∗-EINSTEIN SOLITONS AND LP-SASAKIAN MANIFOLDS

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ABSTRACT. The aim of the present paper is to study LP-Sasakian manifolds admitting ∗-Einstein soliton satisfying certain curvature conditions. Finally, we have constructed a 3-dimensional example of an LP-Sasakian manifold admitting ∗-Einstein soliton.

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1. Introduction

In the beginning of 2016, Catino and Mazzieri [4] proposed a new notion on a Riemannian manifold called “Einstein-soliton”, which generates self-similar solution to Einstein flow \( \frac{\partial g}{\partial t} = 2(\frac{r}{2}g - S) \), and is governed through the equation

\[
(\mathcal{L}_\xi g)(V_1, V_2) + (2\Lambda - r)g(V_1, V_2) + 2S(V_1, V_2) = 0
\]

for any vector fields \( V_1, V_2 \) on \( M \), where \( \mathcal{L}_\xi \) denotes the Lie derivative operator in the direction of vector field \( \xi \), \( S \) is the Ricci tensor, \( r \) is the scalar curvature of the Riemannian metric \( g \) and \( \Lambda \in \mathbb{R} \) (\( \mathbb{R} \) is the set of real numbers). The Einstein soliton is called shrinking, steady or expanding if \( \Lambda < 0, = 0 \) or \( > 0 \), respectively.

In [15], the authors studied ∗-Ricci soliton in real hypersurfaces of complex space forms and defined by the equation

\[
(\mathcal{L}_V g)(V_1, V_2) + 2Ag(V_1, V_2) + 2S^*(V_1, V_2) = 0
\]

where

\[
S^*(V_1, V_2) = \text{Trace}\{\phi \circ R(V_1, \phi V_2)\},
\]

where \( \mathcal{L}_V \) denotes the Lie derivative operator in the direction of vector field \( V \) and \( S^* \) is a tensor field of type (0, 2). It is to be noted that the notion of ∗-Ricci

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tensor was first introduced by Tachibana [27] on almost Hermitian manifolds and further studied by Hamada [10] on real hypersurfaces of non-flat complex space forms.

We introduce a new notion by replacing the Ricci tensor $S$ by the $*$-Ricci tensor $S^*$ and the scalar curvature $r$ by the $*$-scalar curvature $r^*$ in (1), and call as $*$--Einstein soliton (in short, $*$-ES). The $*$-ES is defined by the following equation

\[(\mathcal{L}_\xi g)(V_1, V_2) + (2A - r^*)g(V_1, V_2) + 2S^*(V_1, V_2) = 0,\]  

where the symbols $\mathcal{L}_\xi$ and $A$ are same as defined in (1). Likewise Einstein soliton, the nature of $*$-ES depends on the values of $A$ such that if $A > 0$, $= 0$ or $< 0$, then the soliton is said to be expanding, steady or shrinking, respectively. We remark that the notions of $*$--Ricci soliton and $*$-ES are different for the manifolds of non-constant scalar curvature and if the scalar curvature is constant, then these notions coincide.

Analogously to the Sasakian manifolds, in 1989, Matsumoto [17] introduced the notion of $L^P$-Sasakian manifolds, while in 1992, the same notion was independently studied by Mihai and Rosca [19] and they obtained several results on this manifold. The Lorentzian para-Sasakian manifolds have also been studied by various authors such as [1, 6, 7, 22, 24, 25] and many others. Very recently, the authors Chaubey and Suh [5] studied the properties of almost Ricci solitons and gradient almost Ricci solitons on Lorentzian manifolds; and Haseeb and Almusawa [12] studied Lorentzian para-Kenmotsu manifolds admitting $\eta$-Ricci solitons. Also, we recommend the papers [2, 8, 9, 11, 13, 16, 23, 26, 28] for more details about the related studies in different spaces.

Throughout the manuscript, we denote an $n$-dimensional $L^P$-Sasakian manifold by $M^n(LPS)$ and $*$-Einstein soliton by $*$-ES. The present paper is organized as follows: After preliminaries in section 3, we study $*$-ES in an $M^n(LPS)$. In section 4, we consider the Ricci semi-symmetric $M^n(LPS)$ admitting $*$-ES. In section 5, we consider an $M^n(LPS)$ endowed with $*$-ES, and it is shown that the manifolds with the $*$-ES satisfying the curvature conditions: $P(V_1, \xi)\cdot S = 0$, $R(V_1, \xi)\cdot P = 0$ and $S(V_1, \xi)\cdot P = 0$ are Einstein manifolds; moreover we discussed the conditions for the $*$-ES to be expanding, steady and shrinking in each case. Section 6 deals with the study of Einstein semi-symmetric $M^n(LPS)$ admitting $*$-ES. In section 7, we proved that an $M^n(LPS)$ endowed with $*$-ES satisfying the curvature condition $S(\xi, V_1)\cdot R = 0$ is an $\eta$-Einstein manifold and the $*$-ES is shrinking. In section 8, we proved that an $M^n(LPS)$ admitting $*$-ES with torse-forming vector field is a generalized quasi-Einstein manifold. In the last section 9, we construct an example of an $M^3(LPS)$ which verifies the expanding case of $*$-ES.

2. Preliminaries

Let $M$ be an $n$-dimensional smooth manifold equipped with a triple $(\phi, \xi, \eta)$, where $\phi$ is a tensor field of type $(1,1)$, $\xi$ is the unit timelike vector field, $\eta$ is a
1-form such that [3, 21]
\[ \phi^2 V_1 = V_1 + \eta(V_1)\xi, \quad \eta(\xi) = -1, \] (4)
which implies
\[ \phi \xi = 0, \quad \eta(\phi V_1) = 0, \quad \text{rank}(\phi) = n - 1 \] (5)
for all \( V_1 \in \chi(M) \); where \( \chi(M) \) denotes the set of all smooth vector fields of \( M \).

The manifold \( M \) is called a Lorentzian almost paracontact manifold with structure \( (\phi, \xi, \eta, g) \) if it admits a Lorentzian metric \( g \), such that
\[ g(\phi V_1, \phi V_2) = g(V_1, V_2) + \eta(V_1)\eta(V_2), \quad g(V_1, \xi) = \eta(V_1) \] (6)
for all \( V_1, V_2 \in \chi(M) \), and the manifold with the structure \( (\phi, \xi, \eta, g) \) is called a Lorentzian almost paracontact manifold [17, 18]. A Lorentzian almost paracontact manifold \( M \) equipped with the structure \( (\phi, \xi, \eta, g) \) is called a Lorentzian para-Sasakian manifold [17] if
\[ (\nabla_{V_1}\phi)V_2 = \eta(V_2)\phi^2 V_1 + g(\phi V_1, \phi V_2)\xi, \] (7)
where \( \nabla \) stands for the Levi-Civita connection.

In an \( M^n(LPS) \), the following relations hold [7]:
\[ \nabla_{V_1}\xi = \phi V_1 \iff (\nabla_{V_1}\eta)V_2 = g(\phi V_1, V_2), \] (8)
\[ g(R(V_1, V_2)\xi, \xi) = \eta(R(V_1, V_2)\xi) = g(V_2, V_3)\eta(V_1) - g(V_1, V_3)\eta(V_2), \] (9)
\[ R(\xi, V_1)V_2 = g(V_1, V_2)\xi - \eta(V_2)V_1, \] (10)
\[ R(V_1, V_2)\xi = \eta(V_2)V_1 - \eta(V_1)V_2, \] (11)
\[ R(\xi, V_1)\xi = V_1 + \eta(V_1)\xi, \] (12)
\[ S(V_1, \xi) = (n - 1)\eta(V_1), \] (13)
\[ (\xi g)(V_1, V_2) = 2g(\phi V_1, V_2) \] (14)
for all \( V_1, V_2, V_3 \) on \( M^n(LPS) \), where \( R \) and \( Q \) represent the curvature tensor and the Ricci operator of \( M^n(LPS) \), respectively.

An \( M^n(LPS) \) is said to be generalized \( \eta \)-Einstein manifold if its non-vanishing Ricci tensor \( S \) is of the form [30]
\[ S(V_1, V_2) = d_1g(V_1, V_2) + d_2\eta(V_1)\eta(V_2) + d_3g(\phi V_1, V_2), \] (15)
where \( d_1, d_2 \) and \( d_3 \) are smooth functions on \( M^n(LPS) \). If \( d_3 = 0 \) (resp., \( d_2 = d_3 = 0 \)), then \( M^n(LPS) \) is called an \( \eta \)-Einstein (resp., an Einstein) manifold.

Recently, Haseeb and Chaubey [13] studied the properties of Lorentzian para-Sasakian manifolds equipped with \( * \)-Ricci tensor and proved the following useful lemma:

**Lemma 2.1.** [13] In an \( M^n(LPS) \), the \( * \)-Ricci tensor is given by
\[ S^*(V_2, V_3) = S(V_2, V_3) + (n - 2)g(V_2, V_3) - g(V_1, \phi V_3)a \]
\[ + (2n - 3)\eta(V_2)\eta(V_3) \] (16)
for any \( V_2, V_3 \) on \( M^n(LPS) \), where \( a = \text{trace } \phi \).
By contacting (16) over \( V_2 \) and \( V_3 \), we find
\[
r^* = r + (n - 1)(n - 3) - a^2.
\]

3. \(*-Einstein solitons in \( LP\)-Sasakian manifolds

Let us consider an \( M^n(LPS) \) admitting a \(*\)-ES, then (3) holds. Thus, by using (14) in (3), we have
\[
S^*(V_1, V_2) = (A - \frac{r^*}{2})g(V_1, V_2) - g(\phi V_1, V_2)
\]
which by virtue of (16) takes the form
\[
S(V_1, V_2) = A_1 g(V_1, V_2) + A_2 g(\phi V_1, V_2) + A_3 \eta(V_1) \eta(V_2),
\]
where \( A_1 = -(A + n - 2 - \frac{r^*}{2}) \), \( A_2 = (a - 1) \) and \( A_3 = -(2n - 3) \). Now by fixing \( V_2 = \xi \) in (18), we find
\[
S(V_1, \xi) = A_4 \eta(V_1),
\]
where \( A_4 = (n - A - 1 + \frac{r^*}{2}) \). On comparing (19) and (13) it follows that
\[
A = \frac{r^*}{2},
\]
where \( \eta(V_1) \neq 0 \). By contracting (18), we obtain \( r = \frac{2n A}{n-2} - (n - 1)(n - 3) + a^2 + \frac{2a}{n-2} \). This relation together with (17) leads to \( r^* = \frac{2}{n-2}(nA + a) \). Consequently, from the relation (20) it follows that \( A = -\frac{2}{n-2} \). Thus, we state:

**Theorem 3.1.** An \( M^n(LPS) \) \((n > 3)\) admitting a \(*\)-ES is a generalized \( \eta\)-Einstein manifold. Moreover, the \(*\)-ES is expanding, steady or shrinking according to \( a < 0 \), \( a = 0 \) or \( a > 0 \), respectively.

In particular, for \( a = 1 \), we have \( A = -\frac{2}{3} \) and hence (20) gives \( r^* = -1 \). Now, by using these values of \( A \) and \( r \), (18) reduces to \( S(V_1, V_2) = (n - 2)g(V_1, V_2) - (2n - 3)\eta(V_1)\eta(V_2) \). Thus, we have

**Corollary 3.2.** If an \( M^n(LPS) \) \((n > 3) \) admits \(*\)-ES, then \( M \) is an \( \eta\)-Einstein manifold and the soliton is always shrinking, provided \( a = 1 \).

4. Ricci semi-symmetric \( LP\)-Sasakian manifolds admitting
\(*-Einstein solitons

In 1992, Mirzoyan [20] introduced the notion of Ricci semi-symmetric for Riemann spaces. The geometrical interpretation of the \((0, 4)\)-tensor \( R \cdot S \), obtained by the action of the curvature operator \( R(V_1, V_2) \) on the \((0, 2)\)-symmetric Ricci tensor,

\[
(R \cdot S)(V_3, V_4; V_1, V_2) = (R(V_1, V_2) \cdot S)(V_3, V_4)
\]
\[
= -S(R(V_1, V_2)V_3, V_4) - S(V_3, R(V_1, V_2)V_4),
\]
where \( V_1, V_2, V_3, V_4 \) on \( M \). A Riemannian (semi-Riemannian) manifold \( M \) is said to be Ricci semi-symmetric when \( R \cdot S \) vanishes identically, i.e., \( R \cdot S = 0 \).
Now, consider a vector $v$ at any point $p \in M$ and any coordinate parallelogram $\mathcal{P}$ cornered at $p$ with sides of lengths $\Delta x$ and $\Delta y$ tangent to the linearly independent vectors $x$ and $y$ at $p$. Then, by parallel transport of $v$ around $\mathcal{P}$ we obtain the vector

$$v^* = v + [R(x, y)v]\Delta x \Delta y + O^2(\Delta x, \Delta y),$$

so that

$$\text{Ric}(v^*) = \text{Ric}(v) - [(R \cdot S)(v, v^*, x, y)]\Delta x \Delta y + O^2(\Delta x, \Delta y),$$

where the Ricci curvature in the direction of $v^*$ is denoted by $\text{Ric}(v^*)$.

**Theorem 4.1.** [14] Let $\mathcal{P}$ be any infinitesimal coordinate parallelogram cornered at a point $p$ of $M$ with sides of lengths $\Delta x$ and $\Delta y$, which are tangent to vectors $x$ and $y$ at $p$. Let $v^*$ be the vector obtained from $v$ after parallel transport along $\mathcal{P}$. Then, in second-order approximation,

$$\delta_\mathcal{P}\text{Ric}(v) = -(R \cdot S)(v, v^*, x, y)\Delta x \Delta y,$$

i.e., the $(0,4)$-tensor $R \cdot S$ of $M$ measures the change in Ricci curvature at any point $p$ for any vector $v$ under parallel transport of $v$ around any infinitesimal coordinate parallelogram $\mathcal{P}$ cornered at $p$.

In this section we consider a $*$-ES soliton in an $M^n(LPS)$ which is Ricci semi-symmetric, i.e., $R(V_1, V_2) \cdot S = 0$. This leads to

$$S(R(V_1, V_2)V_4, V_3) + S(V_4, R(V_1, V_2)V_3) = 0 \quad (21)$$

for $V_1, V_2, V_3, V_4$ on $M^n(LPS)$.

Putting $V_2 = \xi$ in (21) and then recalling (10), we arrive at

$$\eta(V_4)S(V_1, V_3) - g(V_1, V_4)S(\xi, V_3) + \eta(V_3)S(V_1, V_4) - g(V_1, V_3)S(V_4, \xi) = 0.$$

Again putting $V_4 = \xi$ and using (19), the foregoing equation takes the form

$$S(V_1, V_3) = (n - A - 1 + \frac{r^*}{2})g(V_1, V_3). \quad (22)$$

By means of the fact that in an $M^n(LPS)$ admitting a $*$-ES equation (20) holds. Thus, (22) turns to

$$S(V_1, V_3) = (n - 1)g(V_1, V_3). \quad (23)$$

On contracting (23) it follows that $r = n(n - 1)$, which in view of (20) and (17), we find $A = \frac{(n-1)(2n-3)-a^2}{2}$. Thus, we have

**Theorem 4.2.** Let an $M^n(LPS)$ be the Ricci semi-symmetric and admits a $*$-ES. Then $M^n(LPS)$ is an Einstein manifold and the $*$-ES is expanding, steady or shrinking according to $a^2 < (n - 1)(2n - 3)$, $= (n - 1)(2n - 3)$ or $> (n - 1)(2n - 3)$, respectively.
5. Projective curvature tensor in $LP$--Sasakian manifolds admitting $*$--Einstein solitons

The projective curvature tensor $P$ in an $M^n(LPS)$ is defined by

$$P(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \frac{1}{n-1}\{S(V_2, V_3)V_1 - S(V_1, V_3)V_2\}$$  \hspace{0.5cm} (24)

for all $V_1, V_2, V_3$ on $M^n(LPS)$.

In this section, we study an $M^n(LPS)$ admitting a $*$-ES satisfying certain curvature conditions on the projective curvature tensor.

First, we consider an $M^n(LPS)$ admitting a $*$-ES which satisfies the condition $P(V_1, \xi)\cdot S = 0$. Thus, we have

$$S(P(V_1, \xi)V_2, V_3) + S(V_2, P(V_1, \xi)V_3) = 0.$$  \hspace{0.5cm} (25)

From (10), (19) and (24), we find

$$P(V_1, \xi)V_2 = -g(V_1, V_2)\xi + (1 - \frac{A_4}{n-1})\eta(V_2)V_1 + \frac{1}{n-1}\{S(V_1, V_3)\xi = 0.$$  \hspace{0.5cm} (26)

Plugging (26) into (25), we obtain

$$\eta(V_2)S(V_1, V_3) + \eta(V_3)S(V_1, V_2) - A_4g(V_1, V_2)\eta(V_3) - A_4g(V_1, V_3)\eta(V_2) = 0.$$  \hspace{0.5cm} (27)

From (27), we obtain $r = n(n-1)$, which in view of (20) and (17), we find

$$S(V_1, V_3) = (n-1)g(V_1, V_3).$$

Thus, we have

**Theorem 5.1.** If an $M^n(LPS)$ admitting a $*$-ES satisfies the condition $P(V_1, \xi)\cdot S = 0$. Then $M^n(LPS)$ is an Einstein manifold. Moreover, the $*$-ES is expanding, steady or shrinking according to $a^2 < (n-1)(2n-3)$, $=(n-1)(2n-3)$ or $>(n-1)(2n-3)$, respectively.

Next, we consider an $M^n(LPS)$ admitting a $*$-ES that satisfies the condition $R(V_1, \xi)\cdot P = 0$. Then we have

$$R(V_1, \xi)P(V_4, V_5)V_6 - P(R(V_1, \xi)V_4, V_5)V_6$$

$$- P(V_4, R(V_1, \xi)V_5)V_6 - P(V_4, V_5)R(V_1, \xi)V_6 = 0$$

for any $V_1, V_4, V_5, V_6$ on $M^n(LPS)$.

By fixing $V_4 = V_6 = \xi$ in (28), we have

$$R(V_1, \xi)P(\xi, V_5)\xi - P(R(V_1, \xi)\xi, V_5)\xi$$

$$- P(\xi, R(V_1, \xi)V_5)\xi - P(\xi, V_5)R(V_1, \xi)\xi = 0.$$  \hspace{0.5cm} (29)

From (11), (19) and (24), we find

$$P(V_1, V_5)\xi = (1 - \frac{A_4}{n-1})(\eta(V_5)V_1 - \eta(V_1)V_5),$$  \hspace{0.5cm} (30)
If an $r$ and (20) it follows that

\[ P(\xi, V_5) V_1 = -(1 - \frac{A_4}{n - 1}) \eta(V_1) V_5 + g(V_1, V_5) \xi - \frac{1}{n - 1} S(V_1, V_5) \xi. \]  

(31)

In view of (10), (30) and (31), after some steps calculation, (29) reduces to $S(V_1, V_5) \xi = A_4 g(V_1, V_5) \xi$, which by taking the inner product with $\xi$ and using (20) gives

\[ S(V_1, V_5) = (n - 1) g(V_1, V_5). \]  

(32)

On contracting (32), we obtain $r = n(n - 1)$, which in view of (20) and (17), we find $A = \frac{(n - 1)(2n - 3) - a^2}{2}$. Thus, we have

**Theorem 5.2.** If an $M^n(LPS)$ admitting a $*\text{-ES}$ satisfies the condition $R(V_1, \xi) \cdot P = 0$. Then $M^n(LPS)$ is an Einstein manifold and the $*\text{-ES}$ is expanding, steady or shrinking according to $a^2 < (n - 1)(2n - 3)$, $= (n - 1)(2n - 3)$ or $> (n - 1)(2n - 3)$, respectively.

Further, we consider an $M^n(LPS)$ admitting a $*\text{-ES}$ that satisfies the condition $S(\xi, V_1) \cdot P = 0$. Then we have

\[ S(V_1, P(V_4, V_5) V_6) \xi - S(\xi, P(V_4, V_5) V_6) V_1 + S(V_1, V_4) P(\xi, V_5) V_6 \]

\[- S(\xi, V_4) P(V_1, V_5) V_6 + S(V_1, V_5) P(V_4, \xi) V_6 - S(\xi, V_5) P(V_4, V_1) V_6 \]

\[ + S(V_1, V_6) P(V_4, V_5) \xi - S(\xi, V_6) P(V_4, V_5) V_1 = 0 \]

(33)

for all $V_1, V_4, V_5, V_6$ on $M^n(LPS)$. Putting $V_4 = V_6 = \xi$ in (33), we have

\[ S(V_1, P(\xi, V_5) \xi) - S(\xi, P(\xi, V_5) \xi) V_1 + S(V_1, \xi) P(\xi, V_5) \xi - S(\xi, \xi) P(V_1, V_5) \xi \]

\[ + S(V_1, V_5) P(\xi, \xi) V_1 + S(V_1, \xi) P(\xi, V_5) \xi - S(\xi, \xi) P(V_1, V_5) V_1 = 0, \]

which in view of (19), (30), (31) and $\eta(P(\xi, V_5) \xi) = 0$ leads to

\[ A_4 g(V_1, V_5) \xi + 2 A_4 (1 - \frac{A_4}{n - 1}) \eta(V_1) \eta(V_5) \xi + (1 - 2 \frac{A_4}{n - 1}) S(V_1, V_5) \xi = 0. \]

By taking the inner product of the foregoing equation with $\xi$ then using (4), (6) and (20) it follows that

\[ S(V_1, V_5) = (n - 1) g(V_1, V_5). \]  

(34)

On contracting (34), we obtain $r = n(n - 1)$, which with the relations (20) and (17) gives $A = \frac{(n - 1)(2n - 3) - a^2}{2}$. Thus, we have

**Theorem 5.3.** If an $M^n(LPS)$ admitting a $*\text{-ES}$ satisfies the condition $S(\xi, V_1) \cdot P = 0$. Then $M^n(LPS)$ is an Einstein manifold and the $*\text{-ES}$ is expanding, steady or shrinking according to $a^2 < (n - 1)(2n - 3)$, $= (n - 1)(2n - 3)$ or $> (n - 1)(2n - 3)$, respectively.
6. Einstein semi-symmetric \( LP \)-Sasakian manifolds admitting 
\(-\)Einstein solitons

**Definition 6.1.** An \( M^n(LPS) \) is called Einstein semi-symmetric if \( R \cdot E = 0 \), where \( E \) is the Einstein tensor given by

\[
E(V_1, V_2) = S(V_1, V_2) - \frac{r}{n}g(V_1, V_2),
\]

where \( r \) is the scalar curvature of the manifold.

Let us consider an \( M^n(LPS) \) admitting a \(-\)ES, which is Einstein semi-symmetric, i.e., \( R \cdot E = 0 \). Thus, we have

\[
E(R(V_1, V_2)V_3, V_4) + E(V_3, R(V_1, V_2)V_4) = 0,
\]

which in view of (35) takes the form

\[
S(R(V_1, V_2)V_3, V_4) + S(V_3, R(V_1, V_2)V_4) = \frac{r}{n}\{g(R(V_1, V_2)V_3, V_4) + g(V_3, R(V_1, V_2)V_4)\}.
\]

By putting \( V_1 = V_3 = \xi \) in (36), we have

\[
S(R(\xi, V_2)\xi, V_4) + S(\xi, R(\xi, V_2)V_4) = \frac{r}{n}\{g(R(\xi, V_2)\xi, V_4) + g(\xi, R(\xi, V_2)V_4)\}.
\]

By making the use of (10), (12) and (19), the foregoing equation leads to

\[
S(V_2, V_4) = A_4g(V_2, V_4),
\]

which by using (20) turns to

\[
S(V_2, V_4) = (n - 1)g(V_2, V_4).
\]

On contracting (37), we obtain \( r = n(n - 1) \). Using this value of \( r \) in (20) and using (17), we obtain \( A = \frac{(n - 1)(2n - 3) - a^2}{2} \). Thus, we have

**Theorem 6.2.** Let an Einstein semi-symmetric \( M^n(LPS) \) admit a \(-\)ES. Then \( M^n(LPS) \) is an Einstein manifold and the \(-\)ES is expanding, steady or shrinking according to \( a^2 < (n - 1)(2n - 3) = (n - 1)(2n - 3) \) or \( (n - 1)(2n - 3) > (n - 1)(2n - 3) \), respectively.

7. \(-\)Einstein solitons in \( LP \)-Sasakian manifolds satisfying

\[
(S(V_1, V_2) \cdot R)(V_4, V_5)V_6 = 0
\]

Let an \( M^n(LPS) \) admitting a \(-\)ES satisfies \( (S(V_1, V_2) \cdot R)(V_4, V_5)V_6 = 0 \). Then we have

\[
S(V_2, R(V_4, V_5)V_6)V_1 - S(V_1, R(V_4, V_5)V_6)V_2 + S(V_2, V_4)R(V_1, V_5)V_6 - S(V_1, V_5)R(V_4, V_2)V_6
\]

\[
- S(V_1, V_4)R(V_2, V_5)V_6 + S(V_2, V_5)R(V_4, V_1)V_6 - S(V_1, V_5)R(V_4, V_2)V_6
\]

\[
+ S(V_2, V_6)R(V_4, V_5)V_1 - S(V_1, V_6)R(V_4, V_5)V_2 = 0,
\]

which by taking the inner product with \( \xi \) takes the form

\[
S(V_2, R(V_4, V_5)V_6)\eta(V_1) - S(V_1, R(V_4, V_5)V_6)\eta(V_2) + S(V_2, V_4)\eta(R(V_1, V_5)V_6)
\]

\[
- S(V_1, V_4)\eta(R(V_2, V_5)V_6) + S(V_2, V_5)\eta(R(V_4, V_1)V_6) - S(V_1, V_5)\eta(R(V_4, V_2)V_6)
\]

\[
+ S(V_2, V_6)\eta(R(V_4, V_5)V_1) - S(V_1, V_6)\eta(R(V_4, V_5)V_2) = 0.
\]
Putting $V_4 = V_6 = \xi$ in (38), then using (10)-(12) and (19) we find
\[ S(V_2, V_5)\eta(V_1) = S(V_1, V_5)\eta(V_2) + A_4 g(V_5, V_1)\eta(V_2) - A_4 g(V_5, V_2)\eta(V_1). \]

Again putting $V_1 = \xi$ in the foregoing equation and using (4), (4), (19), we obtain
\[ S(V_2, V_5) = -A_4 g(V_2, V_5) - 2A_4 \eta(V_2)\eta(V_5), \]
which by using (20) turns to
\[ S(V_2, V_5) = -(n-1)g(V_2, V_5) - 2(n-1)\eta(V_2)\eta(V_5). \]  
(39)

On contracting (39), we obtain $r = -(n-1)(n-2)$. Thus, by virtue of (20) and (17) we obtain $A = -\frac{n-1+n^2}{2}$. This helps us to state

**Theorem 7.1.** If an $M^n(LPS)$ admitting a $\ast$-ES satisfies $S(V_1, \xi) \cdot R = 0$, then $M^n(LPS)$ is an $\eta$-Einstein manifold and the soliton is always shrinking.

### 8. $\ast$-Einstein solitons in $LP$–Sasakian manifolds with torse-forming vector field

**Definition 8.1.** A vector field $U$ in an $M^n(LPS)$ is said to be torse-forming vector field if
\[ \nabla V_1 U = fV_1 + \gamma(V_1)U, \]
where $f$ is a smooth function and $\gamma$ is a 1-form.

Let us consider an $M^n(LPS)$ admitting a $\ast$-ES, further considering the Reeb vector field $\xi$ as a torse-forming vector field. Thus, from (40) we have
\[ \nabla V_1 \xi = fV_1 + \gamma(V_1)\xi \]
for any $V_1$ on $M^n(LPS)$.

Taking the inner product of (41) with $\xi$ we lead to
\[ g(\nabla V_1 \xi, \xi) = f\eta(V_1) - \gamma(V_1). \]

Also from (8), we obtain
\[ g(\nabla V_1 \xi, \xi) = 0. \]  
(43)

Thus, from the last two equations we find $\gamma = f\eta$, and hence (41) turns to
\[ \nabla V_1 \xi = f(V_1 + \eta(V_1)\xi). \]

Now, in view of (44), we have
\[ (\mathcal{L}_\xi g)(V_1, V_2) = 2f\{g(V_1, V_2) + \eta(V_1)\eta(V_2)\}. \]

By virtue of (45), (3) turns to
\[ S^*(V_1, V_2) = -(A - \frac{r}{2} + f)g(V_1, V_2) - f\eta(V_1)\eta(V_2), \]
which by using (18) yields
\[
S(V_1, V_2) = -(A - \frac{r}{2} + f + n - 2)g(V_1, V_2) + ag(V_1, \phi V_2) - (2n - 3 + f)\eta(V_1)\eta(V_2).
\]
By recalling (20) in the foregoing equation, we arrive at
\[ S(V_1, V_2) = -(f + n - 2)g(V_1, V_2) + ag(V_1, \phi V_1) \]
\[ - (2n - 3 + f)\eta(V_1)\eta(V_2), \]
which is a generalized $\eta$-Einstein manifold.

On contracting (46), we obtain $r = -(n - 1)(f + n - 3) + a^2$, and hence from (17) and (20) we obtain $A = -\frac{(n - 1)f}{2}$. Thus, we have

**Theorem 8.2.** Let an $M^n(\text{LPS})$ admit a *-ES with a torse-forming vector field $\xi$. Then $M$ is a generalized $\eta$-Einstein manifold and the soliton is expanding, steady or shrinking according to $f < 0$, $= 0$ or $> 0$, respectively.

9. Examples

We consider the 3-dimensional manifold $M^3 = \{(v_1, v_2, v_3) \in \mathbb{R}^3\}$, where $(v_1, v_2, v_3)$ are the standard coordinates in $\mathbb{R}^3$. Let $\varrho_1$, $\varrho_2$ and $\varrho_3$ be the vector fields on $M^3$ given by
\[ \varrho_1 = e^{-v_3} \frac{\partial}{\partial v_1}, \quad \varrho_2 = e^{-v_3} \left( \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2} \right), \quad \varrho_3 = - \frac{\partial}{\partial v_3} = \xi. \]

Let $g$ be the semi-Riemannian metric defined by
\[ g(\varrho_k, \varrho_l) = \begin{cases} 1, & 1 \leq k = l \leq 2, \\ -1, & k = l = 3, \\ 0, & 1 \leq k \neq l \leq 3. \end{cases} \]

Let $\eta$ be the 1-form on $M$ defined by $\eta(V_1) = g(V_1, \varrho_3)$ for all $V_1 \in \mathfrak{X}(M^3)$. Let $\phi$ be the $(1, 1)$ tensor field on $M^3$ defined by
\[ \phi \varrho_1 = -\varrho_1, \quad \phi \varrho_2 = -\varrho_2, \quad \phi \varrho_3 = 0. \]

By applying the linearity of $\phi$ and $g$, we have
\[ \eta(\xi) = -1, \quad \phi^2 V_1 = V_1 + \eta(V_1)\xi, \quad \eta(\phi V_1) = 0, \]
\[ g(V_1, \xi) = \eta(V_1), \quad g(\phi V_1, \phi V_2) = g(V_1, V_2) + \eta(V_1)\eta(V_2) \]
for $V_1, V_2 \in \chi(M^3)$. Then we have
\[ [\varrho_1, \varrho_2] = 0, \quad [\varrho_1, \varrho_3] = -\varrho_4 \quad [\varrho_2, \varrho_3] = -\varrho_2. \]

By using well-known Koszul’s formula, we can easily calculate
\[ \nabla_{\varrho_1} \varrho_1 = -\varrho_3, \quad \nabla_{\varrho_1} \varrho_2 = 0, \quad \nabla_{\varrho_1} \varrho_3 = -\varrho_1, \]
\[ \nabla_{\varrho_2} \varrho_1 = 0, \quad \nabla_{\varrho_2} \varrho_2 = -\varrho_3, \quad \nabla_{\varrho_2} \varrho_3 = -\varrho_2, \]
\[ \nabla_{\varrho_3} \varrho_1 = 0, \quad \nabla_{\varrho_3} \varrho_2 = 0, \quad \nabla_{\varrho_3} \varrho_3 = 0. \]

It can be easily shown that $M^3$ is an LP-Sasakian manifold. By using the above results, one can easily obtain the following components of the curvature tensor:
\[ R(\varrho_1, \varrho_2) \varrho_2 = \varrho_1, \quad R(\varrho_1, \varrho_3) \varrho_3 = -\varrho_1, \quad R(\varrho_1, \varrho_2) \varrho_1 = -\varrho_2, \]
\[ R(\varrho_2, \varrho_3)\varrho_3 = -\varrho_2, \quad R(\varrho_1, \varrho_3)\varrho_1 = -\varrho_3, \quad R(\varrho_2, \varrho_3)\varrho_2 = -\varrho_3. \]

From these curvature tensors, we can easily calculate
\[ S(\varrho_1, \varrho_1) = S(\varrho_2, \varrho_2) = 2, \quad S(\varrho_3, \varrho_3) = -2. \] (47)

Thus, we find \( r = 6 \). Putting \( V_1 = V_2 = \xi \) in (18) and using the values \( r = 6 \) and \( S(\varrho_3, \varrho_3) = -2 \), we obtain \( \Lambda = 3 \). Thus, an expanding case of \( {}^* \)-Einstein solitons is verified by the given example.

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