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PARACOMPACTNESS IN COC-OPEN SETS

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Abstract. As a new topological property, we introduce paracompact space via co-compact sets. We give some characterizations and implications theorems.

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1. Introduction

The concept of paracompactness is an important concept in topology since it gives a weaker form of compactness, so the focus on it open wide horizons in topology. A new type of open set defined in [4] called coc-compact open set, a subset A of a topological space X is called coc-open set if A is a union of sets of the form $V - C$, when V is open set and C is a compact subset of X.

The family of all coc-open sets of topological space (X, τ) forms a topology denoted by (X, τ^k).

In this paper, no separation axioms to be assumed, for a subset A of X, $\overline{A}^{coc}$ and $int_{coc}(A)$ will denote the closure of A and the interior of A in τ^k, respectively.

For more about coc-compact sets and other topological properties we refer the reader to [1, 2, 3, 7, 5, 6]

First, we need to give the following definitions and theorems :

Definition 1.1. [4] A subset A of a topological space X is called co-compact open set (notation: coc-open) if for every $x \in A$, there exists an open set $U \subseteq X$ and a compact subset K of X such that $x \in U - K \subseteq A$. The complement of a coc-open subset is called coc-closed. The family of all coc-open subsets of a topological space (X, τ) will be denoted by τ^k.

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Definition 1.2. [3] A topological space X is called coc-T_2-space if and only if for all $x, y \in X$ with $x \neq y \in X$, there exist $U, V \in \tau^k$ such that $x \in U, y \in V$ with $U \cap V = \phi$.

Definition 1.3. [4] A family $\mathcal{U} \subseteq X$ is coc-open cover of X if $\mathcal{U}$ covers X and $\mathcal{U}$ is a subfamily of τ^k.

Definition 1.4. [4] A space (X, τ) is coc-compact if every coc-open cover has a finite subcover.

Definition 1.5. A space (X, τ) is coc-Lindelöf if every coc-open cover has a countable subcover.

Definition 1.6. A family $\mathcal{U}$ of (X, τ) is called coc-locally -finite (coc-point-finite) if for each $x \in X$, there coc-open set of x which meets finite members of $\mathcal{U}$ (x belongs to finite members of $\mathcal{U}$).

Definition 1.7. A coc-T_2-space (X, τ) is called coc-paracompact (coc-meta compact) if every coc-open cover $\mathcal{U}$ of X has a coc-locally finite (coc-point finite) coc-open refinement.

Definition 1.8. A coc-T_2-space (X, τ) is called coc-paralindelöf (coc-metaLindelöf) if every coc-open cover has a coc-locally countable (coc-point countable) coc-open refinement.

Clearly, coc-compact $\Rightarrow$ coc-paracompact $\Rightarrow$ coc-metacompact $\Rightarrow$ coc-meta Lindelöf and coc-paracompact $\Rightarrow$ coc-paralindelöf.

Most of the following definitions and theorems are taken from [5].

Definition 1.9. [5] A subset A of a topological space X is coc-regular open if $int_{coc}(\overline{A}^{coc}) = A$, the complement of a coc-regular open is said to be coc-regular closed, or $A = \overline{int_{coc}(A)}^{coc}$.

Definition 1.10. [5] A coc-open cover $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ of X is called coc-regular cover, if for each $\alpha \in \Delta$ there exists a coc-regular closed set F_α such that $F_\alpha \subseteq U_\alpha$ and $X = \bigcup\{int_{coc}(F_\alpha) | \alpha \in \Delta\}$.

Definition 1.11. [5] A space (X, τ) is coc-almost-compact if every coc-open cover of X has a finite collection such that the coc-closure of the union is X.

Definition 1.12. A space (X, τ) is coc-almost-Lindelöf if every coc-open cover of X has a countable collection such that the coc-closure of the union is X.

Definition 1.13. [5] A space (X, τ) is coc-weakly-compact if every coc-regular cover of X has a finite collection such that the coc-closure of the union is X.

Definition 1.14. A space (X, τ) is coc-weakly-Lindelöf if every coc-regular cover of X has a countable collection such that the coc-closure of the union is X.

Definition 1.15. [5] A space (X, τ) is coc-nearly-compact if every coc-open cover of X by coc-regular open sets has a finite subcover.

Definition 1.16. A space (X, τ) is coc-nearly-Lindelöf if every coc-open cover of X by coc-regular open sets has a countable subcover.

And it is clear that coc-compact $\Rightarrow$ coc-nearly-compact $\Rightarrow$ coc-almost-compact $\Rightarrow$ coc-weakly-compact, also coc-nearly-Lindelöf $\Rightarrow$ coc-almost-Lindelöf $\Rightarrow$ coc-weakly-Lindelöf.

Definition 1.17. [5] A space X is called coc-R-compact if every coc-regular cover of X has a finite subcover.

Theorem 1.18. *A coc-almost-compact space X is coc-R-compact space.*

Definition 1.19. [5] A space X is said to be coc-almost regular if for each coc-regular closed F and $x \in X - F$, there exist disjoint coc-open sets U, V such that $F \subseteq U$ and $x \in V$.

Theorem 1.20. [5] *If a space X is coc-weakly-compact and coc-almost-regular, then X is coc-nearly-compact.*

Corollary 1.21. [5] *The following are equivalent for a coc-almost-regular space:*

(1) *X is coc-nearly-compact,*
(2) *X is coc-almost-compact,*
(3) *X is coc-R-compact,*
(4) *X is coc-weakly-compact.*

2. Coc-paracompact Space

Theorem 2.1. *Every coc-closed subspace of a coc-paracompact X is coc-paracompact.*

Proof. Let A be coc-closed subset of X and $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be coc-open cover of A. Then $\mathcal{U} \cup \{X - A\}$ is a coc-open cover of X which has a coc-open, coc-locally finite refinement, say $\mathcal{V}$, now $\mathcal{V} \cap A$ is the needed coc-open coc-locally finite refinement of A (clearly $(\mathcal{V} \cap A) \cap (X - A) = \phi$), hence the result. $\square$

Corollary 2.2. *Every closed subspace of a coc-paracompact X is coc-paracompact.*

Definition 2.3. A space (X, τ^k) is T_3-space if for a coc-closed set A and $x \in A$, there exist disjoint $U, V \in \tau^k$ with $A \subset U, x \in V$.

Definition 2.4. A space (X, τ^k) is normal space if for disjoint coc-closed set A, B, there exist disjoint $U, V \in \tau^k$ with $A \subseteq U, B \subseteq V$.

Lemma 2.5. *Let (X, τ) be a coc-paracompact space. Then (X, τ^k) is T_3- space and hence coc-almost regular space.*

Proof. Let A be coc-closed subset of X and $x \in X - A$. For each $y \in A$, there exist disjoint coc-open sets U_x, V_y contain x, y respectively. Now, the collection $\{U_x | x \in A\} \cup \{X - A\}$ forms a coc-open cover of X, which has coc-locally-finite coc-open refinement $\mathcal{V} = \{V_\gamma | \gamma \in \Gamma\}$. Let $\mathcal{V}_A = \{V_\gamma | V_\gamma \cap A \neq \phi\}$, and $V = \bigcup \{V_\gamma | V_\gamma \in \mathcal{V}_A\}$, then $A \in V$ and $y \notin \bigcup_{\gamma \in \Gamma} \overline{V_\Gamma}^{coc} = \overline{V}^{coc}$, hence the result. $\square$

Definition 2.6. A set $A \subseteq X$ is called coc-dense set if $\overline{A}^{coc} = X$.

Theorem 2.7. *For a coc-paracompact space X, the following are equivalent:*

(1) *X has coc-dense coc-compact space,*
(2) *X is coc-weakly compact space,*
(3) *X is coc-almost compact space,*
(4) *X is coc-R-compact space,*
(5) *X is coc-compact space.*

Proof. The proof comes from the lemma 2.5 and corollary 1.21, and the fact that (see [5]) if (X, τ^k) is T_3-space with X is coc-weakly compact space, then X is coc-compact space. □

Theorem 2.8. *Let X be coc-paralindelöf space. Then the following are equivalent:*

(1) *X has coc-dense coc-Lindelöf space,*
(2) *X is coc-weakly-Lindelöf space,*
(3) *X is coc-almost-Lindelöf space,*
(4) *X is coc-Lindelöf space.*

3. Coc-countable Paracompactness

In this section we introduce a new type of paracompact in coc-open sets and gives equivalent statement to this notion.

Definition 3.1. A coc-T_2-space (X, τ) is called coc-countably paracompact if every countable coc-open cover has a coc-locally finite coc-open refinement.

The following lemma is necessary for the next theorem.

Lemma 3.2. *Let (X, τ^k) be a normal space. Then for all countable coc-point finite cover $\mathcal{G}$ of X, there exist a coc-open refinement cover $\mathcal{W}$ such that $\overline{W_\gamma}^{coc} \subseteq G$ for some $G \in \mathcal{G}$.*

Proof. Let $\mathcal{G} = \{G_m | m \in \Delta, \Delta \quad \text{is countable set}\}$ be coc-locally finite cover for a normal space (X, τ^k).
Define

$$E_1 = X - \bigcup_{m>1} G_m,$$

clearly E_1 is coc-closed set in X and $E_1 \subseteq G_1$, but (X, τ^k) is normal space, so there exist coc-open set W_1 such that

$$E_1 \subseteq W_1 \subseteq \overline{W_1}^{coc} \subseteq G_1.$$

In general, we can define

$$E_\alpha = X - (\bigcup_{s<\alpha} W_s)(\bigcup_{t>\alpha} G_t),$$

clearly E_α is coc-closed set and $E_\alpha \subseteq G_\alpha$, again by normality of (X, τ^k), there exist coc-open set W_α with

$$E_\alpha \subseteq W_\alpha \subseteq \overline{W_\alpha}^{coc} \subseteq G_\alpha.$$

Indeed, the family $\mathcal{W} = \{W_\alpha | \alpha \in \Delta\}$ is coc-open refinement cover. For that let $x \in X$ and x meets finite members of $\mathcal{G}$ say $G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}$, let $\alpha = \max\{\alpha_1, \ldots, \alpha_n\}$, then $x \notin G_t$ for $t > \alpha$ and if $x \notin W_s$ for any $s < \alpha$, then $x \in E_\alpha \subseteq W_\alpha$, so $x \in V_s$ for some $s \leq \alpha$, hence $\mathcal{V}$ is refinement coc-open cover. □

Definition 3.3. A set A in a topological space is called *coc-*G_δ−set if it intersection of coc-open set, the complement of *coc-*G_δ−set is called $coc - F_\sigma$−set.

Theorem 3.4. *Let (X, τ^k) be a normal space. Then the following are equivalent:*

1. *X is coc-countably paracompact,*
2. *Every coc-open cover has coc-locally finite coc-open refinement,*
3. *Every countable coc-open cover $\mathcal{U}$ has a refinement $\mathcal{V}$ such that $\overline{V}^{coc} \subseteq U$ for some $U \in \mathcal{U}$,*
4. *Given a decreasing sequence of coc-closed $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$ with $\bigcap_{\alpha\in\Delta} F_\alpha = \phi$, then there exist a sequence of coc-open family $\mathcal{G} = \{G_\alpha | \alpha \in \Delta\}$ with $\bigcap_{\alpha\in\Delta} G_k = \phi$ such that $F_\alpha \subseteq G_\alpha$,*
5. *Given a decreasing sequence of coc-closed family $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$ with $\bigcap_{\alpha\in\Delta} F_\alpha = \phi$, there exist a sequence of coc-closed (coc-G_δ) family $\mathcal{A} = \{A_\alpha | \alpha \in \Delta\}$ with $\bigcap_{\alpha\in\Delta} A_\alpha = \phi$ and $F_\alpha \subseteq A_\alpha$.*

Proof. (1) → (2) Clearly, since coc-locally finite coc-open cover is coc-point finite coc-open cover.
(2) → (3) Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be a countable coc-open cover of X, so for $x \in X$, by (2) , there is a coc-locally finite coc-open refinement $\mathcal{V}$ such that x belongs to finite members of $\mathcal{V}$. For $V \in \mathcal{V}$ and let $U_V \in \mathcal{U}$ be the first one of $\mathcal{U}$ that contains V. Let

$$G_\alpha = \bigcap_{U_V = U_\alpha} V,$$

then $G_\alpha \subseteq U_\alpha$.
For the collection $\mathcal{G} = \{G_\alpha | \alpha \in \Delta\}$, $\mathcal{G}$ is coc-point finite coc-open cover, and so by Lemma 3.2, there exist a coc-open cover $\mathcal{W}$ such that $\overline{W}^{coc} \subseteq U$ for some $U \in \mathcal{U}$ as required.
(3) → (4) Let $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$ be a countable coc-closed family with $F_{\alpha+1} \subset F_\alpha$ and $\bigcap_{\alpha\in\Delta} F_\alpha = \phi$.
Define $U_\alpha = X - F_\alpha$, then $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be countable coc-open cover, therefore by (3) there exist a coc-open refinement $\mathcal{V}$ with $V_\alpha \subseteq U_\alpha$.
Finally define

$$G_\alpha = X - \overline{V_\alpha}^{coc},$$

then

$$\bigcup_{\alpha\in\Delta} \overline{V_\alpha}^{coc} = X,$$

so

$$\bigcap_{\alpha\in\Delta} G_\alpha = \phi,$$

and

$$\mathcal{G} = \{G_\alpha | \alpha \in \Delta\}$$

is coc-open cover of X, since $\overline{V_\alpha}^{coc} \subseteq U$, so we have $F_\alpha \subseteq G_\alpha$.

(4) $\rightarrow$ (5) Let $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$ be a coc-closed family satisfies (4), so there exists a sequence $\mathcal{G} = \{G_\alpha | \alpha \in \Delta\}$ with $F_\alpha \subseteq G_\alpha$ and $\bigcap\limits_{\alpha\in\Delta} G_\alpha = \phi$, since (X, τ^k) is normal, there exists a continuous function $\phi_\alpha(x) : (X, \tau^k) \rightarrow (\mathbb{R}, \tau_u^\alpha)$ such that $\phi_\alpha(x) = 0$ for $x \in F_\alpha$ and $\phi_\alpha(x) = 1$ for $x \notin G_\alpha$.
Let

$$G_{\alpha_m} = \{x | \phi_\alpha(x) < \frac{1}{m}\},$$

and

$$A_\alpha = \bigcap_m G_{\alpha_m} = \{x | \phi_\alpha(x) = 0\},$$

then G_{α_m} is open set and hence coc-open set, and A_α closed and hence coc-closed set with $F_\alpha \subset A_\alpha \subset G_\alpha$ and $\bigcap\limits_{\alpha\in\Delta} A_\alpha = \bigcap\limits_{\alpha\in\Delta} G_\alpha = \phi$.

(5) $\rightarrow$ (1) Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ be countable coc-open cover and let $F_\alpha = X - \bigcup\limits_{k\leq\alpha} U_k$, so $\mathcal{F} = \{F_\alpha | \alpha \in \Delta\}$ is coc-closed decreasing sequence with $\bigcap\limits_\alpha F_\alpha = \phi$ (since $\bigcup\limits_{\alpha\in\Delta} U_\alpha = X$), so there exists a sequence of coc-closed (coc-G_δ) family $\mathcal{A} = \{A_\alpha | \alpha \in \Delta\}$ with $F_\beta \subseteq A_\beta$ and $\bigcap A_\beta = \phi$, then $X - A_\beta$ is coc-F_α set, so we write $X - A_\beta = \bigcup\limits_\alpha B_{\beta_\alpha}$, where B_{β_α} is coc-closed, but (X, τ^k) is normal, so

$$B_{\beta_\alpha} \subseteq int_{coc}(B_{\beta,\alpha+1}).$$

Define

$$H_{\beta_\alpha} = int_{coc}(B_{\beta_\alpha}),$$

then

$$H_{\beta_\alpha} \subseteq B_{\beta_\alpha} \subseteq H_{\beta,\alpha+1}$$

and

$$X - A_\beta = \bigcup H_{\beta_\alpha},$$

hence

$$B_{\beta_\alpha} \subset X - A_\beta \subset X - F_\beta = \bigcup_{k\leq\beta} U_k.$$

Let

$$V_\alpha = U_\alpha - \bigcup_{\beta<\alpha} B_{\beta_\alpha},$$

then V_α is coc-open set. To end the proof let $\mathcal{V} = \{V_\alpha | \alpha \in \Delta\}$. (1) $\mathcal{V}$ covers X. For $\beta < \alpha$, $B_{\beta_\alpha} \subset \bigcup_{k \leq \beta} U_k \subset \bigcup_{k<\alpha} U_k$, hence $\bigcup_{\beta<\alpha} B_{\beta_\alpha} \subset \bigcup_{k<\alpha} U_k$, and $V_\alpha \supseteq U_\alpha - \bigcup_{k<\alpha} U_k$. So for $x \in X$, there exists $\alpha \in \Delta$ with $x \in U_\alpha$ and hence $x \in V_\alpha$. (2) $\mathcal{V}$ is refinement for $\mathcal{U}$, it is clear. (3) $\mathcal{V}$ coc-locally finite coc-open. Let $x \in X$, for some $\beta \in \Delta$, $x \notin A_\beta$ and for some m, $x \in H_{\beta_k}$, then if $\alpha > \beta$ and $\alpha > k$, $H_{\beta_k} \subset B_{\beta_\alpha}$ and hence $H_{\beta_k} \cap V_\alpha = \phi$. So the coc-open set H_{β_k} contains x meets finite members of $\mathcal{V}$ as required.

□

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