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CHANGING RELATIONSHIP BETWEEN SETS USING CONVOLUTION SUMS OF RESTRICTED DIVISOR FUNCTIONS†

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Abstract. There are real life situations in our lives where the things are changing continuously or from time to time. It is a very important problem for one whether to continue the existing relationship or to form a new one after some occasions. That is, people, companies, cities, countries, etc. may change their opinion or position rapidly. In this work, we think of the problem of changing relationships from a mathematical point of view and think of an answer. In some sense, we comment these changes as power changes. Our number theoretical model will be based on this idea. Using the convolution sum of the restricted divisor function E, we obtain the answer to this problem.

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1. Introduction

Throughout this article, p, $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{Z}$ will denote a prime number, the set of natural numbers, the set of positive integers and the set of integers, respectively.

There are many cases in our lives where the things change from time to time. As a recent unfortunate example, Russia invaded Ukraine after being long term allies with them. Earlier, Russia was allied with Ukraine, and USA was allied with NATO. In some sense, Ukraine and Russia had a relation where there had been a potential for change due to several historical, political, economical, geopolitical reasons appearing in the years. For many, the war seems to have started as Ukraine tried to join NATO which did not make Russia happy. Now, for Ukraine, after all the mean behaviour and invasion by Russia, it is a very

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important problem whether to continue the existing relationship or to form a new one. That is, people, companies, cities, countries, etc. may change their opinion or position rapidly. Let us think of the problem of changing relationships from a mathematical point of view and think of an answer. In some sense, we can comment these changes as power changes. Our number theoretical model will be based on this idea.

Let A, B, C and D be four sets and let the graph potential for change of B be c times as strong as the graph potential for change of A and the graph potential for change of D be d times as strong as the graph potential for change of C. Assuming that each element is a square (x^2, y^2, z^2, w^2) with $x \in A$, $y \in B$, $z \in C$ and $w \in D$, first, suppose that A and B are related and C and D are related. That is, suppose that A (resp. C) and B (resp. D) can produce the same graph potential for change $x^2 + cy^2 = m$ (resp. $z^2 + dw^2 = n$).

Consider the problem of investigating how many new relationships could be created if four are gathered to create graph potential for change $x^2 + cy^2 + z^2 + dw^2 = l$ under the assumption that two sets can create some common graph potential for change. Let us model this to make it a mathematical problem:

Problem 1.1. *To make things easier, we will only deal with the case where $l = p^n$, $c = d = 2$ and m and n are also multiples of p. Let*

$$\mathfrak{R}(p^n) := \{(x, y, z, w) \in \mathbb{Z}^4 \mid x^2 + 2y^2 + z^2 + 2w^2 = p^n\}$$

be the relation set of p^n;

$$\mathfrak{C}(p^n) := \{(x, y, z, w) \in \mathfrak{R}(p^n) \mid x^2+2y^2+0^2+2\cdot 0^2 = p^n, 0^2+2\cdot 0^2+z^2+2w^2 = p^n\}$$

*be the closed relation set of p^n (see (**C**) in Fig. 1);*

$$\mathfrak{I}(p^n) := \{(x, y, z, w) \in \mathfrak{R}(p^n) - \mathfrak{C}(p^n) \mid x^2+2y^2 = pm_1,\ z^2+2w^2 = pm_2,\ m_i\ (i = 1, 2) \in \mathbb{N} \cup \{0\}\}$$

*be the invariant relation set of p^n (see (**I**) in Fig. 1); and*

$$\mathfrak{N}(p^n) := \mathfrak{R}(p^n) - (\mathfrak{C}(p^n) \cup \mathfrak{I}(p^n)) = \{(x_1, y_1, z_1, w_1) \in \mathfrak{R}(p^n) \mid x_1^2+2y_1^2 \neq pm_1, z_1^2 + 2w_1^2 \neq pm_2,\ m_i\ (i = 1, 2) \in \mathbb{N}_0\}$$

*be the new relation set of p^n (see (**N**) in Fig. 1). Here $\#U$ denotes the number of elements in a set U.*

Find the value of $\#\mathfrak{N}$?

Fig. 1 shows the three relationships. In order to solve Problem 1.1 easily by means of convolution sums of divisors, we first need some mathematical notations and properties introduced below:

The Dirichlet convolution of two arithmetic functions f_1 and f_2 is defined by

$$(f_1 * f_2)(n) = \sum_{d|n} f_1(d) f_2(n/d),$$

see [11, p. 301]. An arithmetic function f_2 is called an inverse of f_1 if

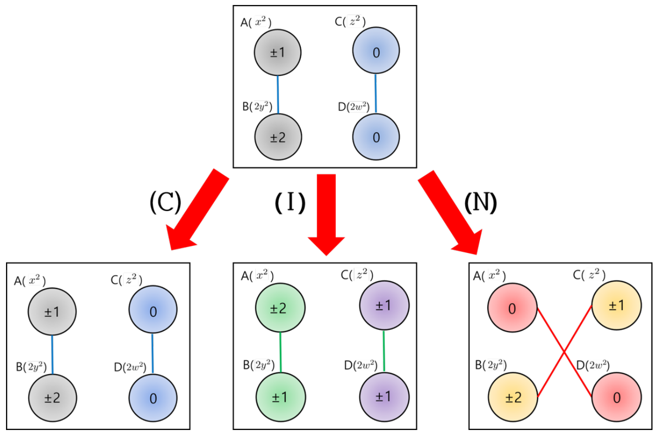

FIGURE 1. Three relationships (C,I,N)

$$(f_1 * f_2)(n) = (f_2 * f_1)(n) = I(n)$$

with

$$I(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In this article, we take $f_2 := f_1^{-1}$. Such an arithmetic inverse function f_1^{-1} of f_1 exists and satisfy the following equality [10, p.6]

$$f_1^{-1}(1) = 1/f_1(1) \text{ and } f_1^{-1}(n) = -\frac{1}{f_1(1)} \sum_{\substack{d|n \\ d>1}} f_1(d) f_1^{-1}(n/d) \tag{1}$$

if $f_1(1) \neq 0$. For more properties of arithmetic functions, see [5, 7, 10, 11, 14].

For $d, n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, we define

$$\sigma_k(n) := \textstyle\sum_{d|n} d^k, \qquad \sigma(n) := \sigma_1(n),$$

$$E(n) := \sum_{\substack{d|n \\ d\equiv 1,3 (\text{mod } 8)}} 1 - \sum_{\substack{d|n \\ d\equiv 5,7 (\text{mod } 8)}} 1, \qquad \mathfrak{E}(n) := \textstyle\sum_{k=1}^{n-1} E(k)E(n-k),$$

$$\overline{\mathfrak{E}}(n) := E(n) + \textstyle\sum_{k=1}^{n-1} E(k)E(n-k), \qquad \widehat{\mathfrak{E}}(n) := \sum_{\substack{1\leq k\leq n-1 \\ gcd(k,n-k)=1}} E(k)E(n-k).$$

Here, we let $\mathfrak{E}(1) = \widehat{\mathfrak{E}}(1) = 0$. Usually, $E(n)$ is often denoted by $E_{1,3}(n; 8)$ [4, p.12]. However, since this symbol appears a lot in this article, it is written as $E(n)$ for brevity.

On the other hand, using Jacobi's identity, we can easily show that

$$\#\{(x, y) \in \mathbb{Z}^2 | x^2 + 2y^2 = n\} = 2E(n) \tag{2}$$

with $n \in \mathbb{N}$. See [4, (31.12)]. By means of Eqn. (2), Problem 1.1 is equivalent to problem of showing $4\widehat{\mathfrak{E}}(p^n) = 4\sum_{\substack{1\le k\le p^n-1\\ gcd(k,p^n-k)=1}} E(k)E(p^n-k)$. In more detail, we have

$$\begin{aligned} \mathfrak{R}(p^n) &= \mathfrak{C}(p^n)\cup\mathfrak{I}(p^n)\cup\mathfrak{N}(p^n),\\ \#\mathfrak{R}(p^n) &= 4\overline{\mathfrak{E}}(p^n),\ \#\mathfrak{C}(p^n) = 4E(p^n),\ \#\mathfrak{N}(p^n) = 4\widehat{\mathfrak{E}}(p^n) \end{aligned}$$

and

$$\#\mathfrak{I}(p^n) = \#\overline{\mathfrak{E}}(p^n) - \#\mathfrak{C}(p^n) - \#\mathfrak{N}(p^n).$$

Using the convolution sum of the restricted divisor function E, we obtain the answer to Problem 1.1 as follows:

Theorem 1.2. *Let* $\epsilon(n) = \begin{cases} 0 & \text{if } n\equiv 1\ (mod\ 2),\\ 1 & \text{if } n\equiv 0\ (mod\ 2). \end{cases}$

(a) If $p\equiv 1,3(mod\ 8)$ is an odd prime, then

$$\#\mathfrak{N}(p^n) = \begin{cases} 4(p-1) & \text{if } n=1,\\ 4(p-1)(p-2) & \text{if } n=2,\\ 4(p-1)(p^2-2p+2) & \text{if } n=3,\\ 4(\sigma(p^n)-4\sigma(p^{n-1})+7\sigma(p^{n-2})+2(-1)^{\epsilon(n)}-8p^{\epsilon(n)}\sigma_2(p^{(n-\epsilon(n)-3)/2})) & \text{if } n\ge 4 \end{cases}$$

and if $p\equiv 5,7(mod\ 8)$ is an odd prime, then $\#\mathfrak{N}(p^n) = 4p^{n-1}(p+1)$.

(b) If $n\in\mathbb{N}$, then

$$\#\mathfrak{N}(2^n) = \begin{cases} 4 & \text{if } n=1,\\ 16 & \text{if } n=2,\\ 0 & \text{if } n\ge 3. \end{cases}$$

If it is assumed that two sets of four satisfy the sum of squares relation, then $4\widehat{\mathfrak{E}}$ will be the value of the new relation set. In other words, using Theorem 1.2, Problem 1.1 is solved.

Example 1.3. For $\mathfrak{R}(3) = \{(\pm1,\pm1,0,0),(\pm1,0,0,\pm1),(0,\pm1,\pm1,0),(0,0,\pm1,\pm1)\}$, we have the following sets $\mathfrak{C}(3) = \{(\pm1,\pm1,0,0),(0,0,\pm1,\pm1)\}$, $\mathfrak{I}(3) = \{\ \}$, $\mathfrak{N}(3) = \{(\pm1,0,0,\pm1),(0,\pm1,\pm1,0),(\pm1,0,0,\mp1),(0,\pm1,\mp1,0)\}$ and hence $\#\mathfrak{R}(3) = 4\overline{\mathfrak{E}}(3) = 16$, $\#\mathfrak{C}(3) = 4E(3) = 8$, $\#\mathfrak{N}(3) = 4\widehat{\mathfrak{E}}(3) = 8$, $\#\mathfrak{I}(3) = 4(\overline{\mathfrak{E}}(3) - \widehat{\mathfrak{E}}(3) - E(3)) = 0$, $\#\mathfrak{R}(9) = 4\overline{\mathfrak{E}}(9) = 52$, $\#\mathfrak{C}(9) = 4E(9) = 12$, $\#\mathfrak{N}(9) = 4\widehat{\mathfrak{E}}(9) = 8$ and $\#\mathfrak{I}(9) = 32$. Fig. 2 (resp. Fig. 3) shows the whole of $\mathfrak{R}(3)$ (resp. $\mathfrak{R}(9)$). In Fig. 2, (A) and (B) belong to the closed relation, and (C) and (D) belong to the new relation. In Fig. 3, (E) ∼(H) belong to the closed relation, (I) and (J) belong to the new relation and (K) and (L) belong to the invariant relation.

Remark. If p is an odd prime and n is big enough, then the number of new relations $\#\mathfrak{N}(p^n)$ is approximately $4p^n$. However, in the case of $p=2$, no matter how large n is, there are no new relations $\#\mathfrak{N}(2^n)$. Here, in the case of $p=2$, it is an example that mathematically informs us that there may be a structure

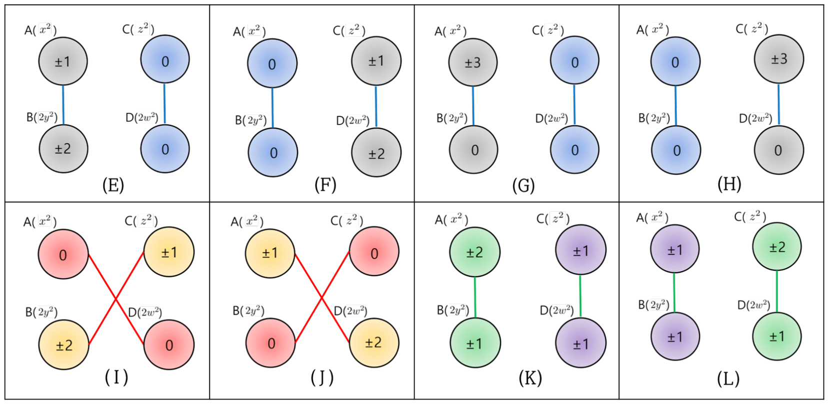

FIGURE 2. Relation set of 3

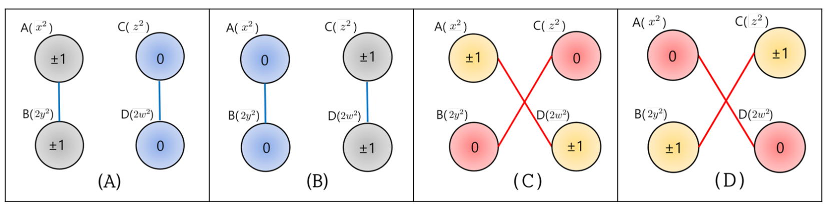

FIGURE 3. Relation set of 9

$\mathfrak{N}(p^\alpha)$	2	3	5	7	11	13
1	4	8	24	32	40	56
2	16	8	120	224	360	728
3	0	40	600	1568	4040	9464
4	0	104	3000	10976	44360	123032
5	0	328	15000	76832	488040	1599416
6	0	968	75000	537824	5368360	20792408
7	0	2920	375000	3764768	59052040	270301304
8	0	8744	1875000	26353376	649572360	3513916952
9	0	26248	9375000	184473632	7145296040	45680920376
10	0	78728	46875000	1291315424	78598256360	593851964888

TABLE 1. Values of $\mathfrak{N}(p^\alpha)(1 \leq \alpha \leq 10)$ with $2 \leq p \leq 13$.

in which a new relationship is not created even if a lot of graph potential for change is given to the four sets. In other words, Theorem 1.2 shows that there is a system in which a new relationship is not created even if a lot of effort is put into it.

In Section 2, for the case where n is odd, we find results related to $\mathfrak{E}(n)$. In Section 3, we obtain the inverses of E and E^2 and find their properties. Finally, in Section 4, we prove Theorem 1.2.

2. Values of $E(n)$ and $\mathfrak{E}(n)$

Let q be a fixed complex number with absolute value less than 1, so that we may write $q = e^{\pi it}$ where $Im(t) > 0$. Fine [4, (9.3),(18.62)] wrote that

$$\prod_{n\geq 1} \frac{(1-q^n)^2}{(1-2q^n\cos 2u + q^{2n})} = 1 - 4\sin u \sum_{n\geq 1} q^n \sum_{w|n} \sin\left(\frac{2n}{w} - w\right) u \tag{3}$$

and

$$\prod_{n\geq 1} \frac{(1-q^n)^4}{(1-2q^n\cos u' + q^{2n})^2} = 1 - 8\sin^2 \frac{u'}{2} \sum_{N\geq 1} q^N \sum_{\substack{nk=N \\ n,k\geq 1}} n\cos(k-n)u'. \tag{4}$$

In (3) and (4), set $u = \frac{\pi}{4}$ and $u' = \frac{\pi}{2}$ to obtain

$$\prod_{n\geq 1} \frac{(1-q^n)^2}{(1+q^{2n})} = 1 - \frac{4}{\sqrt{2}} \sum_{n\geq 1} q^n \sum_{w|n} \sin\left(\frac{2n}{w} - w\right) \frac{\pi}{4} := \sum_{k\geq 0} h_1(k) q^k \tag{5}$$

and

$$\prod_{n\geq 1} \frac{(1-q^n)^4}{(1+q^{2n})^2} = 1 - 4 \sum_{N\geq 1} q^N \sum_{\substack{nk=N \\ n,k\geq 1}} n\cos(k-n)\frac{\pi}{2} := \sum_{i\geq 0} h_2(i) q^i. \tag{6}$$

Thus, by Eqns. (5) and (6),

$$\sum_{k=0}^{n} h_1(k) h_1(n-k) = h_2(n). \tag{7}$$

The study of the convolution sum of arithmetic functions has been studied by many researchers (see [1], [2], [3], [8], [9], [12], [13], [15] and the references therein).The formula for the convolution sum with respect to E is written below as it is necessary to obtain the main result of this paper.

Proposition 2.1. [6] *If* $n = 2^a m \in \mathbb{N}$ *with* $\gcd(2, m) = 1$, *then*

$$h_2(n) = \begin{cases} -4\sigma(n) & \text{if } n \equiv 1 \pmod 4, \\ 4\sigma(n) & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } a = 1, \\ -8\sigma(m) & \text{if } a = 2, \\ 24\sigma(m) & \text{if } a \geq 3. \end{cases}$$

Using Eqn. (5), we will find $E(n)$ term by term.

Lemma 2.2. *If n is a positive integer, then*

$$E(n) = \begin{cases} -\frac{1}{2}h_1(n) & \text{if } n \equiv 1, 5 \ (\text{mod } 8), \\ \frac{1}{2}h_1(n) & \text{if } n \equiv 3, 7 \ (\text{mod } 8). \end{cases}$$

In particular, if $n \equiv 5, 7$ (mod 8) then $E(n) = h_1(n) = 0$.

Proof. First, let $n \equiv 1 \pmod 8$ and $d|n$. Then d is an odd positive integer satisfying $n = d \cdot \frac{n}{d} \equiv 1 \pmod 8$ and $d \equiv \frac{n}{d} \pmod 8$. Hence $\sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} = \sin\frac{d\pi}{4}$. By Eqn. (5), we can write

$$\begin{aligned} -\frac{1}{2}h_1(n) &= \sqrt{2}\left\{ \sum_{\substack{d|n \\ d\equiv 1,3(\text{mod } 8)}} \sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} + \sum_{\substack{d|n \\ d\equiv 5,7(\text{mod } 8)}} \sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} \right\} \\ &= \sqrt{2}\left\{ \sum_{\substack{d|n \\ d\equiv 1,3(\text{mod } 8)}} \sin\frac{d\pi}{4} + \sum_{\substack{d|n \\ d\equiv 5,7(\text{mod } 8)}} \sin\frac{d\pi}{4} \right\} \\ &= \sum_{\substack{d|n \\ d\equiv 1,3(\text{mod } 8)}} 1 - \sum_{\substack{d|n \\ d\equiv 5,7(\text{mod } 8)}} 1. \end{aligned} \tag{8}$$

Secondly, let $n \equiv 3 \pmod 8$ and $d|n$. If $d \equiv 1$ (resp. $3, 5, 7$) (mod 8), then $\frac{d}{n} \equiv 3$(resp., $1, 7, 5$)(mod 8). So, we obtain

$$\sqrt{2}\sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} = \begin{cases} 1 & \text{if } d \equiv 5, 7 \ (\text{mod } 8), \\ -1 & \text{if } d \equiv 1, 3 \ (\text{mod } 8), \end{cases}$$

and

$$\begin{aligned} \frac{1}{2}h_1(n) &= -\sqrt{2}\left\{ \sum_{\substack{d|n \\ d\equiv 1,3(\text{mod } 8)}} \sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} + \sum_{\substack{d|n \\ d\equiv 5,7(\text{mod } 8)}} \sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} \right\} \\ &= \sum_{\substack{d|n \\ d\equiv 1,3(\text{mod } 8)}} 1 - \sum_{\substack{d|n \\ d\equiv 5,7(\text{mod } 8)}} 1. \end{aligned} \tag{9}$$

Thirdly, let $n \equiv 5 \pmod 8$ and $d|n$. Then

$$\begin{aligned} -\frac{1}{2}h_1(n) &= \sqrt{2}\sum_{k=1,3,5,7}\left(\sum_{\substack{d|n\\ d\equiv k (\text{mod } 8)}} \sin\frac{k\pi}{4}\right) \\ &= \sum_{\substack{d|n\\ d\equiv 1,3 (\text{mod } 8)}} 1 - \sum_{\substack{d|n\\ d\equiv 5,7 (\text{mod } 8)}} 1. \end{aligned} \tag{10}$$

As an easy calculation, assuming $d \equiv 1$ or 3 (resp. 5 or 7) $\pmod 8$, we get $\frac{n}{d} \equiv 5$ or 7 (resp. 1 or 3) $\pmod 8$. Therefore

$$\#\{d \mid d \equiv 1, 3 \pmod 8\} = \#\{d \mid d \equiv 5, 7 \pmod 8\}.$$

By Eqn. (10), $-\frac{1}{2}h_1(n) = E(n) = 0$.

Finally, let $n \equiv 7 \pmod 8$ and $d|n$. Then

$$\begin{aligned} \frac{1}{2}h_1(n) &= -\sqrt{2}\sum_{k=1,3,5,7}\left(\sum_{\substack{d|n\\ d\equiv k (\text{mod } 8)}} \sin\frac{(k+4)\pi}{4}\right) \\ &= \sum_{\substack{d|n\\ d\equiv 1,3 (\text{mod } 8)}} 1 - \sum_{\substack{d|n\\ d\equiv 5,7 (\text{mod } 8)}} 1. \end{aligned} \tag{11}$$

We can show that $\frac{1}{2}h_1(n) = E(n) = 0$ in the same way as in the case of $n \equiv 5 \pmod 8$. Therefore, Lemma 2.2 is deduced by Eqns. (8)-(11). $\square$

To compute $h_1(n)$ where n is even, we need the following lemma:

Lemma 2.3. *If n is an even integer, then*

$$\sum_{\substack{d|n\\ d\equiv 0 (\mathit{mod}\ 2)}} \sin\left(\frac{2n}{d} - d\right)\frac{\pi}{4} = 0.$$

Proof. Let $T_1 := \{d \mid d \equiv 0 \pmod 2, d|n, \frac{2n}{d} - d \equiv 2 \pmod 8\}$ and $T_2 := \{d \mid d \equiv 0 \pmod 2, d|n, \frac{2n}{d} - d \equiv 6 \pmod 8\}$. It is easily checked that

$$\frac{2n}{d} - d \equiv 0 \pmod 2 \ \text{ if and only if } \ d \equiv 0 \pmod 2 \tag{12}$$

with $d|n$. If $f_1 : T_1 \to T_2$ via $f_1(d) = \frac{2n}{d}$, then f_1 is bijective and

$$\#T_1 = \#T_2. \tag{13}$$

It is trivial that

$$\sin n\pi = 0 \text{ if } n \text{ is an integer.} \tag{14}$$

By Eqns. (12)-(14),

$$\sum_{\substack{d|n \\ d\equiv 0 (\mathrm{mod}\ 2)}} \sin\left(\frac{2n}{d}-d\right)\frac{\pi}{4} = \sum_{\substack{d|n \\ \frac{2n}{d}-d\equiv 0,4 (\mathrm{mod}\ 8)}} 0 + \sum_{\substack{d|n \\ \frac{2n}{d}-d\equiv 2 (\mathrm{mod}\ 8)}} 1 - \sum_{\substack{d|n \\ \frac{2n}{d}-d\equiv 6 (\mathrm{mod}\ 8)}} 1 = 0.$$

□

Lemma 2.4. *If n is an even positive integer, then*

$$E(n) = \begin{cases} -\frac{1}{2}h_1(n) & \text{if } n \equiv 2,6 \ (\mathit{mod}\ 8), \\ \frac{1}{2}h_1(n) & \text{if } n \equiv 0,4 \ (\mathit{mod}\ 8). \end{cases}$$

Proof. By Lemma 2.3, we only need to consider the odd divisors d. If $n \equiv 2,6 \pmod 8$ is an even integer, then

$$\sqrt{2}\sin\left(\frac{2n}{d}-d\right)\frac{\pi}{4} = \begin{cases} 1 & \text{if } d \equiv 1,3 \pmod 8, \\ -1 & \text{if } d \equiv 5,7 \pmod 8 \end{cases}$$

and

$$-\frac{1}{2}h_1(n) = \sum_{\substack{d|n \\ d\equiv 1,3 (\mathrm{mod}\ 8)}} 1 - \sum_{\substack{d|n \\ d\equiv 5,7 (\mathrm{mod}\ 8)}} 1 = E(n).$$

If $n \equiv 0,4 \pmod 8$ is an even integer, then

$$\sqrt{2}\sin\left(\frac{2n}{d}-d\right)\frac{\pi}{4} = \begin{cases} -1 & \text{if } d \equiv 1,3 \pmod 8, \\ 1 & \text{if } d \equiv 5,7 \pmod 8 \end{cases}$$

and

$$\frac{1}{2}h_1(n) = \sum_{\substack{d|n \\ d\equiv 1,3 (\mathrm{mod}\ 8)}} 1 - \sum_{\substack{d|n \\ d\equiv 5,7 (\mathrm{mod}\ 8)}} 1 = E(n).$$

□

By Lemma 2.2, 2.3 and 2.4, we get

Proposition 2.5. [6] *If $n \in \mathbb{N}$ then*

$$E(n) = \begin{cases} -\frac{1}{2}h_1(n) & \text{if } n \equiv 1,2,5,6 \ (\mathit{mod}\ 8), \\ \frac{1}{2}h_1(n) & \text{if } n \equiv 0,3,4,7 \ (\mathit{mod}\ 8). \end{cases}$$

In particular, if $n \equiv 5,7$ (mod 8) then $E(n) = h_1(n) = 0$.

Proposition 2.6 is a well-known result [4, (31.32)], [9, Theorem 6.5] that can be derived from the theory of the sum of squares, Jacobi theta functions, modular forms, basic hypergeometric series, etc. The necessary case in this paper is to find the case of $\widehat{\mathfrak{E}}(p^n)$, so the result of Proposition 2.6 is sufficient. Proposition 2.6 is revisited using the results obtained in this section.

Proposition 2.6. *Let $n(>1)$ be an odd positive integer. Then $\mathfrak{E}(n) = \sigma(n) - E(n)$. In particular, if $n \equiv 5, 7 \pmod 8$, then $\mathfrak{E}(n) = \sigma(n)$ and*

$$\mathfrak{E}(2^t) = \begin{cases} 1 & \textit{if } t = 1, \\ 5 & \textit{if } t \geq 2. \end{cases}$$

Proof. Let $n \equiv 1 \pmod 8$ be a positive integer. By Proposition 2.5, we obtain

$$\begin{aligned} \mathfrak{E}(n) &= \sum_{k=1}^{n-1} E(k)E(n-k) = \sum_{i=0}^{7} \sum_{\substack{1 \leq k < n \\ k \equiv i \pmod 8}} E(k)E(n-k) \\ &= -\frac{1}{4} \left(\sum_{\substack{1 \leq k < n \\ k \equiv 0,1 \pmod 8}} h_1(k)h_1(n-k) + \sum_{\substack{1 \leq k < n \\ k \equiv 3,6 \pmod 8}} h_1(k)h_1(n-k) \right) \qquad (15) \\ &= -\frac{1}{4} \sum_{k=1}^{n-1} h_1(k)h_1(n-k). \end{aligned}$$

The last identity is obtained from the fact that $h_1(m) = 0$ if $m \equiv 5, 7 \pmod 8$. By Proposition 2.1, Lemma 2.2 and (15),

$$\sum_{k=1}^{n-1} E(k)E(n-k) = -\frac{1}{4} \left(\sum_{k=0}^{n} h_1(k)h_1(n-k) - 2h_1(0)h_1(n) \right) = \sigma(n) - E(n).$$

In the remaining cases $n \equiv 3, 5, 7 \pmod 8$, $\mathfrak{E}(n) = \sum_{k=1}^{n-1} E(k)E(n-k)$ can be obtained using the same method as $n \equiv 1 \pmod 8$. In particular, if $n \equiv 5, 7 \pmod 8$, then $\mathfrak{E}(n) = \sigma(n) - E(n) = \sigma(n)$ because $E(n) = 0$. It is easily obtained that $\sum_{k=1}^{2-1} E(k)E(2-k) = 1$ and $\sum_{k=1}^{4-1} E(k)E(4-k) = 5$. If $t \geq 3$ then $2^t \equiv 0 \pmod 8$.

Finally, we use Proposition 2.1 and Lemma 2.3 to obtain

$$\sum_{k=1}^{2^t-1} E(k)E(2^t-k) = \frac{1}{4} \left(\sum_{k=0}^{2^t} h_1(k)h_1(2^t-k) - 2h_1(0)h_1(2^t) \right) = \frac{1}{4}(24\sigma(1) - 4) = 5$$

in the same way as in (15). □

3. Inverse functions of E and E^2

From the definition of E, we get

$$E(p^t) = \begin{cases} 1 & \text{if } p = 2, \\ t+1 & \text{if } p \equiv 1, 3 \pmod 8, \\ 1 & \text{if } p \equiv 5, 7 \pmod 8 \text{ and } t \equiv 0 \pmod 2, \\ 0 & \text{if } p \equiv 5, 7 \pmod 8 \text{ and } t \equiv 1 \pmod 2. \end{cases} \qquad (16)$$

Here, $t \in \mathbb{N}_0$. It is a well-known fact that E is a multiplicative function, but we briefly prove it again in Lemma 3.1.

Lemma 3.1. *E, E^2, E^{-1} and $(E^2)^{-1}$ are multiplicative functions.*

Proof. Let $n = 2^l m$ with $\gcd(2, m) = 1$. Then, it is easily checked that $2^t \not\equiv 1, 3, 5, 7 \pmod 8$ with $1 \le t \le l$ and $d|m$. By Eqn. (16), $E(2^l m) = E(m) = E(2^l)E(m)$.

To prove Lemma 3.1, we only check that $E(m_1m_2) = E(m_1)E(m_2)$ with $\gcd(m_1, m_2) = 1$ and $m_1 \equiv m_2 \pmod 2$. Let p_i $(1 \le i \le r) \equiv 1, 3 \pmod 8$ and $q_j(1 \le j \le s) \equiv 5, 7 \pmod 8$ be distinct primes. Let $n_1 = p_1^{e_1} \cdots p_r^{e_r} q_1^{f_1} \cdots q_s^{f_s}$. It can be easily proved that

$$E(n_1) = 0 \text{ if and only if there exist at least one } f_j \equiv 1 \pmod 2. \tag{17}$$

For convenience, assume that f_1 is odd. Let $n_1 = m_1m_2$ with $\gcd(m_1, m_2) = 1$. Then either $q_1^{f_1}|m_1$ or $q_1^{f_1}|m_2$. By Eqn. (17), $E(n_1) = 0$ and $E(m_1) = 0$ or $E(m_2) = 0$. So $E(n_1) = 0 = E(m_1)E(m_2)$. By the definition of E, $E(p_1^{e_1} \cdots p_r^{e_r}) = (e_1+1) \cdots (e_r+1)$ and $E(p_1^{e_1} \cdots p_r^{e_r} q_1^{2f_1} \cdots q_u^{2f_u}) = (e_1+1) \cdots (e_r+1) = E(p_1^{e_1} \cdots p_r^{e_r})$.
Thus, if $m_3 = p_1^{e_1} \cdots p_t^{e_t} q_1^{2f_1} \cdots q_v^{2f_v}$ and $m_4 = p_{t+1}^{e_{t+1}} \cdots p_r^{e_r} q_{v+1}^{2f_{v+1}} \cdots q_s^{2f_s}$, then $E(m_3m_4) = (e_1 + 1) \cdots (e_r + 1) = E(m_3)E(m_4)$. Therefore, E is a multiplicative function. By the definition of E^2, $E^2(m_1m_2) = E(m_1m_2)E(m_1m_2) = E(m_1)E(m_2)E(m_1)E(m_2) = E^2(m_1)E^2(m_2)$ with $\gcd(m_1, m_2) = 1$. On the other hand, if O_1 is a multiplicative function, then O_1^{-1} is also a multiplicative function. See [10, p.8]. Using this, the proof of Lemma 3.1 is completed. □

Now, consider $E^{-1}(p^n)$ and $(E^2)^{-1}(p^n)$. It can be expressed a little differently, but for convenience, we will use $E^{-2}(m)$ instead of $(E^2)^{-1}(m)$. That is, we will use $E^2 * (E^2)^{-1}(m) = I(m)$ as $E^2 * E^{-2}(m) = I(m)$. Using Eqns. (1) and (16), we get

$$E^{-i}(2^t) = \begin{cases} 1 & \text{if } t = 0, \\ -1 & \text{if } t = 1, \\ 0 & \text{if } t \ge 2 \end{cases} \quad \textit{and} \quad E^{-i}(p^t) = \begin{cases} 1 & \text{if } t = 0, \\ 0 & \text{if } t = 1, \\ -1 & \text{if } t = 2, \\ 0 & \text{if } t \ge 3 \end{cases} \tag{18}$$

with $p \equiv 5, 7 (\text{mod } 8)$ and $i = 1, 2$. If $p \equiv 1, 3 (\text{mod } 8)$, then

$$E^{-1}(p^t) = \begin{cases} 1 & \text{if } t = 0, \\ -2 & \text{if } t = 1, \\ 1 & \text{if } t = 2, \\ 0 & \text{if } t \ge 3 \end{cases} \quad \textit{and} \quad E^{-2}(p^t) = \begin{cases} 1 & \text{if } t = 0, \\ -4 & \text{if } t = 1, \\ 7 & \text{if } t = 2, \\ (-1)^t 8 & \text{if } t \ge 3. \end{cases} \tag{19}$$

4. Proof of Theorem 1.2

To prove Theorem 1.2, the following Lemma 4.1 is necessary. That is, the following is the result giving the relationship between convolution sum and Dirichlet convolution sum:

Lemma 4.1. *If $a \in \mathbb{N}_0$ and $\widehat{\mathfrak{E}}(1) := 0$, then*

$$\widehat{\mathfrak{E}}(p^a) := \sum_{\substack{1 \le k \le p^a - 1 \\ gcd(k, p^a - k) = 1}} E(k)E(p^a - k) = \begin{cases} 0 & \text{if } a = 0, \\ \mathfrak{E}(p) & \text{if } a = 1, \\ (E^{-2} * \mathfrak{E})(p^a) & \text{if } a \ge 2. \end{cases}$$

Proof. Since p is prime, if $a = 1$, then

$$\widehat{\mathfrak{E}}(p) := \sum_{\substack{1 \le k \le p-1 \\ \gcd(k, p-k) = 1}} E(k)E(p-k) = \sum_{k=1}^{p-1} E(k)E(p-k) = \mathfrak{E}(p)$$

obviously holds. Let a (≥ 2) be a positive integer. By Lemma 3.1,

$$\begin{aligned} \mathfrak{E}(p^a) &= \sum_{k=1}^{p^a-1} E(k)E(p^a - k) = \sum_{i=0}^{a-1} \sum_{\substack{1 \le k < p^a - 1 \\ \gcd(k, p^a - k) = p^i}} E(k)E(p^a - k) \\ &= (E(p^0))^2 \widehat{\mathfrak{E}}(p^a) + (E(p))^2 \widehat{\mathfrak{E}}(p^{a-1}) + \cdots + (E(p^a))^2 \widehat{\mathfrak{E}}(p^0) \\ &= (E^2 * \widehat{\mathfrak{E}})(p^a). \end{aligned} \tag{20}$$

Therefore, Lemma 4.1 is proven. ∎

Theorem 4.2. *(a) If $p \equiv 1, 3 \pmod 8$ and $n \in \mathbb{N}$, then*

$$\widehat{\mathfrak{E}}(p^n) = \begin{cases} p - 1 & \text{if } n = 1, \\ (p-1)(p-2) & \text{if } n = 2, \\ (p-1)(p^2 - 2p + 2) & \text{if } n = 3, \\ \sigma(p^n) - 4\sigma(p^{n-1}) + 7\sigma(p^{n-2}) + (-1)^{\epsilon(n)} 2 - 8p^{\epsilon(n)} \sigma_2(p^{(n-\epsilon(n)-3)/2}) & \text{if } n \ge 4 \end{cases}$$

and if $p \equiv 5, 7 \pmod 8$, then $\widehat{\mathfrak{E}}(p^n) = p^{n-2}(p^2 + 1)$.

(b) If $n \in \mathbb{N}$, then

$$\widehat{\mathfrak{E}}(2^n) = \begin{cases} 1 & \text{if } n = 1, \\ 4 & \text{if } n = 2, \\ 0 & \text{if } n \ge 3. \end{cases}$$

Proof. (a)

$$\begin{aligned} \widehat{\mathfrak{E}}(p) &= \mathfrak{E}(p) = \sigma(p) - E(p) = (p+1) - 2 = (p-1), \\ \widehat{\mathfrak{E}}(p^2) &= (E^{-2} * \mathfrak{E})(p^2) = E^{-2}(1)\mathfrak{E}(p^2) + E^{-2}(p)\mathfrak{E}(p) \\ &= (\sigma(p^2) - E(p^2)) - 4(\sigma(p) - E(p)) \\ &= (p^2 + p + 1 - 3) - 4(p + 1 - 2) = (p-1)(p-2) \end{aligned}$$

and

$$\begin{aligned}\widehat{\mathfrak{E}}(p^3) =&E^{-2}(1)\mathfrak{E}(p^3)+E^{-2}(p)\mathfrak{E}(p^2)+E^{-2}(p^2)\mathfrak{E}(p)\\ =&(\sigma(p^3)-E(p^3))-4(\sigma(p^2)-E(p^2))+7(\sigma(p)-E(p))\\ =&(p-1)(p^2-2p+2).\end{aligned}$$

Now, consider the case where n is a positive integer greater than or equal to 4. Let $n=2a$ with $a\geq 2$. Then we obtain

$$\begin{aligned}\widehat{\mathfrak{E}}(p^{2a}) =&(\sigma(p^{2a})-E(p^{2a}))-4(\sigma(p^{2a-1})-E(p^{2a-1}))\\ &+7(\sigma(p^{2a-2})-E(p^{2a-2}))-8\sum_{k=0}^{2a-3}(-1)^k(\sigma(p^{2a-3-k})-E(p^{2a-3-k}))\\ =&\sigma(p^{2a})-4\sigma(p^{2a-1})+7\sigma(p^{2a-2})+(-E(p^{2a})+4E(p^{2a-1})-7E(p^{2a-2}))\\ &-8\sum_{k=1}^{a-2}(\sigma(p^{2k+1})-\sigma(p^{2k}))+8\sum_{k=1}^{a-2}(E(p^{2k+1})-E(p^{2k}))-8(\sigma(p)-E(p)).\end{aligned}$$

It is easily checked that $-E(p^{2a})+4E(p^{2a-1})-7E(p^{2a-2})=-8a+6$, $\sigma(p^{2k+1})-\sigma(p^{2k})=p^{2k+1}$, $E(p^{2k+1})-E(p^{2k})=1$ and $-8(\sigma(p)-E(p))=-8p+8$. Thus,

$$\begin{aligned}\widehat{\mathfrak{E}}(p^{2a}) =&\sigma(p^{2a})-4\sigma(p^{2a-1})+7\sigma(p^{2a-2})-8a+6-8(p^3+\cdots+p^{2a-3})\\ &+8(a-2)-8p+8\\ =&\sigma(p^{2a})-4\sigma(p^{2a-1})+7\sigma(p^{2a-2})-8p\sigma_2(p^{a-2})-2.\end{aligned}$$

Let $n=2a-1$ be an odd integer with $a\geq 3$. Similarly to the case $n=2a$, we obtain

$$\begin{aligned}\widehat{\mathfrak{E}}(p^{2a-1}) =&\sigma(p^{2a-1})-4\sigma(p^{2a-2})+7\sigma(p^{2a-3})+(-E(p^{2a-1})+4E(p^{2a-2})-7E(p^{2a-3}))\\ &-8\sum_{k=1}^{a-2}(\sigma(p^{2k})-\sigma(p^{2k-1}))+8\sum_{k=1}^{a-2}(E(p^{2k})-E(p^{2k-1}))\\ =&\sigma(p^{2a-1})-4\sigma(p^{2a-2})+7\sigma(p^{2a-3})-8\sigma_2(p^{a-2})+2.\end{aligned}$$

Secondly, consider the case where $n\equiv 5,7 \pmod 8$. Then, by (18) and Lemma 4.1,

$$\widehat{\mathfrak{E}}(p)=\mathfrak{E}(p)=\sigma(p)-E(p)=\sigma(p)=p+1,$$

$$\widehat{\mathfrak{E}}(p^2)=(E^{-2}*\mathfrak{E})(p^2)=E^{-2}(1)\mathfrak{E}(p^2)+E^{-2}(p)\mathfrak{E}(p)=\sigma(p^2)-E(p^2)=p(p+1),$$

$$\widehat{\mathfrak{E}}(p^n)=(E^{-2}*\mathfrak{E})(p^n)=E^{-2}(1)\mathfrak{E}(p^n)+E^{-2}(p^2)\mathfrak{E}(p^{n-2})=p^{n-2}(p^2+1)$$

with $n\geq 3$.

(b) It is easily seen that $\widehat{\mathfrak{E}}(2)=1$ and $\widehat{\mathfrak{E}}(4)=4$. By Proposition 2.6, Lemma 4.1 and Eqn. (18), $\widehat{\mathfrak{E}}(2^n)=E^{-2}(1)\mathfrak{E}(2^n)+E^{-2}(2)\mathfrak{E}(2^{n-1})=5-5=0$ with $n\geq 3$.

Finally, using (1), if we put $\widehat{\mathfrak{E}}(p^n)=\frac{1}{4}\#\mathfrak{N}(p^n)$ in Theorem 4.2, the proof of Theorem 1.2 is completed. □

5. Conclusions

Number theory is a field that has been studied intensively since ancient times. In particular, the four-square problem has been studied by many mathematicians. Finding the inverse divisor function using Dirichlet convolution is not well-known. In this article, the problem related to the changing relationship of four sets by means of the inverse divisor function is introduced and studied as a new challenge. We believe that the results of this article could be a source of inspiration for young mathematicians working in number theory and for fresh researchers thinking to orient their ability in this interseting field of mathematics.

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