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ON ALGEBRA OF LACUNARY STATISTICAL LIMIT OF DOUBLE SEQUENCES IN INTUITIONISTIC FUZZY NORMED SPACE

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Abstract. In 2005, Patterson studied lacunary statistical convergence of double sequences of real numbers and, in 2009, Mursaleen introduced notion of lacunary statistical convergence of single sequences in intuitionistic fuzzy normed space. The current work intends to investigate the lacunary statistical convergence of double sequences and some significant conclusions on the algebra of the lacunary statistical limit of double sequences in intuitionistic fuzzy normed space. In addition, we have studied some examples to support the definitions.

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1. Introduction

In 2004, Park[4] studied the notion of intuitionistic fuzzy metric space. Further, In 2006, Saddati and Park [10] introduced the idea of intuitionistic fuzzy normed space. In recent, the fuzzy theory, which was first investigated by Zadeh In 1965, becomes a wide area for active research. Many researchers examined their work on the idea of fuzzy theory, which have been now using in various areas of science and engineering, such as in chaos control, computer science, non-linear analysis and dynamical system etc. The statistical convergence of single sequences in intuitionistic fuzzy normed space has been studied in 2008, by Karkaus and Dimiric[11] while for double sequences, Mursaleen and Mohiuddine[6] published their findings in 2008. The motivation for the intuitionistic fuzzy normed space is that there are many situations in which it is not possible to find the norm of a vector as usual. In such cases, the intuitionistic fuzzy norm appears appropriate in evaluating the problem. Many authors have used the intuitionistic

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fuzzy norm to deal with the situation where the case of inexactness appears in the norm of a vector. In 1993, Fridy and Orhan[3] studied the concept of lacunary statistical convergence and established some key results on the concept. Later, In 2005, Patterson and Savas[8] examined the lacunary statistical convergence of double sequences in real system and further, in 2009, Mursaleen and Mohiuddine[5] presented some study on lacunary statistical convergence in the intuitionistic fuzzy normed space(IFNS)and investigated its some topological aspects. In recent, several new investigations on convergent sequence spaces in context of intuitionistic fuzzy normed space have been recorded [2, 7, 12, 13, 16].

The current investigation seeks to investigate the lacunary statistical convergence of double sequences in intuitionistic fuzzy normed space (IFNS), as well as the algebra of lacunary statistical limits in IFNS. There are four sections to the work. Section 1 provides the introduction and motivation for the study, Section 2 the fundamentals and preliminary results, Section 3 the lacunary statistical convergence of double sequences, and Section 4 the primary original results related to the algebra of lacunary statistical limits in IFNS. Throughout the article $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}$ denotes the set of real numbers.

2. Basics and preliminaries

Definition 2.1. [5] A continuous t-norm is a binary operation. $\circ : [0, 1]^2 \to [0, 1]$ satisfying the following conditions.

(i) $\circ$ is continuous.
(ii) $\circ$ is commutative.
(iii) $\circ$ is associative.
(iv) $\circ$ is monotonically increasing, i.e. for all $x, y, x_1, y_1 \in [0, 1]$ we have $x \circ x_1 \leq y \circ y_1$ whenever $x \leq x_1$ and $y \leq y_1$
(v) $x \circ 1 = x$ for all $x \in [0, 1]$

Definition 2.2. [5] A continuous t-conorm is a binary operation. $\diamond : [0, 1]^2 \to [0, 1]$ satisfying the following conditions.

(i) $\diamond$ is continuous.
(ii) $\diamond$ is commutative.
(iii) $\diamond$ is associative.
(iv) $\diamond$ is monotonically increasing, i.e. for all $x, y, x_1, y_1 \in [0, 1]$ we have $x \circ x_1 \leq y \circ y_1$ whenever $x \leq x_1$ and $y \leq y_1$.
(v) $x \diamond 0 = x$ for all $x \in [0, 1]$

Definition 2.3. [5] If M is a linear space then the five tuple $(M, \alpha, \beta, \circ, \diamond)$ where α and β are fuzzy sets on $M \times (0, \infty)$, $\circ$ and $\diamond$ denotes the continuous t-norm and continuous t-conorm respectively, is said to be an intuitionistic fuzzy normed space(**IFNS**) if for all $x, y \in M$ and for some $\tau > 0$ and $\xi > 0$ it satisfy the following subsequent requirements.

(i) $0 \leq \alpha(x, \tau) + \beta(x, \tau) \leq 1$
(ii) $\alpha(x, \tau) \geq 0$

(iii) $\alpha(x,\tau) = 1$ iff $x = 0$
(iv) $\alpha(x,.) : (0,\infty) \to [0,1]$ is continuous.
(v) $\alpha(cx,\tau) = \alpha\left(x, \frac{\tau}{|c|}\right)$ for $c \neq 0$
(vi) $\alpha(x+y,\tau+\xi) \geq \alpha(x,\tau) \circ \alpha(y,\xi)$
(vii) $\lim_{\tau\to 0} \alpha(x,\tau) = 0$ and $\lim_{\tau\to\infty} \alpha(x,\tau) = 1$
(viii) $\beta(x,\tau) < 1$
(ix) $\beta(x,\tau) = 0$ iff $x = 0$
(x) $\beta(x,.) : (0,\infty) \to [0,1]$ is continuous.
(xi) $\beta(cx,\tau) = \beta\left(x, \frac{\tau}{|c|}\right)$ for all non-zero c
(xii) $\beta(x+y,\tau+\xi) \leq \beta(x,\tau) \diamond \beta(y,\xi)$
(xiii) $\lim_{\tau\to 0} \beta(x,\tau) = 1$ and $\lim_{\tau\to\infty} \beta(x,\tau) = 0$

Definition 2.4. [6] A double sequence $x = (x_{ij})$ of the elements of IFNS $(M,\alpha,\beta,\circ,\diamond)$ is said to be convergent, with respect Intuitionistic fuzzy norm (α,β) to the number l if there exist some positive integers p and q such that for all $\varepsilon \in (0,1)$ and $\tau > 0$ we have

$$1 - \alpha(x_{ij} - l,\tau) < \varepsilon \text{ and } \beta(x_{ij} - l,\tau) < \varepsilon \text{ for all } i \geq p, j \geq q$$

Symbolically, we denote it as, $(\alpha,\beta) - \lim x = l$ or $x \xrightarrow{(\alpha,\beta)} l$

Definition 2.5. [6] A double sequence $x = (x_{ij})$ of the elements of IFNS $(M,\alpha,\beta,\circ,\diamond)$ is called Cauchy's sequence with respect intuitionistic fuzzy norm (α,β) if there exist some positive integers p and q such that for all $\varepsilon \in (0,1)$ and $\tau > 0$ we have

$$1 - \alpha(x_{ij} - x_{pq},\tau) < \varepsilon \text{ and } \beta(x_{ij} - x_{pq},\tau) < \varepsilon \text{ for all } i \geq p, \; j \geq q$$

3. Lacunary statistical Convergence

Statistical convergence of double sequences in IFNS has been studied by M. Mursaleen[6]. In this section we would study double lacunary sequence and lacunarily statistical convergence of the sequences.

Definition 3.1. [6] Let $\mathbb{N}$ denotes the set of natural numbers and if $A \subseteq \mathbb{N} \times \mathbb{N}$ then natural density of A is defined as

$$\delta(A) = \lim_{m,n\to\infty} \frac{1}{mn} |\{(i,j) \in A : i \leq m, j \leq n\}|$$

Where vertical bar denotes cardinal number of enclosed set.

Definition 3.2. [6] A double sequence $x = (x_{ij})$ of the elements of IFNS $(M,\alpha,\beta,\circ,\diamond)$ is said to be statistically convergent with respect intuitionistic fuzzy norm (α,β) to the number l if for all $\varepsilon \in (0,1)$ and $\tau > 0$ we have natural density of the set

$$A = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ij} - l,\tau) \geq \varepsilon \text{ or } \beta(x_{ij} - l,\tau) \geq \varepsilon\} = 0.$$

Equivalently

$$\lim_{m,n\to\infty}\frac{1}{mn}\left|\{(i,j)\in\mathbb{N}\times\mathbb{N}:1-\alpha\left(x_{ij}-l,\tau\right)\geq\varepsilon \ \text{ or } \ \beta\left(x_{ij}-l,\tau\right)\geq\varepsilon\}\right|=0.$$

Provided that the limit does exist. In notation, we denote it as, $St_2^{(\alpha,\beta)}-\lim x=l$ or $St_2-\lim x\xrightarrow{(\alpha,\beta)}l$

Remark 3.1. [6] Every ordinary convergent sequence is statistically convergent at same limit

Definition 3.3. [14] A lacunary sequence $\theta=(u_k)$ is a sequence of non negative integers such that. $u_0=0$ and $l_k=u_k-u_{k-1}\to\infty$ as $k\to\infty$
Throughout the article I_k will denote the interval $(u_{k-1},u_k]$ and c_k will stand for the quotient $\dfrac{u_k}{u_{k-1}}$.

Definition 3.4. [14] Let $E\subseteq\mathbb{N}$, where $\mathbb{N}$ denotes set of natural numbers,then the $\theta-$ density of E is defined as

$$\delta_\theta(E)=\lim_{k\to\infty}\frac{1}{l_k}\left|\{r\in\mathbb{N}:r\in I_k\cap E\}\right|.$$

Provided that limit does exist.

Definition 3.5. [5] If $\theta=(u_k)$ be a lacunary sequence then a sequence $x=(x_n)$ of the elements of IFNS $(M,\alpha,\beta,\circ,\diamond)$ is said to be lacunarylly statistical convergent, with respect intuitionistic fuzzy norm (α,β), to the number l if for all $\varepsilon\in(0,1)$ and $\tau>0$ the θ-density of the set

$$\{r\in\mathbb{N}:1-\alpha(x_r-l,\tau)\geq\varepsilon \ \text{ or } \ \beta(x_r-l,\tau)\geq\varepsilon\} \quad \text{equals } \ 0.$$

Equivalentaly,

$$\lim_{k\to\infty}\frac{1}{l_k}\left|\{r\in I_k:1-\alpha(x_r-l,\tau)\geq\varepsilon \quad \text{or} \quad \beta(x_r-l,\tau)\geq\varepsilon\}\right|=0.$$

Provided that the limit does exist. In notation, we denote it as, $S_\theta^{(\alpha,\beta)}-\lim x=l$ or $x_n\xrightarrow{(\alpha,\beta)}l(S_\theta)$

Definition 3.6. [4] A double lacunary sequence $\theta=\{(u_k,v_s)\}$ is an integer sequence of doublets such that

$$u_0=0; \quad l_k=u_k-u_{k-1} \quad \text{as} \quad k\to\infty \quad \text{and} \quad v_0=0; \quad l_s=v_s-v_{s-1} \quad \text{as} \quad s\to\infty$$

We use, $l_{k,s}=l_kl_s$ where,

$$I_k=\{p:p\in(u_{k-1},u_k]\} \quad \text{and} \quad I_s=\{q:q\in(v_{s-1},v_s]\}$$

$I_{k,s}=I_k\times I_s$; $c_k=\dfrac{u_k}{u_{k-1}}$ and $c_s=\dfrac{v_s}{v_{s-1}}$; $c_{k,s}=c_kc_s$
and by $N_{\theta_{k,s}}$, we denote the class of all double lacunary sequences.

Definition 3.7. Let $\theta = \{(u_k, v_s)\}$ be a double lacunary sequence then a double sequence $x = (x_{ij})$ of the elements of IFNS $(M, \alpha, \beta, \circ, \diamond)$ is said to be lacunarily statistical convergent, with respect intuitionistic fuzzy norm (α, β) to the number l if for all $\varepsilon \in (0,1)$ and $\tau > 0$ the $\theta-$ density of the set

$$\{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ij} - l, \tau) \geq \varepsilon \text{ and } \beta(x_{ij} - l, \tau) \geq \varepsilon\} \text{ is } 0.$$

Equivalently,

$$\lim_{k,s \to \infty} \frac{1}{l_k l_s} |\{(i,j) \in I_k \times I_s : 1 - \alpha(x_{ij} - l, \tau) \geq \varepsilon \text{ or } \beta(x_{ij} - l, \tau) \geq \varepsilon\}| = 0.$$

Provided that the limit does exist. In notation, we denote it as, $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$ or $x_{ij} \xrightarrow{(\alpha,\beta)} l(S_\theta)$

Theorem 3.8. *Let θ be a double lacunary sequence. If a double sequence $x = (x_{ij})$ of the elements of IFNS $(M, \alpha, \beta, \circ, \diamond)$ is statistical convergent to the number l then $x = (x_{ij})$ is lacunarily statistical convergent to the same limit l, with respect to intuitionistic fuzzy norm (α, β).*

Proof. For all $e \in (0,1)$, we have, $\delta(E) = 0$, where

$$E = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ij} - l, \tau) \geq \varepsilon \text{ or } \beta(x_{ij} - l, \tau) \geq \varepsilon\}$$

Also, $\theta = \{(u_k, v_s)\}$ now using the inequiality, $0 \leq \delta_\theta(E) \leq \delta(E)$ we get. $\delta_\theta(E) = 0$. □

Theorem 3.9. *Let $x = (x_{ij})$ be a double sequence of the elements of IFNS $(M, \alpha, \beta, \circ, \diamond)$ and θ is a double lacunary sequence. If $(\alpha, \beta) - \lim x = l$ then $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$, but converse is not true.*

Proof. We have, if $(\alpha, \beta) - \lim x = l$ then $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l$ (see [6]) Again, $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l$ then $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$ (by theorem 3.8), consequently we get $(\alpha, \beta) - \lim x = l \Rightarrow St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$

For converse part, we would construct an example which leads contradiction ahead. Let us consider an IFNS defined as (M=$\mathbb{R}^2, \alpha, \beta, \circ, \diamond$), where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ denotes the set of real numbers, and let $x \circ y = xy$ and $x \diamond y = min\{x+y, 1\}$ for all $x, y \in [0,1]$.

$$\alpha(u, \tau) = \frac{\tau}{\tau + \|u\|_2} \text{ and } \beta(u, \tau) = \frac{\|u\|_2}{\tau + \|u\|_2} \text{ for all } u \in \mathbb{R} \text{ and } \tau > 0.$$

We now define a double sequence of the elements of (M=$\mathbb{R}^2, \alpha, \beta, \circ, \diamond$) defined as

$$x = x_{ks} = \begin{cases} (k,s)^t : k \in I_{k^*}, s \in I_{s^*} \\ (1,1)^t : otherwise \end{cases}$$

where

$$I_{k^*} = \left\{k : u_{k-1} < k \leq u_{k-1} + [\sqrt{l_k}] + 1\right\}$$

and

$$I_{s^*} = \left\{s : u_{s-1} < s \leq u_s + [\sqrt{l_s}] + 1\right\}$$

where, [x] denotes greatest integral value staying below x, now for $\varepsilon > 0$ and $\tau > 0$ we put

$$K(\varepsilon, \tau) = \{(k, s) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ks} - l, \tau) \geq \varepsilon \text{ or } \beta(x_{ks} - l, \tau) \geq \varepsilon\}$$

therefore for every $\varepsilon \in (0, 1)$ and $(k, s) \in I_{k^*} \times I_{s^*}$ we obtain

$$1 - \alpha(x_{ks} - (1, 1), \tau) = 1 - \lim_{k,s} \frac{\tau}{\tau + \sqrt{(k-1)^2 + (s-1)^2}} \to 1 \geq \varepsilon$$

on the other hand

$$\beta(x_{ks} - (1, 1), \tau) \to 1 \geq \varepsilon$$

Thus from equations (3) and (3) we reach, $K(\varepsilon, \tau) = I_{k^*} \times I_{s^*}$ Now

$$\delta_\theta (K(\varepsilon, \tau)) = \lim_{k,s\to\infty} \frac{1}{l_k l_s} |K(\varepsilon, \tau)| = \lim_{k,s\to\infty} \frac{[\sqrt{l_k}][\sqrt{l_s}]}{l_k l_s}$$

$$\Rightarrow St_2 - \lim x \xrightarrow{(\alpha,\beta)} (1, 1)(S_\theta).$$

On the other hand, with respect to IFN (α, β) we have $(\alpha, \beta) - \lim x \neq (1, 1)^t$ If possible, we assume there exist some n_0 and $s_0 \in \mathbb{N}$ such that for all $k \geq k_0$ and $s \geq s_0$ for every $0 < \varepsilon < 1$ we have,

$$1 - \alpha(x_{ks} - (1, 1), \tau) < \varepsilon \text{ and } \beta(x_{ks} - (1, 1), \tau) < \varepsilon$$

Now taking some $k_1 \in I_{k+1}$ and $s_1 \in I_{s+1}$ such that

$$u_k < k_1 \leq u_{k+1} + [\sqrt{l_{k+1}}] \text{ and } u_s < s_1 \leq u_{s+1} + [\sqrt{l_{s+1}}]$$

then, we get

$$1 - \alpha(x_{k_1 s_1} - (1, 1), \tau) > \varepsilon \text{ and } \beta(x_{k_1 s_1} - (1, 1), \tau) > \varepsilon$$

whereas $k_1 > k_0$ and $s_1 > s_0$, that leads contradiction. we thus conclude that there does not exist any positive integers k_0, s_0 such that for every $0 < \varepsilon < 1$ we have,

$$1 - \alpha(x_{ks} - (1, 1), \tau) < \varepsilon \text{ and } \beta(x_{ks} - (1, 1), \tau) < \varepsilon$$

□

Lemma 3.10. *Let θ be double lacunary sequence and $x = (x_{ij})$ be a double lacunarlly statistical convergent sequence of the elements of IFNS $(M, \alpha, \beta, \circ, \diamond)$ then for every $0 < \varepsilon < 1$ and $\tau > 0$ the following are equivalents.*

(i) $t_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$
(ii) $\delta_\theta \{(i, j) : 1 - \alpha(x_{ij} - l, \tau) \geq \varepsilon\} = \delta_\theta \{(i, j) : \beta(x_{ij} - l, \tau) \geq \varepsilon\} = 0.$
(iii) $\delta_\theta \{(i, j) : 1 - \alpha(x_{ij} - l, \tau) < \varepsilon \text{ and } \beta(x_{ij} - l, \tau) < \varepsilon\} = 1$

(iv) $\delta_\theta\{(i,j) : 1-\alpha(x_{ij}-l,\tau) < \varepsilon\} = \delta_\theta\{(i,j) : \beta(x_{ij}-l,\tau) < \varepsilon\} = 1$

(v) $S_\theta - \lim \alpha(x_{ij}-l,\tau) = 1$ *and* $S_\theta - \lim \beta(x_{ij}-l,\tau) = 0.$

Theorem 3.11. *Let* $x = (x_{ij})$ *be a double sequence of the elements of IFNS* $(M, \alpha, \beta, \circ, \diamond)$ *and* θ *is a double lacunary sequence. If* $(\alpha, \beta) - \lim x = l$ *then* $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$*. But converse is not true.*

Proof.

$$\text{If } (\alpha,\beta) - \lim x = l \text{ then } St_2 - \lim x \xrightarrow{(\alpha,\beta)} l \quad \text{(see [9])}$$

Also, by theorem 3.9, we have

$$St_2 - \lim x \xrightarrow{(\alpha,\beta)} l \text{ then } St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$$

Hence we reach

$$(\alpha,\beta) - \lim x = l \Rightarrow St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$$

For converse part, we would construct an example which leads the contradiction ahead. Let us consider an IFNS defined as ($M = \mathbb{R}^2, \alpha, \beta, \circ, \diamond$), Where $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ and $\mathbb{R}$ denotes the set of real numbers, and let $x \circ y = xy$ and $x \diamond y = min\{x+y, 1\}$ for all $x, y \in [0,1]$.

$$\alpha(u,\tau) = \frac{\tau}{\tau + \|u\|_2} \text{ and } \beta(u,\tau) = \frac{\|u\|_2}{\tau + \|u\|_2} \text{ for all } u \in \mathbb{R} \text{ and } \tau > 0.$$

We now define a double sequence $x = (x_{ks})$ of the elements of (M=$\mathbb{R}^2, \alpha, \beta, \circ, \diamond$) defined as $x = (x_{ks})$ as

$$x = x_{ks} = \begin{cases} (k,s)^t : k \in I_{k^*}, s \in I_{s^*} \\ (1,1)^t : otherwise \end{cases}$$

Where

$$I_{k^*} = \Big\{k : u_{k-1} < k \leq u_{k-1} + [\sqrt{l_k}] + 1\Big\}$$

And

$$I_{s^*} = \Big\{s : u_{s-1} < s \leq u_s + [\sqrt{l_s}] + 1\Big\}$$

$[x]$ denotes greatest integral value staying below x

Now for $\varepsilon > 0$ and $\tau > 0$, we put

$$K(\varepsilon,\tau) = \{(k,s) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ks}-l,\tau) \geq \varepsilon \text{ or } \beta(x_{ks}-l,\tau) \geq \varepsilon\}$$

then for every $\varepsilon \in (0,1)$ and $(k,s) \in I_{k^*} \times I_{s^*}$ we have

$$1 - \alpha\left(x_{ks} - (1,1), \tau\right) = 1 - \lim_{k,s} \frac{\tau}{\tau + \sqrt{(k-1)^2 + (s-1)^2}} \to 1 \geq \varepsilon$$

Also, $\beta(x_{ks}-(1,1),\tau)\to 1\geq\varepsilon$, we thus obtain

$$K(\varepsilon,\tau)=I_{k^*}\times I_{s^*}$$

then

$$\delta_\theta\left(K(\varepsilon,\tau)\right)=\lim_{k,s\to\infty}\frac{1}{l_kl_s}|K\left(\varepsilon,\tau\right)|=\lim_{k,s\to\infty}\frac{[\sqrt{l_k}][\sqrt{l_s}]}{l_kl_s}$$

$$\leq\lim_{k,s\to\infty}\frac{\sqrt{l_kl_s}}{l_kl_s}=0.$$

$$\Rightarrow St_2-\lim x\xrightarrow{(\alpha,\beta)}(1,1)(S_\theta).$$

On the other hand, we have $(\alpha,\beta)-\lim x\neq(1,1)^t$ with respect to IFN (α,β) If possible, we assume there exist some n_0 and $s_0\in\mathbb{N}$ such that for all $k\geq k_0$ and $s\geq s_0$ for every $0<\varepsilon<1$ we have, $1-\alpha(x_{ks}-(1,1),\tau)<\varepsilon$ and $\beta(x_{ks}-(1,1),\tau)<\varepsilon$, now taking some $k_1\in I_{k+1}$ and $s_1\in I_{s+1}$ such that

$$u_k<k_1\leq u_{k+1}+[\sqrt{l_{k+1}}]$$

And

$$u_s<s_1\leq u_{s+1}+[\sqrt{l_{s+1}}]$$

then we get

$$1-\alpha(x_{k_1s_1}-(1,1),\tau)>\varepsilon\ \text{ and }\ \beta(x_{k_1s_1}-(1,1),\tau)>\varepsilon$$

whereas $k_1>k_0$ and $s_1>s_0$, which leads contradiction, we thus conclude that there does not exist any positive integers k_0, and s_0 such that for every $0<\varepsilon<1$ we can have

$$1-\alpha(x_{ks}-(1,1),\tau)<e\ \text{ and }\ \beta(x_{ks}-(1,1),\tau)<e$$

This completes the proof. □

4. Main results

In this section, we shall make some fundamental conclusions on the lacunarily statistical convergence of double sequences in intuitionistic fuzzy normed spaces.

Theorem 4.1. *Lacunary statistical limit of $x=(x_{ij})$ of the elements of IFNS $(M,\alpha,\beta,\circ,\diamond)$ is unique.*

Proof. Let $\theta=\{(u_k,v_s)\}$ be double lacunary sequence, we suppose that. $St_2-\lim x\xrightarrow{(\alpha,\beta)}l_1(S_\theta)$ and $St_2-\lim x\xrightarrow{(\alpha,\beta)}l_2(S_\theta)$ then by (ii) of lemma (3.10), we have

$$\delta_\theta(E)=\delta_\theta(F)=0$$

where

$$E=\{(i,j)\in\mathbb{N}\times\mathbb{N}:1-\alpha\left(x_{ij}-l,\tau\right)\geq\varepsilon\}$$

And

$$F=\{(i,j)\in\mathbb{N}\times\mathbb{N}:1-\alpha(x_{ij}-l,\tau)\geq\varepsilon\}$$

$$\delta_\theta(E^c) = 1 - \delta_\theta(E) = 1 - 0 = 1 \text{ and } \delta_\theta(F^c) = 1 - \delta_\theta(F) = 1 - 0 = 1$$

where

$$E^c = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\,(x_{ij} - l, \tau) < \varepsilon\}$$

And

$$F^c = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ij} - l, \tau) < \varepsilon\}$$

Also, we have

$$\delta_\theta(E^c \cap F^c) = 1 - \delta_\theta(E \cup F) = 1 - 0 = 1.$$

Let $(p,q) \in E^c \cap F^c$ and use (vi) of definition (2.3), now for every $0 < \varepsilon < 1$ we can find ε_1 and ε_2 in $(0,1)$ such that $(1-\varepsilon_1) \circ (1-\varepsilon_2) > 1 - \varepsilon$ and $\varepsilon_1 \diamond \varepsilon_2 < \varepsilon$ and then

$$\begin{aligned} 1 - \alpha\,((l_1 - l_2)\,, \tau) = 1 - \alpha\,((l_1 - x_{pq} + x_{pq} - l_2)\,, \tau) \leq \\ 1 - \alpha\left(l_1 - x_{pq}, \frac{\tau}{2}\right) \circ \alpha\left(x_{pq} - l_2, \frac{\tau}{2}\right) < 1 - (1-\varepsilon_1) \circ (1-\varepsilon_2) \\ < 1 - (1-\varepsilon) = \varepsilon \Rightarrow 1 - \alpha\,(l_1 - l_2, \tau) < \varepsilon. \end{aligned}$$

We thus obtained

$$\delta_\theta\,\{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(l_1 - l_2, \tau) < \varepsilon\} = 1 \Rightarrow l_1 = l_2.$$

□

Theorem 4.2. *Let $x = (x_{ij})$ and $y = (y_{ij})$ are two double sequences defined in IFNS $(M, \alpha, \beta, \circ, \diamond)$, which converges and let θ is a double lacunary sequence such that $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l_1(S_\theta)$ and $St_2 - \lim y \xrightarrow{(\alpha,\beta)} l_2(S_\theta)$ then $St_2 - \lim(x + y) \xrightarrow{(\alpha,\beta)} (l_1 + l_2)(S_\theta)$.*

Proof. For every $\varepsilon \in (0,1)$ and $\tau > 0$ we can find ε_1 and ε_2 in $(0,1)$ such that $(1-\varepsilon_1) \circ (1-\varepsilon_2) > 1 - \varepsilon$ and $\varepsilon_1 \diamond \varepsilon_2 < \varepsilon$ and by hypothesis, we have

$$\delta_\theta(E) = \delta_\theta(F) = 0.$$

Where

$$E = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\,(x_{ij} - l_1, \tau) \geq \varepsilon \text{ or } \beta\,(x_{ij} - l_1, \tau) \geq \varepsilon\}$$

And

$$F = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\,(y_{ij} - l_2, \tau) \geq \varepsilon \text{ or } \beta\,(y_{ij} - l_2, \tau) \geq \varepsilon\}$$

Therefore,

$$E^c = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\,(x_{ij} - l_1, \tau) < \varepsilon \text{ and } \beta\,(x_{ij} - l_1, \tau) < \varepsilon\}$$

And

$$F^c = \{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\,(y_{ij} - l_2, \tau) < \varepsilon \text{ and } \beta\,(y_{ij} - l_2, \tau) < \varepsilon\}$$

With

$$\delta_\theta(E^c) = \delta_\theta(F^c) = 1 \text{ (by lemma 3.10)}$$

Let $(p,q) \in E^c \cap F^c$, then (by using the (vi) of definition 2.3) we get,

$$1 - \alpha\left(x_{pq} + y_{pq} - (l_1 + l_2)\right), \tau) \leq 1 - \alpha\left(x_{pq} - l_1, \frac{\tau}{2}\right) \circ \alpha\left(y_{pq} - l_2, \frac{\tau}{2}\right)$$
$$< 1 - (1 - \varepsilon_1) \circ (1 - \varepsilon_2) < 1 - (1 - \varepsilon) = \varepsilon.$$

And

$$\beta\left(x_{pq} + y_{pq} - (l_1 + l_2), \tau\right) \leq \beta\left(x_{pq} - l_1, \frac{\tau}{2}\right) \diamond \beta\left(y_{pq} - l_2, \frac{\tau}{2}\right) < \varepsilon_1 \diamond \varepsilon_2 < \varepsilon.$$

which implies

$$\delta_\theta(E^c \cap F^c) = 1 - \delta_\theta(E \cup F) = 1 - 0 = 1$$

Hence

$$\delta_\theta\left\{(p,q) : 1 - \alpha(x_{pq} + y_{pq} - (l_1 + l_2) \geq \varepsilon \text{ or } \beta(x_{pq} + y_{pq} - (l_1 + l_2) \geq \varepsilon\right\} = 0.$$

□

Theorem 4.3. *Let* $x = (x_{ij})$ *is a double sequences defined in IFNS* $(M, \alpha, \beta, \circ, \diamond)$, *and let* θ *is a double lacunary sequence such that* $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$ *and if* λ *is any scalar then* $St_2 - \lim(\lambda x) \xrightarrow{(\alpha,\beta)} (\lambda l)(S_\theta)$.

Proof. For the case, when $\lambda \neq 0$ we have to prove the theorem. For every $\varepsilon \in (0,1)$ and $\tau > 0$, we have

$$\delta_\theta(E) = 0.$$

where

$$E = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\left(x_{ij} - l, \frac{\tau}{|\lambda|}\right) \geq \varepsilon \quad \text{or} \quad \beta\left(x_{ij} - l, \frac{\tau}{|\lambda|}\right) \geq \varepsilon\right\}$$

therefore

$$E^c = \left\{(i,j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha\left(x_{ij} - l, \frac{\tau}{|\lambda|}\right) < \varepsilon \quad \text{and} \quad \beta\left(x_{ij} - l, \frac{\tau}{|\lambda|}\right) < \varepsilon\right\}$$
$$\text{and} \quad \delta_\theta(E^c) = 1.$$

Let $(p,q) \in E^c$ then

$$1 - \alpha\left(\lambda x_{pq} - \lambda l, \tau\right) = 1 - \alpha\left(x_{pq} - l, \frac{\tau}{|\lambda|}\right) < \varepsilon$$

and

$$\beta\left(\lambda x_{pq} - \lambda l, \tau\right) = \beta\left(x_{pq} - l, \frac{\tau}{|\lambda|}\right) < \varepsilon$$

hence

$$\delta_\theta\left\{(p,q) \in \mathbb{N} \times \mathbb{N} \ : 1 - \alpha(\lambda x_{pq} - \lambda l, \tau) \geq \varepsilon \quad \text{or} \quad \beta(\lambda x_{pq} - \lambda l, \tau) \geq \varepsilon\right\} = 0.$$

□

Theorem 4.4. *If $x = (x_{ij})$ is bounded double sequences defined in IFNS $(M, \alpha, \beta, \circ, \diamond)$, and let θ is a double lacunary sequence such that $St_2 - \lim x \xrightarrow{(\alpha,\beta)} l(S_\theta)$ then $St_2 - \lim(x^2) \xrightarrow{(\alpha,\beta)} l^2(S_\theta)$.*

Proof. Since, $x = (x_{ij})$ is bounded then there exist and number L such that $|x_{ij}| \leq L$ for all i, j
Now for all $0 < \varepsilon < 1$ and $\tau > 0$ we have

$$\delta_\theta(E) = 0.$$

Where

$$E = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{ij} - l, \tau) \geq \varepsilon \text{ or } \beta\,(x_{ij} - l, \tau) \geq \varepsilon\}$$

Now we put.

$$F = \left\{(p, q) \in \mathbb{N} \times \mathbb{N} : 1 - \alpha(x_{pq}^2 - l^2, t) \geq \varepsilon \text{ or } \beta\left(x_{pq}^2 - l^2, t\right) \geq \varepsilon\right\}$$

We now wish to prove

$$\delta_\theta(F) = 0.$$

Let $(p, q) \in F$ and then

$$1 - \alpha\left(x_{pq}^2 - l^2, t\right) \geq \varepsilon \Rightarrow 1 - \alpha\left(\left(x_{pq} - l\right)\left(x_{pq} + l\right), t\right) \geq \varepsilon$$

which gives

$$1 - \alpha\left(x_{pq} - l, \tau\right) \geq e, \text{ where } \tau = \frac{t}{L + l} > 0$$

Again

$$\beta\left(x_{pq}^2 - l^2, t\right) \geq \varepsilon \rightarrow \beta\left((x_{pq} - l)(x_{pq} + l), t\right) \geq \varepsilon$$

which implies

$$\beta(x_{pq} - l, \tau) \geq \varepsilon, \text{ where } \tau = \frac{t}{L + l} > 0$$

$\Rightarrow (p, q) \in E.$ or , $\phi \subseteq F \subseteq E. \Rightarrow 0 \leq \delta_\theta(F) \leq \delta_\theta(E) = 0$

□

Conclusion

Theorems on the algebra of limit that are conserved in the case of ordinary convergence are extended in our work to the case of the lacunary statistical limit with regard to intuitionistic fuzzy normed space. Which are more general than the algebra of ordinary limits.

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