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$*$-EINSTEIN SOLITONS AND LP-SASAKIAN MANIFOLDS

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ABSTRACT. The aim of the present paper is to study LP-Sasakian manifolds admitting $*$-Einstein soliton satisfying certain curvature conditions. Finally, we have constructed a 3-dimensional example of an LP-Sasakian manifold admitting $*$-Einstein soliton.

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1. Introduction

In the beginning of 2016, Catino and Mazzieri [4] proposed a new notion on a Riemannian manifold called "Einstein-soliton", which generates self-similar solution to Einstein flow $\frac{\partial}{\partial t}g = 2(\frac{r}{2}g - S)$, and is governed through the equation

$$(\pounds_\xi g)(V_1, V_2) + (2\Lambda - r)g(V_1, V_2) + 2S(V_1, V_2) = 0 \tag{1}$$

for any vector fields V_1, V_2 on M, where $\pounds_\xi$ denotes the Lie derivative operator in the direction of vector field ξ, S is the Ricci tensor, r is the scalar curvature of the Riemannian metric g and $\Lambda \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers). The Einstein soliton is called shrinking, steady or expanding if $\Lambda < 0, = 0$ or > 0, respectively.

In [15], the authors studied $*$-Ricci soliton in real hypersurfaces of complex space forms and defined by the equation

$$(\pounds_V g)(V_1, V_2) + 2\Lambda g(V_1, V_2) + 2S^*(V_1, V_2) = 0, \tag{2}$$

where

$$S^*(V_1, V_2) = \text{Trace}\left\{\phi \circ R(V_1, \phi V_2)\right\},$$

where $\pounds_V$ denotes the Lie derivative operator in the direction of vector field V and S^* is a tensor field of type $(0, 2)$. It is to be noted that the notion of $*$-Ricci

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tensor was first introduced by Tachibana [27] on almost Hermitian manifolds and further studied by Hamada [10] on real hypersurfaces of non-flat complex space forms.

We introduce a new notion by replacing the Ricci tensor S by the $*$-Ricci tensor S^* and the scalar curvature r by the $*$-scalar curvature r^* in (1), and call as $*-$Einstein soliton (in short, $*$-ES). The $*$-ES is defined by the following equation

$$(\pounds_\xi g)(V_1,V_2)+(2\Lambda-r^*)g(V_1,V_2)+2S^*(V_1,V_2)=0, \tag{3}$$

where the symbols $\pounds_\xi$ and Λ are same as defined in (1). Likewise Einstein soliton, the nature of $*$-ES depends on the values of Λ such that if $\Lambda>0,=0$ or <0, then the soliton is said to be expanding, steady or shrinking, respectively. We remark that the notions of $*-$Ricci soliton and $*$-ES are different for the manifolds of non-constant scalar curvature and if the scalar curvature is constant, then these notions coincide.

Analogously to the Sasakian manifolds, in 1989, Matsumoto [17] introduced the notion of LP-Sasakian manifolds, while in 1992, the same notion was independently studied by Mihai and Rosca [19] and they obtained several results on this manifold. The Lorentzian para-Sasakian manifolds have also been studied by various authors such as [1, 6, 7, 22, 24, 25] and many others. Very recently, the authors Chaubey and Suh [5] studied the properties of almost Ricci solitons and gradient almost Ricci solitons on Lorentzian manifolds; and Haseeb and Almusawa [12] studied Lorentzian para-Kenmotsu manifolds admitting η-Ricci solitons. Also, we recommend the papers [2, 8, 9, 11, 13, 16, 23, 26, 28] for more details about the related studies in different spaces.

Throughout the manuscript, we denote an n-dimensional LP-Sasakian manifold by $M^n(LPS)$ and $*$-Einstein soliton by $*$-ES. The present paper is organized as follows: After preliminaries in section 3, we study $*$-ES in an $M^n(LPS)$. In section 4, we consider the Ricci semi-symmetric $M^n(LPS)$ admitting $*$-ES. In section 5, we consider an $M^n(LPS)$ endowed with $*$-ES, and it is shown that the manifolds with the $*$-ES satisfying the curvature conditions: $P(V_1,\xi)\cdot S=0$, $R(V_1,\xi)\cdot P=0$ and $S(V_1,\xi)\cdot P=0$ are Einstein manifolds; moreover we discussed the conditions for the $*$-ES to be expanding, steady and shrinking in each case. Section 6 deals with the study of Einstein semi-symmetric $M^n(LPS)$ admitting $*$-ES. In section 7, we proved that an $M^n(LPS)$ endowed with $*$-ES satisfying the curvature condition $S(\xi,V_1)\cdot R=0$ is an η-Einstein manifold and the $*$-ES is shrinking. In section 8, we proved that an $M^n(LPS)$ admitting $*$-ES with torse-forming vector field is a generalized quasi-Einstein manifold. In the last section 9, we construct an example of an $M^3(LPS)$ which verifies the expanding case of $*$-ES.

2. Preliminaries

Let M be an n-dimensional smooth manifold equipped with a triple (ϕ,ξ,η), where ϕ is a tensor field of type $(1,1)$, ξ is the unit timelike vector field, η is a

1-form such that [3, 21]

$$\phi^2 V_1 = V_1 + \eta(V_1)\xi, \quad \eta(\xi) = -1, \tag{4}$$

which implies

$$\phi\xi = 0, \quad \eta(\phi V_1) = 0, \quad rank(\phi) = n - 1 \tag{5}$$

for all $V_1 \in \chi(M)$; where $\chi(M)$ denotes the set of all smooth vector fields of M.

The manifold M is called a Lorentzian almost paracontact manifold with structure (ϕ, ξ, η, g) if it admits a Lorentzian metric g, such that

$$g(\phi V_1, \phi V_2) = g(V_1, V_2) + \eta(V_1)\eta(V_2), \quad g(V_1, \xi) = \eta(V_1) \tag{6}$$

for all $V_1, V_2 \in \chi(M)$, and the manifold with the structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact manifold [17, 18]. A Lorentzian almost paracontact manifold M equipped with the structure (ϕ, ξ, η, g) is called a Lorentzian para-Sasakian manifold [17] if

$$(\nabla_{V_1}\phi)V_2 = \eta(V_2)\phi^2 V_1 + g(\phi V_1, \phi V_2)\xi, \tag{7}$$

where ∇ stands for the Levi-Civita connection.

In an $M^n(LPS)$, the following relations hold [7]:

$$\nabla_{V_1}\xi = \phi V_1 \iff (\nabla_{V_1}\eta)V_2 = g(\phi V_1, V_2) = g(V_1, \phi V_2), \tag{8}$$

$$g(R(V_1, V_2)V_3, \xi) = \eta(R(V_1, V_2)V_3) = g(V_2, V_3)\eta(V_1) - g(V_1, V_3)\eta(V_2), \tag{9}$$

$$R(\xi, V_1)V_2 = g(V_1, V_2)\xi - \eta(V_2)V_1, \tag{10}$$

$$R(V_1, V_2)\xi = \eta(V_2)V_1 - \eta(V_1)V_2, \tag{11}$$

$$R(\xi, V_1)\xi = V_1 + \eta(V_1)\xi, \tag{12}$$

$$S(V_1, \xi) = (n-1)\eta(V_1), \tag{13}$$

$$(\pounds_\xi g)(V_1, V_2) = 2g(\phi V_1, V_2) \tag{14}$$

for all V_1, V_2, V_3 on $M^n(LPS)$, where R and Q represent the curvature tensor and the Ricci operator of $M^n(LPS)$, respectively.

An $M^n(LPS)$ is said to be generalized η-Einstein manifold if its non-vanishing Ricci tensor S is of the form [30]

$$S(V_1, V_2) = d_1 g(V_1, V_2) + d_2\eta(V_1)\eta(V_2) + d_3 g(\phi V_1, V_2), \tag{15}$$

where d_1, d_2 and d_3 are smooth functions on $M^n(LPS)$. If $d_3 = 0$ $(resp., d_2 = d_3 = 0)$, then $M^n(LPS)$ is called an η-Einstein (resp., an Einstein) manifold.

Recently, Haseeb and Chaubey [13] studied the properties of Lorentzian para-Sasakian manifolds equipped with $*$-Ricci tensor and proved the following useful lemma:

Lemma 2.1. [13] *In an $M^n(LPS)$, the $*$-Ricci tensor is given by*

$$\begin{aligned} S^*(V_2, V_3) \;=\; & S(V_2, V_3) + (n-2)g(V_2, V_3) - g(V_1, \phi V_3)a \\ & +(2n-3)\eta(V_2)\eta(V_3) \end{aligned} \tag{16}$$

for any V_2, V_3 on $M^n(LPS)$, where $a = trace\ \phi$.

By contacting (16) over V_2 and V_3, we find

$$r^* = r + (n-1)(n-3) - a^2. \tag{17}$$

3. $*-$Einstein solitons in $LP-$Sasakian manifolds

Let us consider an $M^n(LPS)$ admitting a $*$-ES, then (3) holds. Thus, by using (14) in (3), we have

$$S^*(V_1, V_2) = (\Lambda - \frac{r^*}{2})g(V_1, V_2) - g(\phi V_1, V_2)$$

which by virtue of (16) takes the form

$$S(V_1, V_2) = A_1 g(V_1, V_2) + A_2 g(\phi V_1, V_2) + A_3 \eta(V_1)\eta(V_2), \tag{18}$$

where $A_1 = -(\Lambda + n - 2 - \frac{r^*}{2}), A_2 = (a-1)$ and $A_3 = -(2n-3)$. Now by fixing $V_2 = \xi$ in (18), we find

$$S(V_1, \xi) = A_4 \eta(V_1), \tag{19}$$

where $A_4 = (n - \Lambda - 1 + \frac{r^*}{2})$. On comparing (19) and (13) it follows that

$$\Lambda = \frac{r^*}{2}, \tag{20}$$

where $\eta(V_1) \neq 0$. By contracting (18), we obtain $r = \frac{2n\Lambda}{n-2} - (n-1)(n-3) + a^2 + \frac{2a}{n-2}$. This relation together with (17) leads to $r^* = \frac{2}{n-2}(n\Lambda + a)$. Consequently, from the relation (20) it follows that $\Lambda = -\frac{a}{2}$. Thus, we state:

Theorem 3.1. *An $M^n(LPS)$ $(n > 3)$ admitting a $*-ES$ is a generalized η-Einstein manifold. Moreover, the $*$-ES is expanding, steady or shrinking according to $a < 0$, $a = 0$ or $a > 0$, respectively.*

In particular, for $a = 1$, we have $\Lambda = -\frac{1}{2}$ and hence (20) gives $r^* = -1$. Now, by using these values of Λ and r, (18) reduces to $S(V_1, V_2) = (n-2)g(V_1, V_2) - (2n-3)\eta(V_1)\eta(V_2)$. Thus, we have

Corollary 3.2. *If an $M^n(LPS)$ $(n > 3)$ admits $*$-ES, then M is an η-Einstein manifold and the soliton is always shrinking, provided $a = 1$.*

4. Ricci semi-symmetric $LP-$Sasakian manifolds admitting $*-$Einstein solitons

In 1992, Mirzoyan [20] introduced the notion of Ricci semi-symmetric for Riemann spaces. The geometrical interpretation of the $(0,4)$-tensor $R \cdot S$, obtained by the action of the curvature operator $R(V_1, V_2)$ on the $(0,2)$-symmetric Ricci tensor,

$$\begin{aligned}(R \cdot S)(V_3, V_4; V_1, V_2) &= (R(V_1, V_2) \cdot S)(V_3, V_4) \\ &= -S(R(V_1, V_2)V_3, V_4) - S(V_3, R(V_1, V_2)V_4),\end{aligned}$$

where V_1, V_2, V_3, V_4 on M. A Riemannian (semi-Riemannian) manifold M is said to be Ricci semi-symmetric when $R \cdot S$ vanishes identically, i.e., $R \cdot S = 0$.

Now, consider a vector v at any point $p \in M$ and any coordinate parallelogram $\mathcal{P}$ cornered at p with sides of lengths Δx and Δy tangent to the linearly independent vectors x and y at p. Then, by parallel transport of v around $\mathcal{P}$ we obtain the vector

$$v^* = v + [R(x, y)v]\Delta x\Delta y + O^{>2}(\Delta x, \Delta y),$$

so that

$$Ric(v^*) = Ric(v) - [(R \cdot S)(v, v^*, x, y)]\Delta x\Delta y + O^{>2}(\Delta x, \Delta y),$$

where the Ricci curvature in the direction of v^* is denoted by $Ric(v^*)$.

Theorem 4.1. [14] *Let $\mathcal{P}$ be any infinitesimal coordinate parallelogram cornered at a point p of M with sides of lengths Δx and Δy, which are tangent to vectors x and y at p. Let v^* be the vector obtained from v after parallel transport along $\mathcal{P}$. Then, in second-order approximation,*

$$\delta_{\mathcal{P}} Ric(v) = -(R \cdot S)(v, v^*, x, y)\Delta x\Delta y,$$

i.e., the $(0, 4)$-tensor $R \cdot S$ of M measures the change in Ricci curvature at any point p for any vector v under parallel transport of v around any infinitesimal coordinate parallelogram $\mathcal{P}$ cornered at p.

In this section we consider a ∗-ES soliton in an $M^n(LPS)$ which is Ricci semi-symmetric, i.e., $R(V_1, V_2) \cdot S = 0$. This leads to

$$S(R(V_1, V_2)V_4, V_3) + S(V_4, R(V_1, V_2)V_3) = 0 \tag{21}$$

for V_1, V_2, V_3, V_4 on $M^n(LPS)$.
Putting $V_2 = \xi$ in (21) and then recalling (10), we arrive at

$$\eta(V_4)S(V_1, V_3) - g(V_1, V_4)S(\xi, V_3) + \eta(V_3)S(V_1, V_4) - g(V_1, V_3)S(V_4, \xi) = 0.$$

Again putting $V_4 = \xi$ and using (19), the foregoing equation takes the form

$$S(V_1, V_3) = (n - \Lambda - 1 + \frac{r^*}{2})g(V_1, V_3). \tag{22}$$

By means of the fact that in an $M^n(LPS)$ admitting a $*-$ES equation (20) holds. Thus, (22) turns to

$$S(V_1, V_3) = (n - 1)g(V_1, V_3). \tag{23}$$

On contracting (23) it follows that $r = n(n-1)$, which in view of (20) and (17), we find $\Lambda = \frac{(n-1)(2n-3)-a^2}{2}$. Thus, we have

Theorem 4.2. *Let an $M^n(LPS)$ be the Ricci semi-symmetric and admits a $*-$ES. Then $M^n(LPS)$ is an Einstein manifold and the ∗-ES is expanding, steady or shrinking according to $a^2 < (n-1)(2n-3)$, $= (n-1)(2n-3)$ or $> (n-1)(2n-3)$, respectively.*

5. Projective curvature tensor in $LP-$Sasakian manifolds admitting $*-$Einstein solitons

The projective curvature tensor P in an $M^n(LPS)$ is defined by

$$P(V_1, V_2)V_3 = R(V_1, V_2)V_3 - \frac{1}{n-1}\{S(V_2, V_3)V_1 - S(V_1, V_3)V_2\} \tag{24}$$

for all V_1, V_2, V_3 on $M^n(LPS)$.

In this section, we study an $M^n(LPS)$ admitting a $*$-ES satisfying certain curvature conditions on the projective curvature tensor.

First, we consider an $M^n(LPS)$ admitting a $*$-ES which satisfies the condition $P(V_1, \xi) \cdot S = 0$. Thus, we have

$$S(P(V_1, \xi)V_2, V_3) + S(V_2, P(V_1, \xi)V_3) = 0. \tag{25}$$

From (10), (19) and (24), we find

$$P(V_1, \xi)V_2 = -g(V_1, V_2)\xi + (1 - \frac{A_4}{n-1})\eta(V_2)V_1 + \frac{1}{n-1}S(V_1, V_2)\xi. \tag{26}$$

Plugging (26) into (25), we obtain

$$\eta(V_2)S(V_1, V_3) + \eta(V_3)S(V_1, V_2) - A_4 g(V_1, V_2)\eta(V_3) - A_4 g(V_1, V_3)\eta(V_2) = 0.$$

By putting $V_2 = \xi$ in the foregoing equation then using (4) and (19), we obtain $S(V_1, V_3) = A_4 g(V_1, V_3)$, which by using (20) takes the form

$$S(V_1, V_3) = (n-1)g(V_1, V_3). \tag{27}$$

On contracting (27), we obtain $r = n(n-1)$, which in view of (20) and (17), we find $\Lambda = \frac{(n-1)(2n-3)-a^2}{2}$. Thus, we have

Theorem 5.1. *If an $M^n(LPS)$ admitting a $*$-ES satisfies the condition $P(V_1, \xi) \cdot S = 0$. Then $M^n(LPS)$ is an Einstein manifold. Moreover, the $*$-ES is expanding, steady or shrinking according to $a^2 < (n-1)(2n-3)$, $= (n-1)(2n-3)$ or $> (n-1)(2n-3)$, respectively.*

Next, we consider an $M^n(LPS)$ admitting a $*$-ES that satisfies the condition $R(V_1, \xi) \cdot P = 0$. Then we have

$$\begin{aligned} R(V_1, \xi)P(V_4, V_5)V_6 - P(R(V_1, \xi)V_4, V_5)V_6 \\ -P(V_4, R(V_1, \xi)V_5)V_6 - P(V_4, V_5)R(V_1, \xi)V_6 = 0 \end{aligned} \tag{28}$$

for any V_1, V_4, V_5, V_6 on $M^n(LPS)$.
By fixing $V_4 = V_6 = \xi$ in (28), we have

$$\begin{aligned} R(V_1, \xi)P(\xi, V_5)\xi - P(R(V_1, \xi)\xi, V_5)\xi \\ -P(\xi, R(V_1, \xi)V_5)\xi - P(\xi, V_5)R(V_1, \xi)\xi = 0. \end{aligned} \tag{29}$$

From (11), (19) and (24), we find

$$P(V_1, V_5)\xi = (1 - \frac{A_4}{n-1})(\eta(V_5)V_1 - \eta(V_1)V_5), \tag{30}$$

$$P(\xi, V_5)V_1 = -(1 - \frac{A_4}{n-1})\eta(V_1)V_5 + g(V_1, V_5)\xi - \frac{1}{n-1}S(V_1, V_5)\xi. \tag{31}$$

In view of (10), (30) and (31), after some steps calculation, (29) reduces to $S(V_1, V_5)\xi = A_4 g(V_1, V_5)\xi$, which by taking the inner product with ξ and using (20) gives

$$S(V_1, V_5) = (n-1)g(V_1, V_5). \tag{32}$$

On contracting (32), we obtain $r = n(n-1)$, which in view of (20) and (17), we find $\Lambda = \frac{(n-1)(2n-3)-a^2}{2}$. Thus, we have

Theorem 5.2. *If an $M^n(LPS)$ admitting a $*$-ES satisfies the condition $R(V_1, \xi)\cdot P = 0$. Then $M^n(LPS)$ is an Einstein manifold and the $*$-ES is expanding, steady or shrinking according to $a^2 < (n-1)(2n-3)$, $= (n-1)(2n-3)$ or $> (n-1)(2n-3)$, respectively.*

Further, we consider an $M^n(LPS)$ admitting a $*$-ES that satisfies the condition $S(\xi, V_1)\cdot P = 0$. Then we have

$$\begin{aligned} S(V_1, P(V_4, V_5)V_6)\xi - S(\xi, P(V_4, V_5)V_6)V_1 + S(V_1, V_4)P(\xi, V_5)V_6 \\ -S(\xi, V_4)P(V_1, V_5)V_6 + S(V_1, V_5)P(V_4, \xi)V_6 - S(\xi, V_5)P(V_4, V_1)V_6 \\ +S(V_1, V_6)P(V_4, V_5)\xi - S(\xi, V_6)P(V_4, V_5)V_1 = 0 \end{aligned} \tag{33}$$

for all V_1, V_4, V_5, V_6 on $M^n(LPS)$. Putting $V_4 = V_6 = \xi$ in (33), we have

$$\begin{aligned} S(V_1, P(\xi, V_5)\xi)\xi - S(\xi, P(\xi, V_5)\xi)V_1 + S(V_1, \xi)P(\xi, V_5)\xi - S(\xi, \xi)P(V_1, V_5)\xi \\ +S(V_1, V_5)P(\xi, \xi)\xi - S(\xi, V_5)P(\xi, V_1)\xi + S(V_1, \xi)P(\xi, V_5)\xi - S(\xi, \xi)P(\xi, V_5)V_1 = 0, \end{aligned}$$

which in view of (19), (30), (31) and $\eta(P(\xi, V_5)\xi) = 0$ leads to

$$A_4 g(V_1, V_5)\xi + 2A_4(1 - \frac{A_4}{n-1})\eta(V_1)\eta(V_5)\xi + (1 - 2\frac{A_4}{n-1})S(V_1, V_5)\xi = 0.$$

By taking the inner product of the foregoing equation with ξ then using (4), (6) and (20) it follows that

$$S(V_1, V_5) = (n-1)g(V_1, V_5). \tag{34}$$

On contracting (34), we obtain $r = n(n-1)$, which with the relations (20) and (17) gives $\Lambda = \frac{(n-1)(2n-3)-a^2}{2}$. Thus, we have

Theorem 5.3. *If an $M^n(LPS)$ admitting a $*$-ES satisfies the condition $S(\xi, V_1)\cdot P = 0$. Then $M^n(LPS)$ is an Einstein manifold and the $*$-ES is expanding, steady or shrinking according to $a^2 < (n-1)(2n-3)$, $= (n-1)(2n-3)$ or $> (n-1)(2n-3)$, respectively.*

6. Einstein semi-symmetric $LP-$Sasakian manifolds admitting $*-$Einstein solitons

Definition 6.1. An $M^n(LPS)$ is called Einstein semi-symmetric if $R \cdot E = 0$, where E is the Einstein tensor given by

$$E(V_1, V_2) = S(V_1, V_2) - \frac{r}{n} g(V_1, V_2), \tag{35}$$

where r is the scalar curvature of the manifold.

Let us consider an $M^n(LPS)$ admitting a $*$-ES, which is Einstein semi-symmetric, i. e., $R \cdot E = 0$. Thus, we have

$$E(R(V_1, V_2)V_3, V_4) + E(V_3, R(V_1, V_2)V_4) = 0,$$

which in view of (35) takes the form

$$\begin{aligned} S(R(V_1, V_2)V_3, V_4) + S(V_3, R(V_1, V_2)V_4) &= \frac{r}{n}\{g(R(V_1, V_2)V_3, V_4) \\ &\quad + g(V_3, R(V_1, V_2)V_4)\}. \end{aligned} \tag{36}$$

By putting $V_1 = V_3 = \xi$ in (36), we have

$$S(R(\xi, V_2)\xi, V_4) + S(\xi, R(\xi, V_2)V_4) = \frac{r}{n}\{g(R(\xi, V_2)\xi, V_4) + g(\xi, R(\xi, V_2)V_4)\}.$$

By making the use of (10), (12) and (19), the foregoing equation leads to $S(V_2, V_4) = A_4 g(V_2, V_4)$, which by using (20) turns to

$$S(V_2, V_4) = (n-1)g(V_2, V_4). \tag{37}$$

On contracting (37), we obtain $r = n(n-1)$. Using this value of r in (20) and using (17), we obtain $\Lambda = \frac{(n-1)(2n-3)-a^2}{2}$. Thus, we have

Theorem 6.2. *Let an Einstein semi-symmetric $M^n(LPS)$ admit a $*-ES$. Then $M^n(LPS)$ is an Einstein manifold and the $*$-ES is expanding, steady or shrinking according to $a^2 < (n-1)(2n-3)$, $= (n-1)(2n-3)$ or $> (n-1)(2n-3)$, respectively.*

7. $*-$Einstein solitons in $LP-$Sasakian manifolds satisfying $(S(V_1, V_2) \cdot R)(V_4, V_5)V_6 = 0$

Let an $M^n(LPS)$ admitting a $*$-ES satisfies $(S(V_1, V_2) \cdot R)(V_4, V_5)V_6 = 0$. Then we have

$$\begin{aligned} &S(V_2, R(V_4, V_5)V_6)V_1 - S(V_1, R(V_4, V_5)V_6)V_2 + S(V_2, V_4)R(V_1, V_5)V_6 \\ &- S(V_1, V_4)R(V_2, V_5)V_6 + S(V_2, V_5)R(V_4, V_1)V_6 - S(V_1, V_5)R(V_4, V_2)V_6 \\ &+ S(V_2, V_6)R(V_4, V_5)V_1 - S(V_1, V_6)R(V_4, V_5)V_2 = 0, \end{aligned}$$

which by taking the inner product with ξ takes the form

$$\begin{aligned} &S(V_2, R(V_4, V_5)V_6)\eta(V_1) - S(V_1, R(V_4, V_5)V_6)\eta(V_2) + S(V_2, V_4)\eta(R(V_1, V_5)V_6) \\ - &S(V_1, V_4)\eta(R(V_2, V_5)V_6) + S(V_2, V_5)\eta(R(V_4, V_1)V_6) - S(V_1, V_5)\eta(R(V_4, V_2)V_6) \\ + &S(V_2, V_6)\eta(R(V_4, V_5)V_1) - S(V_1, V_6)\eta(R(V_4, V_5)V_2) = 0. \end{aligned} \tag{38}$$

Putting $V_4 = V_6 = \xi$ in (38), then using (10)-(12) and (19) we find

$$S(V_2, V_5)\eta(V_1) = S(V_1, V_5)\eta(V_2) + A_4 g(V_5, V_1)\eta(V_2) - A_4 g(V_5, V_2)\eta(V_1).$$

Again putting $V_1 = \xi$ in the foregoing equation and using (4), (4), (19), we obtain $S(V_2, V_5) = -A_4 g(V_2, V_5) - 2A_4\eta(V_2)\eta(V_5)$, which by using (20) turns to

$$S(V_2, V_5) = -(n-1)g(V_2, V_5) - 2(n-1)\eta(V_2)\eta(V_5). \tag{39}$$

On contracting (39), we obtain $r = -(n-1)(n-2)$. Thus, by virtue of (20) and (17) we obtain $\Lambda = -\frac{n-1+a^2}{2}$. This helps us to state

Theorem 7.1. *If an $M^n(LPS)$ admitting a ∗-ES satisfies $S(V_1, \xi)\cdot R = 0$, then $M^n(LPS)$ is an η-Einstein manifold and the soliton is always shrinking.*

8. ∗−Einstein solitons in $LP-$Sasakian manifolds with torse-forming vector field

Definition 8.1. A vector field U in an $M^n(LPS)$ is said to be torse-forming vector field if [29]

$$\nabla_{V_1} U = fV_1 + \gamma(V_1)U, \tag{40}$$

where f is a smooth function and γ is a 1-form.

Let us consider an $M^n(LPS)$ admitting a ∗-ES, further considering the Reeb vector field ξ as a torse-forming vector field. Thus, from (40) we have

$$\nabla_{V_1}\xi = fV_1 + \gamma(V_1)\xi \tag{41}$$

for any V_1 on $M^n(LPS)$.
Taking the inner product of (41) with ξ we lead to

$$g(\nabla_{V_1}\xi, \xi) = f\eta(V_1) - \gamma(V_1). \tag{42}$$

Also from (8), we obtain

$$g(\nabla_{V_1}\xi, \xi) = 0. \tag{43}$$

Thus, from the last two equations we find $\gamma = f\eta$, and hence (41) turns to

$$\nabla_{V_1}\xi = f(V_1 + \eta(V_1)\xi). \tag{44}$$

Now, in view of (44), we have

$$(\pounds_\xi g)(V_1, V_2) = 2f\{g(V_1, V_2) + \eta(V_1)\eta(V_2)\}. \tag{45}$$

By virtue of (45), (3) turns to

$$S^*(V_1, V_2) = -(\Lambda - \frac{r}{2} + f)g(V_1, V_2) - f\eta(V_1)\eta(V_2),$$

which by using (18) yields

$$\begin{aligned} S(V_1, V_2) \quad &= \quad -(\Lambda - \frac{r}{2} + f + n - 2)g(V_1, V_2) \\ &\quad +ag(V_1, \phi V_2) - (2n - 3 + f)\eta(V_1)\eta(V_2). \end{aligned}$$

By recalling (20) in the foregoing equation, we arrive at

$$\begin{aligned} S(V_1, V_2) &= -(f+n-2)g(V_1, V_2) + ag(V_1, \phi V_1) \\ &\quad -(2n-3+f)\eta(V_1)\eta(V_2), \end{aligned} \tag{46}$$

which is a generalized $\eta-$Einstein manifold.
On contracting (46), we obtain $r = -(n-1)(f+n-3) + a^2$, and hence from (17) and (20) we obtain $\Lambda = -\frac{(n-1)f}{2}$. Thus, we have

Theorem 8.2. *Let an $M^n(LPS)$ admit a $*$-ES with a torse-forming vector field ξ. Then M is a generalized η-Einstein manifold and the soliton is expanding, steady or shrinking according to $f < 0$, $= 0$ or > 0, respectively.*

9. Examples

We consider the 3-dimensional manifold $M^3 = \{(v_1, v_2, v_3) \in \mathbb{R}^3\}$, where (v_1, v_2, v_3) are the standard coordinates in $\mathbb{R}^3$. Let ϱ_1, ϱ_2 and ϱ_3 be the vector fields on M^3 given by

$$\varrho_1 = e^{-v_3}\frac{\partial}{\partial v_1},\ \varrho_2 = e^{-v_3}(\frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_2}),\ \varrho_3 = -\frac{\partial}{\partial v_3} = \xi.$$

Let g be the semi-Riemannian metric defined by

$$g(\varrho_k, \varrho_l) = \begin{cases} 1, & 1 \le k = l \le 2, \\ -1, & k = l = 3, \\ 0, & 1 \le k \ne l \le 3. \end{cases}$$

Let η be the 1-form on M defined by $\eta(V_1) = g(V_1, \varrho_3)$ for all $V_1 \in \mathfrak{X}(M^3)$. Let ϕ be the $(1,1)$ tensor field on M^3 defined by

$$\phi\varrho_1 = -\varrho_1, \quad \phi\varrho_2 = -\varrho_2, \quad \phi\varrho_3 = 0.$$

By applying the linearity of ϕ and g, we have

$$\eta(\xi) = -1,\ \phi^2 V_1 = V_1 + \eta(V_1)\xi,\ \eta(\phi V_1) = 0,$$

$$g(V_1, \xi) = \eta(V_1),\ g(\phi V_1, \phi V_2) = g(V_1, V_2) + \eta(V_1)\eta(V_2)$$

for $V_1, V_2 \in \chi(M^3)$. Then we have

$$[\varrho_1, \varrho_2] = 0, \quad [\varrho_1, \varrho_3] = -\varrho_1 \quad [\varrho_2, \varrho_3] = -\varrho_2.$$

By using well-known Koszul's formula, we can easily calculate

$$\begin{aligned} &\nabla_{\varrho_1}\varrho_1 = -\varrho_3, \quad \nabla_{\varrho_1}\varrho_2 = 0, \quad \nabla_{\varrho_1}\varrho_3 = -\varrho_1, \\ &\nabla_{\varrho_2}\varrho_1 = 0, \quad \nabla_{\varrho_2}\varrho_2 = -\varrho_3, \quad \nabla_{\varrho_2}\varrho_3 = -\varrho_2, \\ &\nabla_{\varrho_3}\varrho_1 = 0, \quad \nabla_{\varrho_3}\varrho_2 = 0, \quad \nabla_{\varrho_3}\varrho_3 = 0. \end{aligned}$$

It can be easily shown that M^3 is an LP-Sasakian manifold. By using the above results, one can easily obtain the following components of the curvature tensor:

$$R(\varrho_1, \varrho_2)\varrho_2 = \varrho_1,\ R(\varrho_1, \varrho_3)\varrho_3 = -\varrho_1,\ R(\varrho_1, \varrho_2)\varrho_1 = -\varrho_2,$$

$$R(\varrho_2,\varrho_3)\varrho_3=-\varrho_2,\ R(\varrho_1,\varrho_3)\varrho_1=-\varrho_3,\ R(\varrho_2,\varrho_3)\varrho_2=-\varrho_3.$$

From these curvature tensors, we can easily calculate

$$S(\varrho_1,\varrho_1)=S(\varrho_2,\varrho_2)=2,\ S(\varrho_3,\varrho_3)=-2. \tag{47}$$

Thus, we find $r=6$. Putting $V_1=V_2=\xi$ in (18) and using the values $r=6$ and $S(\varrho_3,\varrho_3)=-2$, we obtain $\Lambda=3$. Thus, an expanding case of $*-$Einstein solitons is verified by the given example.

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