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L(4, 3, 2, 1)-PATH COLORING OF CERTAIN CLASSES OF GRAPHS

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Abstract. An $L(p_1,\ p_2,\ p_3,\ \ldots,\ p_m)$-labeling of a graph G is an assignment of non-negative integers, called as labels, to the vertices such that the vertices at distance i should have at least p_i as their label difference. If $p_1 = 4,\ p_2 = 3,\ p_3 = 2,\ p_4 = 1$, then it is called a $L(4,\ 3,\ 2,\ 1)$-labeling which is widely studied in the literature. A $L(4,\ 3,\ 2,\ 1)$-path coloring of graphs, is a labeling $g : V(G) \to Z^+$ such that there exists at least one path P between every pair of vertices in which the labeling restricted to this path is a $L(4,\ 3,\ 2,\ 1)$-labeling. This concept was defined and results for some simple graphs were obtained by the same authors in an earlier article. In this article, we study the concept of $L(4,\ 3,\ 2,\ 1)$-path coloring for complete bipartite graphs, 2-edge connected split graph, Cartesian product and join of two graphs and prove an existence theorem for the same.

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1. Introduction

All graphs studied in this paper are simple, connected, undirected and finite. The standard graph theory terminologies are considered from [1]. The Frequency Assignment Problem (FAP) has a wide range of significance as there is a extensive growth in wireless communication networks. To avoid the interference in FAP, the radio stations or the transmitters are to be assigned with frequencies which differ by certain minimum value. But this would mean a requirement of a huge frequency spectrum. Optimal use of frequency spectrum in order to minimize interference is the challenging aspect of the problem. In 1980, Hale et al., turned this FAP as a graph labeling problem which is defined as follows: An $L(p_1,\ p_2,\ \ldots..,\ p_m)$- labeling of a graph G, is labeling of vertices with non negative integers, called labels, such that the vertices at distance i are assigned

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with the labels whose difference is at least p_i [2, 3]. The distance two labeling or $L(2,\ 1)$-labeling is the most explored one in the literature [4, 5, 6, 7, 8]. In this labeling, adjacent vertices are assigned labels that differ by at least two and vertices at distance two assigned labels that differ by at least one. The largest integer used in any such labeling was called as the span of the labeling. The least value of span taken over all such labelings was called as $L(2,\ 1)$- number or λ- number of G, denoted by $\lambda(G)$.

Practically, interference can occur at more than two levels also. Jean Clipperton et al., studied $L(3,\ 2,\ 1)$-labeling problems and defined $L(3,\ 2,\ 1)$-labeling as an assignment of non negative integers to each vertex of G such that the vertices at distance $1,\ 2,\ 3$ are labelled with integers that differ by at least $3,\ 2,\ 1$ respectively [9]. Later, Soumen Atta et al., defined the $L(4,\ 3,\ 2,\ 1)$-labeling as an assignment of non negative integers to each vertex of G such that the vertices at distance $1,\ 2,\ 3,\ 4$ are labelled with a difference of at least $4,\ 3,\ 2,\ 1$ respectively. Analogous to the $L(2,\ 1)$- number, the $L(3,\ 2,\ 1)$-labeling number and $L(4,\ 3,\ 2,\ 1)$-labeling number of G respectively, denoted by $k(G)$ were defined. Minimizing the span, in terms of FAP meant the optimal use of frequency spectrum with minimal interference. In [10], $k(G)$ for paths, cycles, complete graphs and complete bipartite graphs was obtained. In [11], R. Sweetly and J. Paulraj Joseph obtained an upper bound for $k(G)$ in terms of maximum degree of G. Sk Amanathulla et al., discussed the $L(3,\ 2,\ 1)$ and $L(4,\ 3,\ 2,\ 1)$-labeling problem on interval graphs [12]. Variations of the FAP for higher levels of interference has been studied in [13, 14, 15, 16].

According to Ruxandra Marinescu-Ghemeci [17], in order to have a safe communication in security networks, existence of a path satisfying $L(2,\ 1)$- labeling condition is crucial. Restricting the interference level to two, the author studied $L(2,\ 1)$-path coloring in [17]. This definition seeks a path between every pair of vertices in the graph G such that, vertices at distance 1 and 2 on that path should be assigned labels that differ by at least 2 and 1 respectively. Again the highest integer used in such a labeling was called span and the minimum span over all such labelings was called 2-radio connection number, denoted by $\lambda_c(G)$. A graph G was called 2-radio connected if there was a $L(2,\ 1)$-path between every pair of vertices. Upper and lower bound for $\lambda_c(G)$ where G is connected graph with $n \geq 5$ vertices has been obtained in [17]. $\lambda_c(G)$ for graphs having Hamiltonian path, complete graphs, cycles, complete bipartite graphs, 2-edge connected split graphs, graph obtained by Cartesian product or join of two graphs were also addressed in this paper.

Applying the above idea of path coloring to the $L(3,\ 2,\ 1)$ labeling we conceived the idea of $L(3,\ 2,\ 1)$-path coloring of graph G in [18]. Analogous to 2-connection number of G, we defined 3-connection number, $k_c(G)$ in [18] and completely determined $k_c(G)$ for $G \simeq C_n$ or K_n, $K_{m,n}$, 2-edge connected Split graph, Cartesian product of two graphs, Join of two graphs. In our earlier work [19], we extended this concept to four levels by defining $L(4,\ 3,\ 2,\ 1)$-path coloring of graphs as follows :

A $L(4,\ 3,\ 2,\ 1)$-path coloring of graphs, is a labeling $g : V(G) \to Z^+$ such that there exists at least one path P between every pair of vertices in which the labeling restricted to this path is a $L(4,\ 3,\ 2,\ 1)$-labeling. This means that between every pair of vertices in the graph there should be at least one path such that, vertices on that path at distances $1, 2, 3$ and 4 should be assigned labels that differ by at least $4, 3, 2$ and 1 respectively. The maximum label given to any vertex of G under such a labeling g was called as span of g. The minimum value of span of g taken over all such labelings g was called $L(4,\ 3,\ 2,\ 1)$-connection number or 4-connection number of G, again denoted by $k_c(G)$. The value of $k_c(G)$ for graphs containing Hamiltonian path was obtained in [19]. In this paper, we find $k_c(G)$ of complete bipartite graphs, 2-edge connected split graph, Cartesian product of graphs, join of graphs and also prove an existence theorem. Some Remarks:

Remark 1.1. If 1 is not assigned to any vertex in a $L(4,\ 3,\ 2,\ 1)$-path coloring of a graph G, then we can get another $L(4,\ 3,\ 2,\ 1)$-path coloring of G by reducing the label of every vertex by 1. As a result of which the span is decreased by 1. Therefore, in a minimal $L(4,\ 3,\ 2,\ 1)$-path coloring of G, 1 must be assigned to some vertex of G.

Remark 1.2. For a graph G, let $g : V(G) \to Z^+$ be a $L(4,\ 3,\ 2,\ 1)$-path coloring of G with $k = span(g)$. Then the complementary coloring of g, denoted by g', defined as $g'(v) = k+1-g(v)$, for every $v \in V(G)$ is also an $L(4,\ 3,\ 2,\ 1)$-path coloring of G with span k.

Remark 1.3. Let T be any tree. Then, $k_c(T) = k(T)$. The definition of $L(4,\ 3,\ 2,\ 1)$-labeling and $L(4,\ 3,\ 2,\ 1)$-path coloring coincide as there is a unique path between every pair of vertices in T.

Remark 1.4. An optimal $L(4,\ 3,\ 2,\ 1)$-path coloring of G is a $L(4,\ 3,\ 2,\ 1)$-path coloring of a graph G with $span(g) = k_c(G)$ where $g : V(G) \to Z^+$.

We will be using the following notations in this paper:

(1) Let P be a path in G and u, v be the vertices in P, the sub path of P from u to v will be denoted by $u \overset{P}{-} v$.
(2) $G[C]$ is the subgraph induced by C in G where C is a subset of the vertex set of G.

2. Preliminary Results

In this section we discuss some preliminary results. At first we recall the following two Theorems for ready reference from [11].

Theorem 2.1. For $n \geq 2$, P_n be a path, then $k(P_n) = \begin{cases} 5, & \text{if } n = 2 \\ 8, & \text{if } n = 3 \\ 9, & \text{if } n = 4 \\ 11, & \text{if } 5 \leq n \leq 7 \\ 12, & \text{if } 8 \leq n \leq 12 \\ 13, & \text{if } n \geq 13 \end{cases}$

Theorem 2.2. For a star graph $k(K_{1,n}) = 3n + 2$.

Proposition 2.3. Let G be a connected graph with $n \geq 2$ vertices.

(1) $5 \leq k_c(P_{diam(G)+1}) \leq k_c(P_r) \leq k_c(G)$ for $r \geq diam(G) + 1$.

Proof. Let g be an optimal $L(4,\ 3,\ 2,\ 1)$-path coloring of G. Let u and v be vertices such that $d(u,\ v) = diam(G)$. Then any $L(4,\ 3,\ 2,\ 1)$-path P_r between them has at least $diam(G) + 1$ vertices and contains a sub path isomorphic to $P_{diam(G)+1}$. From Theorem 2.1 and Remark 1.3 we have the result. □

(2) If H is any induced subgraph of G containing a Hamiltonian path with $m<n$ vertices, $k_c(H) \leq k_c(G) \leq k_c(H) + 4(n - m)$

Proof. Since H is an induced subgraph of G, $k_c(H) \leq k_c(G)$.
The inequality $k_c(G) \leq k_c(H) + 4(n - m)$ holds by the definition of $L(4,\ 3,\ 2,\ 1)$-path coloring of graphs. □

(3) If H is a spanning connected subgraph of G, then $k_c(G) \leq k_c(H)$.

Proof. As H contains all the vertices of G, any $L(4,\ 3,\ 2,\ 1)$-path coloring of H is also a $L(4,\ 3,\ 2,\ 1)$-path coloring of G, hence the result. □

The following results are proved by us in [19]. We recall them here for ready reference.

Theorem 2.4. For any connected graph G with n vertices,

(1) $k_c(G) = 5$ if $n = 2$
(2) $k_c(G) = 8$ if $n = 3$
(3) $k_c(G) = 9$ if $n = 4$ and $G \ncong K_{1,3}$

Theorem 2.5. Let G be a connected graph with $5 \leq n \leq 7$ vertices and containing Hamiltonian path. Then $k_c(G) = 11$.

Theorem 2.6. Let G be a connected graph with n vertices. Then

(1) $k_c(G) = 12$ if $8 \leq n \leq 12$ and G contains a Hamiltonian path.
(2) $k_c(G) \geq 13$, otherwise

3. Main results

In this section, we focus on graphs that do not contain a Hamiltonian path such as Complete bipartite graph. Also we prove the result for 2-edge connected split graph.

Proposition 3.1. Let $1 \leq m \leq n$. Then

$$k_c(K_{m,n}) = \begin{cases} 3n+2, & \text{if } m = 1 \\ n+8, & \text{if } m = 2,\ n \geq 6 \\ 9, & \text{if } m, n = 2 \\ 11, & \text{if } m = 2, n = 3,\ m = n = 3 \text{ and } m = 3, n = 4 \\ 12, & \text{if } 4 \leq m = n \leq 6,\ m = 4, n = 5 \text{ and } m = 5, n = 6 \\ 13, & \text{otherwise} \end{cases}$$

Proof. **Case 1:** By Theorem 2.2 and Remark 1.3, $k(K_{1,n}) = k_c(K_{1,n}) = 3n+2$.
Case 2: For $m = 2,\ n \geq 6$, let $V(K_{2,n}) = \{x_1,\ x_2\} \cup \{y_1,\ y_2,\ y_3,\ \ldots,\ y_n\}$.

$$\text{Define } g \text{ as follows: } g(v) = \begin{cases} 1, & \text{if } v = x_1 \\ 4, & \text{if } v = x_2 \\ 6, & \text{if } v = y_1 \\ 8, & \text{if } v = y_2 \\ 9, & \text{if } v = y_3 \\ j+8, & \text{if } v = y_j,\ 4 \leq j \leq n \end{cases}$$

It can be easily seen that between each pair of vertices of $K_{2,n}$ there exists a $L(4,\ 3,\ 2,\ 1)$-path coloring as follows:

(1) The path from x_1 to x_2 is $[x_1, y_j, x_2]$ where $4 \leq j \leq n$.
(2) The path from x_1 to y_j, $1 \leq j \leq n$ is $[x_1, y_j]$.
(3) The path from x_2 to y_j , $2 \leq j \leq n$ is $[x_2, y_j]$.
(4) The path from x_2 to y_1 is $[x_2, y_j, x_1, y_1]$ where $3 \leq j \leq n$.
(5) The path from y_1 to y_j, $3 \leq j \leq n$ is $[y_1, x_1, y_j]$.
(6) The path from y_1 to y_2 is $[y_1, x_1, y_j, x_2, y_2]$ where $4 \leq j \leq n$.
(7) The path from y_2 to y_3 is $[y_2, x_1, y_j, x_2, y_3]$ where $4 \leq j \leq n$.
(8) The path from y_2 to y_j, $4 \leq j \leq n$ is $[y_2, x_1, y_j]$.
(9) The path from y_3 to y_j, $4 \leq i \leq n$ is $[y_3, x_1, y_j]$.
(10) The path from y_i to y_j, $4 \leq i < j \leq n$, is $[y_i, x_2, y_3, x_1, y_j]$.

It remains to show that $k_c(K_{2,n}) \nless n+8$.
Suppose $k_c(K_{2,n}) = n+7$, there exists two vertices between x_i and x_j such that $g(x_i) = 1$ and $g(x_j) = n+7$ or $g(x_i) = 1$ and $g(y_j) = n+7$ or $g(y_i) = 1$ and $g(y_j) = n+7$.
Subcase 2.1: Let $g(x_1) = 1$ and $g(x_2) = n+7$.
Then $g(y_1) = 5$ and $g(y_2) = 9$ implies that $g(y_3) = 4$, $g(y_4) = 8$ and $g(y_i) = i+5$, $5 \leq i \leq n-1$. As $g(x_2) = n+7$, let $g(y_n) = n+7+4 = n+11$ or

$g(y_n) = n + 7 - 4 = n + 3$.
If $n = 1$, then $g(x_2) = 8$ which is not possible as $g(y_4) = 8$.
If $n = 2$ then $g(x_2) = 9$, not possible as $g(y_2) = 9$.
If $n = 3$ then $g(x_2) = 10$, as $g(y_2) = 9$, there is only a path of length 3 from x_2 to y_2, so the labeling condition fails here.
If $n = 4$ then $g(x_2) = 11$. Since $g(y_i) = i + 5$, for $i = 5$, $g(y_i) = 10$. So, there is no path from x_2 to y_i.
Hence for all $n \geq 5$, similar contradiction occurs. Therefore, $g(x_2) \neq n + 7$.
Similar contradiction appears if x_1 and x_2 are interchanged.
Subcase 2.2: Let $g(x_1) = 1$ and $g(y_1) = n + 7$.
Then, $g(x_2) = 4$ and $g(y_2) = 6$ implies that $g(y_3) = 8$, $g(y_4) = 9$ and $g(y_i) = i+5$, $5 \leq i \leq n - 1$.
Since y_n is at distance two from y_1, $g(y_n) = n + 7 + 3 = n + 10$ or $g(y_n) = n + 7 - 3 = n + 4$ which implies that there is no path from y_n to y_i, a similar contradiction occurs as in Subcase 2.1. Again a similar contradiction occurs in the case $g(y_i) = 1$ and $g(y_j) = n + 7$.
Thus, it is not possible to have a $L(4,\ 3,\ 2,\ 1)$-path coloring of $K_{2,n}, n \geq 6$ with $n + 7$ or fewer colors. $\therefore\ k_c(K_{2,n}) = n + 8$.
A $L(4,\ 3,\ 2,\ 1)$-path coloring of $m = 2$, $n = 6$ is shown in Figure 1.
Case 3: For $n = m = 2$, result holds by the Theorem 2.4, point 3.

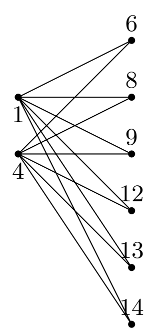

FIGURE 1. $L(4,\ 3,\ 2,\ 1)$-path coloring of $K_{2,6}$

Case 4: In this case G is a graph with $5 \leq m+n \leq 7$ and contains a Hamiltonian path. Then by Theorem 2.5, $k_c(K_{m,n}) = 11$.
Case 5: In this case G is a graph with $8 \leq n \leq 12$ and contains a Hamiltonian path. Then, by Theorem 2.6, point 1, $k_c(K_{m,n}) = 12$.
Case 6: From the Theorem 2.6, point 2, $k_c(G) \geq 13$. By the labeling shown in Figures 2 and 3, $k_c(K_{2,4}) = k_c(K_{2,5}) = 13$.

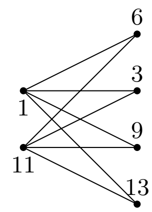

FIGURE 2. $L(4,3,2,1)$-path coloring of $K_{2,4}$

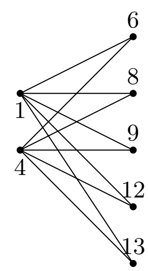

FIGURE 3. $L(4,\ 3,\ 2,\ 1)$-path coloring of $K_{2,5}$

For all other m, n vertices, let $V(K_{m,n}) = \{x_1, x_2, x_3, \ldots x_m\} \cup \{y_1, y_2, y_3, \ldots, y_n\}$ where $3 \leq m < n$ such that $n \geq m+2$.

Now, define the labelling g as follows: $g(v) = \begin{cases} 1, & \text{if } v = x_1 \\ 11, & \text{if } v = x_2 \\ 6, & \text{if } v = y_1 \\ 3, & \text{if } v = y_2 \\ 8, & \text{if } v = x_i, 3 \leq i \leq m \\ 13, & \text{if } v = y_j, 3 \leq j \leq n \end{cases}$

It can be easily seen that between each pair of vertices of $K_{m,n}$ there exists a $L(4,\ 3,\ 2,\ 1)$-path coloring as follows:

(1) The path from x_1 to x_2 is $[x_1, y_1, x_2]$.
(2) The path from x_1 to x_i, $\forall i$, $3 \leq i \leq m$, is $[x_1, y_i, x_i]$.
(3) The path from x_2 to x_i $\forall i$, $3 \leq i \leq m$ is $[x_2, y_2, x_i]$.
(4) The path from x_i to x_j, $3 \leq i < j \leq n$, is $[x_i, y_2, x_2, y_1, x_1, y_j, x_j]$.
(5) The path from y_1 to y_2 is $[y_1, x_2, y_2]$.
(6) The path from y_2 to y_i, $\forall i$, $3 \leq i \leq n$ is $[y_2, x_2, y_1, x_1, y_i]$.
(7) The path from y_1 to y_i, $\forall i$, $3 \leq i \leq m$, is $[y_1, x_2, y_2, x_i, y_i]$.
(8) The path from y_i to y_j, $3 \leq i < j \leq n$, is $[y_i, x_1, y_1, x_2, y_2, x_i, y_j]$.
(9) The path from x_i to y_j, $3 \leq i \leq m$, $3 \leq j \leq n$ is $[x_i, y_j]$.

A $L(4,\ 3,\ 2,\ 1)$-path coloring of $m = 4$, $n = 6$ is shown in the Figure 4 below.

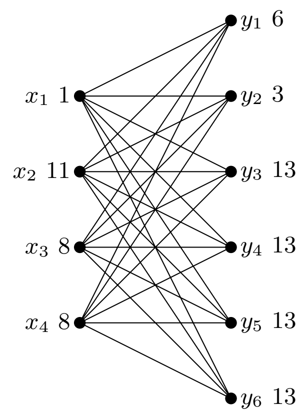

FIGURE 4. $L(4,\ 3,\ 2,\ 1)$-path coloring of $K_{4,6}$

□

Theorem 3.2. For 2-edge connected split graph G with $n \geq 13$, $k_c(G) = 13$.

Proof. Let G be a 2-edge connected split graph with $V(G)$ as vertex set, $E(G)$ as a edge set. Then $V(G)$ can be partitioned into a subset C such that $G[C]$ is a clique with at least 11 vertices and an independent set S with at least two vertices.

By Theorem 2.6, point 2, $k_c(G) \geq 13$. Now, it is enough to prove for equality that there exists an $L(4,\ 3,\ 2,\ 1)$-path with 13 colors. Define a labeling g as follows:

Step 1: Label vertices of C arbitrarily with $\{1,\ 6,\ 8\}$ such that each label is used at least once and the labels $\{3,\ 11,\ 13\}$ are used exactly once.

Step 2: Let $s \in S$ and g_s and $g_s^{'}$ be two neighbors of s in C. Then we say s is of type 1 if $c(g_s) \neq c(g_s^{'})$,
of type 2 if $c(g_s) = c(g_s^{'}) = 1$,
of type 3 if $c(g_s) = c(g_s^{'}) = 6$,
of type 4 if $c(g_s) = c(g_s^{'}) = 8$
Label s as follows:

$$c(s)=\begin{cases}\{1,\ 6,\ 11, 3,\ 8,\ 13\}-\{c(g_s),\ c(g_s^{'})\} & \text{if} s \text{is of type 1 and if} c(g_s)=1, c(g_s')=6 \text{then} c(s)\neq 3\\ \{1,\ 6,\ 11, 3,\ 8,\ 13\}-\{c(g_s),\ c(g_s^{'})\} & \text{if} s \text{is of type 1 and if} c(g_s)=6, c(g_s')=11 \text{then} c(s)\neq 8\\ \{1,\ 6,\ 11, 3,\ 8,\ 13\}-\{c(g_s),\ c(g_s^{'})\} & \text{if} s \text{is of type 1 and if} c(g_s)=8, c(g_s')=3 \text{then} c(s)\neq 6\\ \{1,\ 6,\ 11, 3,\ 8,\ 13\}-\{c(g_s),\ c(g_s^{'})\} & \text{if} s \text{is of type 1 and if} c(g_s)=8, c(g_s')=13 \text{then} c(s)\neq 11\\ 9, & \text{if} s \text{is of type 2}\\ 10, & \text{if} s \text{is of type 3}\\ 4 & \text{if} s \text{is of type 4}\end{cases}$$

Now, to prove that c is an $L(4,\ 3,\ 2,\ 1)$-path coloring, find an $L(4,\ 3,\ 2,\ 1)$-path P between each pair (x, y) of distinct vertices of G. So, we consider the following cases.

Case 1: $x,\ y \in C$. If $c(x) \neq c(y)$ then either $e = xy$ is a $L(4,\ 3,\ 2,\ 1)$-path between x and y or there exist $p \in C$ such that $x \overset{P}{-} y = [x,\ p,\ y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y or there exist $p \in C$ such that $x \overset{P}{-} y = [p,\ x,\ y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y or there exist $p, q \in C$ such that $x \overset{P}{-} y = [x,\ p,\ y,\ q]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y or there exists $p,\ q \in C$ such that $x \overset{P}{-} y = [p,\ x,\ q,\ y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y where $g(p),\ g(q) \in \{1,\ 6,\ 11,\ 3,\ 8,\ 13\} - \{c(x), c(y)\}$.

If $c(x) = c(y) = 1$, then there is a $L(4,\ 3,\ 2,\ 1)$-path from x to y such that $x \overset{P}{-} y = \{1,\ 6,\ 11,\ 3,\ 8,\ 13,\ 1\}$.

If $c(x) = c(y) = 6$, then there is a $L(4,\ 3,\ 2,\ 1)$-path from x to y such that $x \overset{P}{-} y = \{6,\ 11,\ 3,\ 8,\ 13,\ 1,\ 6\}$.

If $c(x) = c(y) = 8$, then there is a $L(4,\ 3,\ 2,\ 1)$-path from x to y such that $x \overset{P}{-} y = \{8,\ 13,\ 1,\ 6,\ 11,\ 3,\ 8\}$.

Case 2: $x \in S$, $y \in C$.

Case 2.1: x is of type 2, 3 or 4. Choose $g_x \in C$. By case 1, if x is of type 2 or 4, then we can find an $L(4,\ 3,\ 2,\ 1)$- path Q from g_x to y. Let $P = [x,\ Q]$ be the path obtained by adding x at the beginning of Q. If x is of type 3, then we can find an $L(4,\ 3,\ 2,\ 1)$- path Q from g_x to y such that $P = [Q,\ x]$. Since $c(x) \notin \{1, 6, 11, 3, 8, 13\}$, P is an $L(4,\ 3,\ 2,\ 1)$-path.

Case 2.2: x is of type 1. Then there exists $g_x \in C$ with $c(g_x) \neq c(y)$. If $c(x) \neq c(y)$, consider $P = [x,\ g_x,\ y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y or $P = [g_x,\ x,\ y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y or consider $P = [x,\ g_x,\ y,\ g_y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y or $P = [g_x,\ x,\ g_y,\ y]$ is the $L(4,\ 3,\ 2,\ 1)$-path between x and y where $g(g_x),\ g(g_y) \in \{1,\ 6,\ 11,\ 3,\ 8,\ 13\} - \{c(x),\ c(y)\}$.

If $c(x) = c(y) = 1$ or $c(x) = c(y) = 8$, choose $x_1,\ x_2,\ x_3,\ x_4,\ x_5$ in C unique colors from $\{1,\ 6,\ 11,\ 3,\ 8,\ 13\}$-$\{c(g_x),\ c(y)\}$ and let $P = [x\ , g_x,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ y]$.

If $c(x) = c(y) = 6$, choose $x_1,\ x_2,\ x_3,\ x_4,\ x_5$ in C unique colors from

$\{1,\ 6,\ 11,\ 3,\ 8,\ 13\}$-$\{c(g_x),\ c(y)\}$ and let $P = [g_x,\ x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ y,\ x]$.
Case 3: $x,\ y \in S$.
Case 3.1: Both x and y are of type 2, 3 or 4. Choose $g_x \in C$ and $g_y \in C$ with $g_x \neq g_y$ and x is adjacent to both g_x and g_y. Let Q be a $L(4,\ 3,\ 2,\ 1)$-path in C from g_x to g_y (exists from case 1). If x and y are of type 2 or 4, then $P = [x,\ Q,\ y]$ is a $L(4,\ 3,\ 2,\ 1)$-path. If x and y are of type 3, then $P = [y,\ Q,\ x]$ is a $L(4,\ 3,\ 2,\ 1)$-path.
Case 3.2: x is of type 2,3 or 4 and y is of type 1. Let $g_x \in C$. By case 2 there is Q a $L(4,\ 3,\ 2,\ 1)$-path from g_x to y, having only vertices of colors $\{1,\ 6,\ 11,\ 3,\ 8,\ 13\}$. Consider $P = [x,\ Q]$.
Case 3.3: Both x and y are of type 1. We then have $\{c(x)\} \cup c(g_x) = \{c(y)\} \cup c(g_y) = \{1,\ 6,\ 11,\ 3,\ 8,\ 13\}$. If $c(x) = c(y)$, choose $g_m,\ g_n,\ g_o,\ g_p \in C$ such that $c(g_m) \neq c(g_n) \neq c(g_o) \neq c(g_p)$. Then $|c(g_i) - c(g_j)| \geq 4$, $m \leq i<j \leq p$ and $P = [x,\ g_m,\ g_n,\ g_o,\ g_p,\ g_y,\ y]$ is a $L(4,\ 3,\ 2,\ 1)$-path.
If $c(x) \neq c(y)$, let $g_x \in C$ of color $c(y)$ and $g_y \in C$ of color $c(x)$. Let $z_1,\ z_2,\ z_3,\ z_4$ be a vertices from Q of color $\{1,\ 6,\ 11,\ 3,\ 8,\ 13\} - \{c(x),\ c(y)\}$. Consider $P = [x,\ g_x,\ z_1,\ z_2,\ z_3,\ z_4,\ g_y,\ y]$.
In all cases, it is easily proved that path P is a $L(4,\ 3,\ 2,\ 1)$-path.
Hence c is a $L(4,\ 3,\ 2,\ 1)$-path coloring. □

4. Graph Operation

Cartesian Product of two graphs G and H, $G \times H$ is the graph with $V(G) \times V(H)$ as the disjoint vertex set and $E(G \times H)$ as the edge set such that $\{((u,v),(u^{'},v^{'}))|(u = u^{'}$ and $vv^{'} \in E(H))or(uu^{'} \in E(G)$ and $v = v^{'})\}$.

Theorem 4.1. Let G and H be the two connected non trivial graphs.

$$\text{Then } k_c(G\times H) = \begin{cases} 9, & \text{if } |V(G)| = |V(H)| = 2 \\ 11, & \text{if } 5 \leq |V(G)| + |V(H)| \leq 7 \text{ where } G \text{ and } H \text{ contains Hamiltonian path} \\ 12, & \text{if } 8 \leq |V(G)| + |V(H)| \leq 12 \text{ where } G \text{ and } H \text{ contains Hamiltonian path} \\ 13, & \text{if } |V_1| \geq 13 \text{ and } |V_2| \geq 13 \text{ where } G \text{ and } H \text{ contains Hamiltonian path} \end{cases}$$

Proof. **Case 1:** This case holds by Theorem 2.4, point 3.
Case 2: Since Cartesian product of two graphs containing Hamiltonian path is also a graph that contains a Hamiltonian path, by Theorem 2.5, result holds.
Case 3: Similar to Case 2, result holds by Theorem 2.6, point 1.
Case 4: For all $i,\ j,\ 1 \leq i \leq m,\ 1 \leq j \leq n$ and $i \leq j$, define g as follows:

$$\text{If } i \equiv 1 \pmod 2, \text{ then } g(u_i, v_j) = \begin{cases} 1, & \text{if } j \equiv 1 \pmod 6 \\ 6, & \text{if } j \equiv 2 \pmod 6 \\ 11, & \text{if } j \equiv 3 \pmod 6 \\ 3, & \text{if } j \equiv 4 \pmod 6 \\ 8, & \text{if } j \equiv 5 \pmod 6 \\ 13, & \text{if } j \equiv 0 \pmod 6 \end{cases} \quad \text{If } i \equiv 0 \pmod 2,$$

$$\text{then } g(u_i, v_j) = \begin{cases} 13, & \text{if } j \equiv 1 \pmod 6 \\ 8, & \text{if } j \equiv 2 \pmod 6 \\ 3, & \text{if } j \equiv 3 \pmod 6 \\ 11, & \text{if } j \equiv 4 \pmod 6 \\ 6, & \text{if } j \equiv 5 \pmod 6 \\ 1, & \text{if } j \equiv 0 \pmod 6 \end{cases}$$

By the Theorem 2.6, point 2, $k_c(G \times H) \geq 13$. By the labelling defined above $k_c(G \times H) \leq 13$. It can be easily seen that between each pair of vertices of G there is a $L(4,\ 3,\ 2,\ 1)$-path coloring as follows:

(1) For $i \equiv 1 \pmod 2$, the path from $u_iv_j - u_iv_p$, $1 \leq p \leq n$ is $[u_iv_j, u_iv_{j+1}, u_iv_{j+2}, \ldots, u_iv_p]$.
Also, in $i \equiv 0 \pmod 2$ case, the same path $[u_iv_j, u_iv_{j+1}, u_iv_{j+2}, \ldots, u_iv_p]$, $1 \leq p \leq n$ holds
(2) For $i \equiv 1 \pmod 2$, $q \equiv 0 \pmod 2$, the path from $u_iv_j - u_qv_r$ is $[u_iv_n, u_iv_{n-1}, u_iv_{n-2}, u_iv_1, u_qv_1, u_qv_2, \ldots, u_qv_r]$.
(3) For $i \equiv 1 \pmod 2$, the path from $u_iv_j - u_qv_r$ is obtained by combining the paths from 1 and 2.
Same path occurs in the case $q \equiv 1 \pmod 2$

□

Join of two graphs G and H, $G \vee H$, is the graph with $V(G) \cup V(H)$ as the disjoint vertex set and $E(G \vee H)$= $E(G) \cup E(H) \cup \{uv | u \in V(G)$ and $v \in V(H)$ $\}$ as the edge set.

Theorem 4.2. If G and H are the two connected non trivial graphs,

$$\text{then } k_c(G \vee H) = \begin{cases} 9, & \text{if } |V(G)| = |V(H)| = 2 \\ 11, & \text{if } 5 \leq |V(G)| + |V(H)| \leq 7 \text{ where } G \text{ and } H \text{ contains Hamiltonian path} \\ 12, & \text{if } 8 \leq |V(G)| + |V(H)| \leq 12 \text{ where } G \text{ and } H \text{ contains Hamiltonian path} \\ 13, & \text{otherwise} \end{cases}$$

Proof. If $|V(G)| = |V(H)| = 2$ then G and H are isomorphic to P_2, hence $k_c(G \vee H)$=$k_c(K_4) = 9$, by Theorem 2.4, point 3.
If G and H contains a Hamiltonian path, then the $G \vee H$ also contains a Hamiltonian path in it where $5 \leq |V(G)| + |V(H)| \leq 7$ and $8 \leq |V(G)| + |V(H)| \leq 12$. Thus, the result holds from Theorem 2.5 and Theorem 2.6, point 1. Thus, $k_c(G \vee H) = 11$ for $5 \leq n \leq 7$ and $k_c(G \vee H) = 12$ for $8 \leq n \leq 12$ respectively.
Otherwise, $G \vee H$ contains a spanning connected graph isomorphic to $K_{m,n}$ where $n = |V(G)|$ and $m = |V(H)|$. By Theorem 2.6, point 2 we have $13 \leq k_c(G \vee H) \leq k_c(K_{m,n}) = 13$. □

5. Existence Theorem

In the previous sections we have seen that $k_c(G) \geq 5$ and $k_c(G) \neq 6, 7, 10$ for any graph G. In this section we answer the reverse question i. e, "Given any positive integer, does there always exist a graph G, whose $k_c(G)$ is the given integer?" The following Theorem 5.1 answers this question in the affirmative.

Theorem 5.1. For any integer $a \geq 13$, there always exists a graph G with $k_c(G) = a$.

Proof. Construct a graph G with vertex set, $V(G) = \{u, v,\ u_1, u_2, \ldots, u_6\} \cup \{v_1, v_2, \ldots, v_n\}$ and edge set, $E(G) = \{uv_i, vv_i | 1 \leq i \leq n\} \cup \{v_1u_1, u_iu_{i+1} | 1 \leq i \leq 5\}$ with $n = a - 8 \geq 5$. Then G has a sub graph isomorphic to $K_{2,n}$ and hence $k_c(G) \geq k_c(K_{2,n}) = n + 8$ (By Proposition 2.3 point 2, Remark 1.3 and Theorem 2.5)
We exhibit a labelling g below with span $n+8$ which proves the required result.

$$\text{Let } g(x) = \begin{cases} 1, & \text{if } x = u \\ 4, & \text{if } x = v \\ 6, & \text{if } x = v_1 \\ 11, & \text{if } x = u_1 \\ 3, & \text{if } x = u_2 \\ 8, & \text{if } x = u_3 \\ 13, & \text{if } x = u_4 \\ 5, & \text{if } x = u_5 \\ 8, & \text{if } x = v_2 \\ 9, & \text{if } x = v_3 \\ 10, & \text{if } x = u_6 \\ i+8, & \text{if } x = v_i,\ 4 \leq i \leq n \end{cases}$$

It can be easily seen that between each pair of vertices of G there is a $L(4,\ 3,\ 2,\ 1)$-path coloring as follows:
Since u is adjacent to v_1, v_1 is adjacent to u_i and u_i is adjacent to u_{i+1}, $\forall i$, $1 \leq i \leq 5$, $u - v_1 - u_1 - u_2 - u_3 - u_4 - u_5 - u_6$ forms a path of length 7 which is a $L(4,3,2,1)$-path. Its enough to show that there is a $L(4,3,2,1)$-path from u and v to all the v_i's of G.

(1) For every i, $1 \leq i \leq n$, the path from u to v_i is $[u, v_i]$.
(2) The path from u to v is $[u, v_i, v]$, $2 \leq i \leq n$ respectively.
(3) The path from v to v_1 is $[v, v_3, u, v_1]$.
(4) For every i, $2 \leq i \leq n$, the path from v to v_i is $[v, v_i]$.
(5) The path from v to u_6 is $[v, v_n, u, v_1, u_1, u_2, u_3, u_4, u_5, u_6]$.
(6) The path from v_1 to v_2 is $[v_1, u, v_4, v, v_2]$.
(7) The path from v_1 to v_j is $[v_1, u, v_j]$, $3 \leq j \leq n$.
(8) The path from v_2 to v_3 is $[v_2, u, v_4, v, v_3]$.
(9) The path from v_2 to v_j is $[v_2, u, v_j]$, $4 \leq j \leq n$.
(10) The path from v_i to v_j is $[v_i, u, v_j]$, $3 \leq i < j \leq n$.

□

From the above Theorem 5.1 it is clear that there is no gap in the $k_c(G)$ number line except at 6, 7 and 10.

6. Conclusion

In this paper, $k_c(G)$ of certain classes of graphs were obtained which could be very helpful in providing a safe communication by reducing the interference in wireless networks. The authors are working towards generalising the idea of path coloring to levels equal to diameter of the graph and developing the mathematical theory governing it.

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