-1}(1-p^m)(r^{1+m}-1)B_{m+\ell}^{(\ell)}(\alpha). \end{aligned} \tag{3.10}$$

From Theorem 3.4 and Theorem 3.5, and the relation (3.9) we have the following result.

Theorem 3.6. *There exists a continuous extension of the function*

$$\varepsilon^x(1-p^{-s})(r^{1-s}-1)\zeta_\ell(s,x)$$

from the dense subset $\{0,-1,-2,\ldots\}$ *to the entire* $\mathcal{X}_p$ *as well as an integral representation*

$$\zeta_{p,\ell}(s,x) = \frac{1}{\varepsilon^x(r^{1-s}-1)} \int_{\substack{\mathbb{Z}_p^\ell\\ |\mathfrak{x}|\in\mathbb{Z}_p^\times}} (x+|\mathfrak{x}|)^{-s} d\mu(\mathfrak{x}),$$

where $\mu = \sum_{\varepsilon^r=1,\varepsilon\neq 1} \varepsilon^\alpha \mu_\varepsilon/(1-\varepsilon)^\ell$.

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