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$\mathcal{H}$-SIMULATION FUNCTIONS AND Ω_b-DISTANCE MAPPINGS IN THE SETTING OF G_b-METRIC SPACES AND APPLICATION

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Abstract. The conceptions of generalized b-metric spaces or G_b-metric spaces and a generalized Ω-distance mappings play a key role in proving many important theorems in existence and uniqueness of fixed point theory. In this manuscript, we establish a new type of contraction namely, $\Omega_b(\mathcal{H}, \theta, s)$-contraction, this contraction based on the concept of a generalized Ω-distance mappings which established by Abodayeh et.al. in 2017 together with the concept of $\mathcal{H}$-simulation functions which established by Bataihah et.al [10] in 2020. By utilizing this new notion we prove new results in existence and uniqueness of fixed point. On the other hand, examples and application were established to show the importance of our results.

1. Introduction

In the past century, the developing of fixed point theory has taken a wide range in the field of mathematics certainly after the original result of Banach [6]. The study was divided into many directions; one of these directions by modifying the contraction condition for examples see [2], [3], [8], [9], [14]–[23] and the other by modifying the setting of the distance spaces, the pioneer mathematicians established new distance spaces such as b-metric space, extended b-metric spaces and extended quasi b-metric spaces, etc. Qawasmeh et.al. [21] and Shatanawi [29] discuss and unify the existence and uniqueness

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of fixed point results by utilizing the concept of extended b-metric spaces and extended quasi b-metric spaces respectively, for more examples in distance spaces see [24], [25], [27]–[30] and others by modifying both see [7], [23], [28], [31].

In 2014, Aghajani et.al. [4] established the concept of generalized b-metric spaces or G_b-metric spaces as a generalization of the standard concept of both spaces b-metric spaces which investigated by Bakhtin [5] and G-metric spaces which investigated by Mustafa and Sims [19] as follows:

2. Preliminaries

Definition 2.1. ([4]) Let D be a nonempty set and $s \in [1, +\infty)$. Assume that the function
$G_b : D \times D \times D \to [0, +\infty)$ satisfy:

(1) $G_b(d, d^{'}, d^{''}) = 0$ if and only if $d = d^{'} = d^{''}$;
(2) $G_b(d, d, d^{'}) \geq 0$ for all $d, d^{'} \in D$ with $d \neq d^{'}$;
(3) $G_b(d, d^{'}, d^{'}) \leq G_b(d, d^{'}, d^{''})$ for all $d, d^{'}, d^{''} \in D$ with $d^{'} \neq d^{''}$;
(4) $G_b(d, d^{'}, d^{''}) = G_b(p\{d, d^{'}, d^{''}\})$ where p is a permutation of $d, d^{'}, d^{''}$;
(5) $G_b(d, d^{'}, d^{''}) \leq s[G_b(d, a, a) + G_b(a, d^{'}, d^{''})]$ for all $d, d^{'}, d^{''}, a \in D$.

Then the function G_b is called a generalized b-metric on D and the pair (D, G_b) is a generalized b-metric space or G_b-metric space.

Example 2.2. ([4]) Suppose that (D, G) is a G-metric space and $p \in (1, +\infty)$. Define $G_b : D \times D \times D \to [0, +\infty)$ via $G_b(d_1, d_2, d_3) = (G(d_1, d_2, d_3))^p$. Then G_b is a generalized b metric space with base $s = 2^{p-1}$.

From now on, (D, G_b) refers to G_b-metric space. The notions of G_b convergence and G_b completeness are given as follows:

Definition 2.3. ([4]) Let $\{d_n\}$ be a sequence in (D, G_b). Then

(1) $\{d_n\}$ is G_b Cauchy sequence if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m, l \geq N, G(d_n, d_m, d_l) < \epsilon$;
(2) $\{d_n\}$ is G_b convergent to $d \in D$ if for all $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n, m \geq N, G(d, d_n, d_m) < \epsilon$;
(3) (D, G_b) is complete if every G_b Cauchy sequence is G_b convergent.

Remark 2.4. A sequence $\{d_n\}$ in (D, G_b) is G_b convergent to $d \in D$ if one of the following conditions hold:

(i) $G_b(d_n, d_n, d) \to 0$ as $n \to +\infty$;
(ii) $G_b(d_n d, d) \to 0$ as $n \to +\infty$.

The notion of Generalized Ω distance mappings or Ω_b was introduced by Abodayeh et.al. [1] and they employed this notion to discuss some fixed point results in the literature.

Definition 2.5. ([1]) A generalized Ω-distance mapping on (D, G_b) (denoted by Ω_b) is a function $\Omega_b : D \times D \times D \to [0, +\infty)$ satisfy:

(1) $\Omega_b(d, d^{'}, d^{''}) \leq s[\Omega_b(d, a, a) + \Omega_b(a, d^{'}, d^{''})]$ for all $d, d^{'}, d^{''}, a \in D,\ s \in [0, +\infty)$;
(2) for all $d, d^{'} \in D,\ \Omega_b(d, d^{'}, .), \Omega_b(d, ., d^{'}) : D \to D$ are lower semi-continuous;
(3) for all $\epsilon > 0$ there exists $\alpha > 0$ such that if $\Omega_b(d, a, a) \leq \alpha$ and $\Omega_b(a, d^{'}, d^{''}) \leq \alpha$, then $G_b(d, d^{'}, d^{''}) \leq \epsilon$ for all $d, d^{'}, d^{''} \in D$.

Example 2.6. ([1]) Let $D = \mathbb{R}$. Define $G_b : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ and $\Omega_b : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times \to [0, +\infty)$ via $G_b(d_1, d_2, d_3) = (|d_1-d_2|+|d_2-d_3|+|d_1-d_3|)^2$, $\Omega_b(d_1, d_2, d_3) = (|d_1 - d_2| + |d_1 - d_3|)^2$, respectively. Then G_b is a generalized b metric space with $s = 2$ and Ω_b is a generalized Ω-distance mapping.

Definition 2.7. ([10, 16]) Let Θ denotes the class of all continuous and none decreasing functions $\theta : [0, +\infty) \to [1, +\infty)$ that satisfies the condition : for all a sequence $\{r_n\}$ in $[0, +\infty)$,

$$\lim_{n\to+\infty} \theta(r_n) = 1 \iff \lim_{n\to+\infty} r_n = 0.$$

Remark 2.8. If $\theta \in \Theta$, then θ^{-1} ($\{1\}$)=0.

The notion of the class of functions namely, $\mathcal{H}$-simulation functions which investigated by Bataihah et.al. in 2020 as follows:

Definition 2.9. ([10]) A class of functions $H : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ is called a $\mathcal{H}$-simulation function if

$$H(d, d^{'}) \leq \frac{d^{'}}{d},\ \ \forall\, d, d^{'} \in [1, +\infty). \tag{2.1}$$

Remark 2.10. ([10]) Suppose $H \in \mathcal{H}$ and $(d_n), (d^{'}_n)$ are sequences in $[1, +\infty)$ with $1 \leq \lim_{n\to+\infty} d^{'}_n < \lim_{n\to+\infty} d_n$. Then

$$\limsup_{n\to+\infty} H(d_n, d^{'}_n) < 1. \tag{2.2}$$

In the subsequence, D refers to nonempty set, Φ_f stands for the class of all fixed points of f in the set D, $\mathbb{R}$ and $\mathbb{N}$ refer to the set of reals and the set of natural numbers, respectively.

3. Main results

Before we introduce our new results, it is necessary to introduce the following definitions.

Definition 3.1. Suppose that (D, G_b) is equipped with generalized Ω-distance mapping Ω_b. we say that D is bounded with respect to Ω_b if there is $K > 0$ with $\Omega_b(d_1, d_2, d_3) \leq K$ for all $d_1, d_2, d_3 \in D$.

Definition 3.2. Suppose that (D, G_b) is equipped with generalized Ω-distance mapping Ω_b. A self mapping f on D is called $\Omega_b(\mathcal{H}, \theta, s)$-contraction if there exist $s \in [1, +\infty)$, $\delta \in [0, 1)$, $\theta \in \Theta$ and $H \in \mathcal{H}$ such that for all $d_1, d_2, d_3 \in D$ we have

$$1 \leq H\bigg(\theta s \Omega_b(fd_1, f^2d_1, fd_2), \theta \delta W(d_1, d_2, d_3)\bigg), \tag{3.1}$$

where

$$W(d_1, d_2, d_3) = \max\bigg\{\Omega_b(d_1, fd_1, d_2), \Omega_b(d_1, fd_1, fd_1), \Omega_b(d_2, fd_2, fd_2)\bigg\}.$$

Lemma 3.3. *Suppose that that the function $f : D \to D$ satisfies the conditions of $\Omega_b(\mathcal{H}, \theta, s)$-contraction. Then*

(1) *if $0 < W(d_1, d_2, d_3)$, then $\Omega_b(fd_1, f^2d_1, fd_2) \leq \dfrac{\delta}{s} W(d_1, d_2, d_3)$;*
(2) *if $0 = W(d_1, d_2, d_3)$, then $\Omega_b(fd_1, f^2d_1, fd_2) = 0$.*

Proof. (1) If $W(d_1, d_2, d_3) > 0$, then

$$\begin{aligned} 1 &\leq H(\theta s \Omega_b(fd_1, f^2d_1, fd_2), \theta \delta \Omega_b(d_1, fd_1, d_2)) \\ &\leq \frac{\theta \delta W(d_1, d_2, d_3)}{\theta s \Omega_b(fd_1, f^2d_1, fd_2)}. \end{aligned}$$

This implies that, $\theta s \Omega_b(fd_1, f^2d_1, fd_2) \leq \theta \delta W(d_1, d_2, d_3)$. Since the class Θ is non-decreasing function, we get

$$\Omega_b(fd_1, f^2d_1, fd_2) \leq \frac{\delta}{s} W(d_1, d_2, d_3).$$

(2) If $0 = W(d_1, d_2, d_3)$, then by utilizing condition (1), we have

$$1 \leq \theta s \Omega_b(fd_1, f^2d_1, fd_2) \leq \theta \delta W(d_1, d_2, d_3) = 1.$$

Thus, $\Omega_b(fd_1, f^2d_1, fd_2) = 0$. □

Lemma 3.4. *Suppose that that the function $f : D \to D$ satisfies $\Omega_b(\mathcal{H}, \theta, s)$-contraction. Then Φ_f has at most one element.*

Proof. First, if $\omega \in \Phi_f$, we claim that $\Omega_b(\omega,\omega,\omega) = 0$. If not, then by Lemma 3.3, we get that

$$\begin{aligned}\Omega_b(f\omega, f^2\omega, f\omega) &\leq \frac{\delta}{s} W(\omega,\omega,\omega)\\ &= \frac{\delta}{s}\max\{\Omega_b(\omega, f\omega, \omega), \Omega_b(\omega, f\omega, f\omega), \Omega_b(\omega, f\omega, f\omega)\}\\ &= \frac{\delta}{s}[\Omega_b(\omega,\omega,\omega)]\\ &< \Omega_b(\omega,\omega,\omega),\end{aligned}$$

which is a contradiction.

Now, assume that there are $\omega, v \in \Phi_f$. Then $\Omega_b(\omega, v, v) > 0$ by utilizing Lemma 3.3, we get that

$$\begin{aligned}\Omega_b(\omega, v, v) = \Omega_b(f\omega, f^2 v, fv) &\leq \frac{\delta}{s} W(\omega, v, v)\\ &= \frac{\delta}{s}\max\{\Omega_b(\omega, fv, v)\Omega_b(\omega,\omega,\omega), \Omega_b(v,v,v)\}\\ &= \frac{\delta}{s}\Omega_b(\omega, v, v)\\ &< \Omega_b(\omega, v, v),\end{aligned}$$

which is a contradiction. Therefore, $\Omega_b(\omega, v, v) = 0$ and by the condition (3) of the definition of of Ω_b and since $\Omega_b(\omega,\omega,\omega) = 0$, we have $G_b(\omega, v, v) = 0$ and so $\omega = v$. □

Theorem 3.5. *Suppose (D, G_b) is complete equipped with a generalized Ω-distance mapping Ω_b with base $s \in [1, +\infty)$ and D is bounded with respect to Ω_b. Suppose there exist $\theta \in \Theta$, $H \in \mathcal{H}$, $\delta \in [0, 1)$ such that the self-mapping $f : D \to D$ is a $\Omega_b(\mathcal{H}, \theta, s)$-contraction and satisfies one of the followings:*

(i) *f is a continuous mapping;*
(ii) *for all $\beta \in D$ if $f\beta \neq \beta$, then $0 < inf\{\Omega_b(d, fd, \beta) : d \in D\}$,*

then Φ_f contains only one element.

Proof. Construct a Picard sequence $\{d_n\}$ by choosing an arbitrary point $d_0 \in D$ and by letting $d_{n+1} = f^{n+1}(d_0) = f(d_n)$ for $n \in \mathbb{N}$.

Since $\Omega_b(d_n, d_{n+1}, d_{n+1}) = \Omega_b(fd_{n-1}, f^2 d_{n-1}, fd_n)$,

$$\begin{aligned}1 &\leq H(\theta s\Omega_b(fd_{n-1}, f^2 d_{n-1}, fd_n), \theta\delta W(d_{n-1}, d_n, d_n))\\ &= H(\theta s\Omega_b(d_n, d_{n+1}, d_{n+1}), \theta\delta W(d_{n-1}, d_n, d_n))\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\theta\delta W(d_{n-1},d_n,d_n)}{\theta s\Omega_b(d_n,d_{n+1},d_{n+1})} \\
&= \frac{\theta\delta \max\{\Omega_b(d_{n-1},d_n,d_n),\Omega_b(d_{n-1},d_n,d_n),\Omega_b(d_n,d_{n+1},d_{n+1}))\}}{\theta s\Omega_b(d_n,d_{n+1},d_{n+1})}.
\end{aligned} \tag{3.2}$$

Therefore, we have

$$\theta s\Omega_b(d_n,d_{n+1},d_{n+1}) \leq \theta\delta \max\{\Omega_b(d_{n-1},d_n,d_n)+\Omega_b(d_n,d_{n+1},d_{n+1}).$$

Since the class Θ is a non-decreasing function, we get

$$\Omega_b(d_n,d_{n+1},d_{n+1}) \leq \frac{\delta}{s}\max\{\Omega_b(d_{n-1},d_n,d_n),\Omega_b(d_n,d_{n+1},d_{n+1}))\}.$$

If $\max\{\Omega_b(d_{n-1},d_n,d_n),\Omega_b(d_n,d_{n+1},d_{n+1}))\} = \Omega_b(d_n,d_{n+1},d_{n+1})$, then $\Omega_b(d_n,d_{n+1},d_{n+1}) < \Omega_b(d_n,d_{n+1},d_{n+1})$ which is a contradiction. Therefore, we have

$$\begin{aligned}
\Omega_b(d_n,d_{n+1},d_{n+1}) &\leq \frac{\delta}{s}\Omega_b(d_{n-1},d_n,d_n) \\
&\leq (\frac{\delta}{s})^2\Omega_b(d_{n-2},d_{n-1},d_{n-1}) \\
&\vdots \\
&\leq (\frac{\delta}{s})^n\Omega_b(d_0,d_1,d_1).
\end{aligned} \tag{3.3}$$

Since D is bounded with respect to Ω_b, there is $K>0$ such that

$$\Omega_b(d_n,d_{n+1},d_{n+1}) \leq (\frac{\delta}{s})^n K. \tag{3.4}$$

Next, by utilizing Equation (3.4) and condition (1) of the the definition of Ω_b for all $n<m\leq l$, we have

$$\begin{aligned}
\Omega_b(d_n,d_m,d_l) &\leq s\Omega_b(d_n,d_{n+1},d_{n+1}) + s\Omega_b(d_{n+1},d_m,d_l) \\
&\leq s\Omega_b(d_n,d_{n+1},d_{n+1}) + s^2\Omega_b(d_{n+1},d_{n+2},d_{n+2}) \\
&\quad + s^2\Omega_b(d_{n+2},d_m,d_l) \\
&\vdots \\
&\leq s\Omega_b(d_n,d_{n+1},d_{n+1}) + s^2\Omega_b(d_{n+1},d_{n+2},d_{n+2}) + \cdots \\
&\quad + s^{m-n-1}\Omega_b(d_{m-2},d_{m-1},d_{m-1}) + s^{m-n-1}\Omega_b(d_{m-1},d_m,d_l)
\end{aligned}$$

$$
\begin{aligned}
&\leq s(\frac{\delta}{s})^n K + s^2(\frac{\delta}{s})^{n+1} K + \cdots + s^{m-n-1}(\frac{\delta}{s})^{m-1} K \\
&= s(\frac{\delta}{s})^n K \left[1 + \delta + \delta^2 + \cdots + \delta^{m-n-1}\right] \\
&= sK(\frac{1-\delta^{m-n}}{1-\delta})(\frac{\delta}{s})^n.
\end{aligned}
\tag{3.5}
$$

By taking the limit as $n \to +\infty$, we get

$$\lim_{n\to+\infty} \Omega_b(d_n, d_m, d_l) = 0. \tag{3.6}$$

Consequently, $\{d_n\}$ is G_b Cauchy sequence, so there is $\beta \in D$ such that the sequence $\{d_n\}$ is G_b convergent to β. If f is a continuous function, then $f\beta = \beta$. But if f is any mapping by using the lower semi continuity of Ω_b, we have

$$\Omega_b(d_n, d_m, \beta) \leq \lim_{t\to+\infty} \Omega_b(d_n, d_m, d_t) < \epsilon, \ \ \forall\, n, m \geq N. \tag{3.7}$$

Assume that $m = n + 1$. Then

$$\Omega_b(d_n, d_{n+1}, \beta) \leq \lim_{t\to+\infty} \Omega_b(d_n, d_{n+1}, d_t) < \epsilon$$

for all $n \geq N$. Now, if $f\beta \neq \beta$, we have

$$0 < \inf\{\Omega_b(d, fd, \beta) : d \in D\} \leq \inf\{\Omega_b(d_n, d_{n+1}, \beta) : n \in \mathbb{N}\} < \epsilon \tag{3.8}$$

for each $\epsilon > 0$, which is a contradiction. Hence, $f\beta = \beta$. To prove the uniqueness, Lemma 3.4 ensures that Φ_f contains only one element and this completes the proof. □

Corollary 3.6. *Suppose (D, G_b) is complete equipped with a generalized Ω-distance mapping Ω_b with base $s \in [1, +\infty)$ and D is bounded with respect to Ω_b. Suppose there exist $\theta \in \Theta$, $\delta \in [0, 1)$ such that the self-mapping $f : D \to D$ satisfies the condition:*

$$2^{s\Omega_b(fd, f^2d, ft)} \leq 2^{\delta\Omega_b(d, fd, t)}, \ \ \forall\, d, t, l \in D.$$

If one of the following conditions satisfied

(1) *f is a continuous mapping;*
(2) *for all $\beta \in D$ if $f\beta \neq \beta$, then $0 < inf\{\Omega_b(d, fd, \beta) : d \in D\}$,*

then Φ_f contains only one element.

Proof. Define $H : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$, $\theta : [0, \infty) \to [1, \infty)$ by

$$H(v_1, v_2) = 1 + \ln(\frac{v_2}{v_1}), \lambda \in (0, 1) \ \text{ and } \ \theta(v) = 2^v, \ \ \forall\, v \in D,$$

respectively, then $H \in \mathcal{H}$ *and* $\theta \in \Theta$. Now,

$$2^{s\Omega_b(fd,f^2d,ft)} \leq 2^{\delta\Omega_b(d,fd,t)},$$

then

$$\theta s\Omega_b(fd, f^2d, ft) \leq \theta\delta\Omega_b(d, fd, t) \leq \theta\delta W(d, t, l).$$

Therefore, we have

$$1 \leq \frac{\theta\delta W(d,t,l)}{\theta s\Omega_b(fd, f^2d, ft)}.$$

It means that

$$1 \leq 1 + \ln\left(\frac{\theta\delta W(d,t,l)}{\theta s\Omega_b(fr, f^2r, ft)}\right),$$

it implies

$$1 \leq H(\theta\delta W(d,t,l), \theta s\Omega_b(fd, f^2d, ft)).$$

□

Definition 3.7. Let $\mathcal{L}$ denotes the class of all continuous functions:

$$\mathcal{L} := \{l : [0, +\infty) \to [1, +\infty) : l^{-1}(\{1\}) = 0\}.$$

Corollary 3.8. *Suppose (D, G_b) is complete equipped with generalized Ω-distance mapping Ω_b with base $s \in [1, +\infty)$ and D is bounded with respect to Ω_b. Suppose there exist $\theta \in \Theta$, $\delta \in [0, 1)$. Assume there are $l_1, l_2 \in \mathcal{L}$ with $l_2(t) \leq t \leq l_1(t)$ with $t \in [0, +\infty)$ such that the self-mapping $f : D \to D$ satisfies the condition:*

$$1 \leq \frac{l_2(\theta\delta W(d_1, d_2, d_3))}{l_1(\theta s\Omega_b(fd_1, f^2d_1, fd_2))}. \tag{3.9}$$

If one of the following conditions satisfied:

(1) *f is a continuous mapping;*
(2) *for all $\beta \in D$ if $f\beta \neq \beta$, then $0 < inf\{\Omega_b(d, fd, \beta) : d \in D\}$,*

then Φ_f contains only one element.

Proof. Define $H : [1, +\infty) \times [1, +\infty) \to \mathbb{R}$ by $H(v_1, v_2) = \dfrac{l_2(v_2)}{l_1(v_1)}$, it is obviously that $H \in \mathcal{H}$. Theorem 3.5 ensures that Φ_f has only one element. □

Next, assume that the class $\mathcal{L}$ is a non-decreasing function and by using Corollary 3.8, we get the following corollaries:

Corollary 3.9. Suppose (D, G_b) is complete equipped with generalized Ω-distance mapping Ω_b with base $s \in [1, +\infty)$ and D is bounded with respect to Ω_b. Suppose there exist $\theta \in \Theta$, $\delta \in [0, 1)$. Assume there are $l_1, l_2 \in \mathcal{L}$ with

$l_2 \leq t \leq l_1$ and $t \in [0, +\infty)$ such that the self-mapping $f : D \to D$ satisfies the condition:

$$l_1(\theta s \Omega_b(fd_1, f^2 d_1, fd_2)) \leq l_2(\Omega_b(d_1, fd_1, d_2)). \tag{3.10}$$

If one of the following conditions satisfied:

(1) f is a continuous mapping;
(2) for all $\beta \in D$ if $f\beta \neq \beta$, then $0 < \inf\{\Omega_b(d, fd, \beta) : d \in D\}$,

then Φ_f contains only one element.

Next, let $l_1(t) = t$, $l_2(t) = \lambda t$ with $\lambda \in [0, 1)$ by employing Corollary 3.8, we get the following corollary:

Corollary 3.10. Suppose (D, G_b) is complete equipped with generalized Ω-distance mapping Ω_b with base $s \in [1, +\infty)$ and D is bounded with respect to Ω_b. Suppose there exist $\theta \in \Theta$, $\lambda, \delta \in [0, 1)$ such that the self-mapping $f : D \to D$ satisfies the condition:

$$\theta s \Omega_b(fd_1, f^2 d_1, fd_2) \leq \lambda \theta \delta W(d_1, d_2, d_3), \ \ \forall\, d_1, d_2, d_3 \in D. \tag{3.11}$$

If one of the following conditions satisfied:

(1) f is a continuous mapping;
(2) for all $\beta \in D$ if $f\beta \neq f\beta$, then $0 < \inf\{\Omega_b(d, fd, \beta) : d \in D\}$,

then Φ_f contains only one element.

Example 3.11. Let $D = I = [0, 1]$. Then Φf of the following mapping

$$fd = 1 - \frac{d^2}{8 + d^2}$$

has only one element on I. To show this, define the following mapping:
$H : [1, +\infty) \times [1, +\infty) \to [0, +\infty)$ and $\theta : [0, +\infty) \to [1, +\infty)$ by

$$H(d_1, d_2) = \frac{d_2^{\lambda}}{d_1} \text{ with } \lambda = \frac{1}{2} \ \text{ and } \ \theta(\omega) = 2^{\omega}, \ \ \forall\, \omega \in I,$$

respectively, then $H \in \mathcal{H}$ *and* $\theta \in \Theta$. Also, define $G_b : I \times I \times I \to [0, +\infty)$ by

$$G_b(d_1, d_2, d_3) = (|d_1 - d_2| + |d_2 - d_3| + |d_1 - d_3|)^2,$$

then (D, G_b) is a complete G_b-metric space with $s = 2$.

Furthermore, define $\Omega_b : I \times I \times I \to [0, +\infty)$ by

$$\Omega_b(d_1, d_2, d_3) = (|d_1 - d_2| + |d_1 - d_3|)^2,$$

then Ω_b is a generalized Ω-distance mapping.

Now, for all $d, t, l \in D$, assume $fd = u$. Then we have

$$\begin{aligned}
s\Omega_b(fd, f^2d, ft) &= 2\Omega_b(fd, fu, ft) \\
&= 2\bigg(|1 - \frac{d^2}{8+d^2} - (1 - \frac{u^2}{8+u^2})| \\
&\quad + |1 - \frac{d^2}{8+d^2} - (1 - \frac{t^2}{8+t^2})| \bigg)^2 \\
&\leq \frac{2}{(64)^2}\bigg[|(d^2)(8+u^2) - (u^2)(8+d^2)| \\
&\quad + |(d^2)(8+t^2) - (t^2)(8+d^2)| \bigg]^2 \\
&= \frac{2}{64}\bigg[|d^2 - u^2| + |d^2 - t^2| \bigg]^2 \\
&\leq \frac{1}{8}\bigg[|d - u| + |d - t| \bigg]^2 \\
&= \frac{1}{8}\bigg[|d - fd| + |d - t| \bigg]^2 \\
&= \lambda\delta\Omega_b(d, fd, t),
\end{aligned}$$

with $\lambda = \frac{1}{2},\ \delta = \frac{1}{4}$. Now,

$$s\Omega_b(fd, f^2d, ft) \leq \lambda\delta\Omega_b(d, fd, t) \leq \lambda\delta W(d, t, l),$$

then

$$2^{s\Omega_b(fd, f^2d, ft)} \leq (2^{\delta W(d,t,l)})\lambda.$$

Therefore, we have

$$1 \leq \frac{(2^{\delta W(d,t,l)})^{\lambda}}{2^{s\Omega_b(fd, f^2d, ft)}}.$$

It means that

$$1 \leq H\bigg(\theta s\Omega_b(fd, f^2d, ft), \theta\delta W(d, t, l)\bigg).$$

Hence, f satisfy the conditions of $\Omega_b(\mathcal{H}, \theta, s)$-contraction. Theorem 3.5 ensures that Φ_f has only one element.

By employing MATLAB simulation we find out that the fixed point of f is $d \approx 0.9067953030328075$.

Example 3.12. Consider the following mapping

$$fr = \frac{1-d^m}{\Gamma + d^m}, \ \ where\ m, \Gamma \in \mathbb{R} \ \ and \ \ m > \frac{1}{\sqrt{2}}, \ \Gamma \geq \sqrt{2}\, m.$$

Then Φf has only one element on $[0,1]$. To show this, define the following mappings: $H : [1,+\infty) \times [1,+\infty) \to [0,+\infty)$, $\theta : [0,+\infty) \to [1,+\infty)$ by $H(d_1, d_2) = 1 + \ln \dfrac{d_2}{d_1}$, $\theta(\omega) = e^{\omega}$ for all $\omega \in D$, respectively, then $H \in \mathcal{H}$ *and* $\theta \in \Theta$. Also, define: $G_b : D \times D \times D \to [0,+\infty)$ via $G_b(d_1, d_2, d_3) = (|d_1 - d_2| + |d_2 - d_3| + |d_1 - d_3|)^2$. Then (D, G_b) is a complete G_b metric space with $s = 2$.

Furthermore, define $\Omega_b : D \times D \times D \to [0,+\infty)$ via

$$\Omega_b(d_1, d_2, d_3) = (|d_1 - d_2| + |d_1 - d_3|)^2,$$

then Ω_b is a generalized Ω-distance mapping.

Now, for all $d, t, l \in R$, assume $fd = u$. Then we have

$$\begin{aligned}
s\Omega_b(fd, f^2d, ft) &= 2\Omega_b(fd, fu, ft)\\
&= 2\left(|\frac{1-d^m}{\Gamma+d^m} - \frac{1-u^m}{\Gamma+u^m}| + |\frac{1-d^m}{\Gamma+d^m} - \frac{1-t^m}{\Gamma+t^m}|\right)^2\\
&\leq \frac{2}{\Gamma^4}\Big[|(1-d^m)(\Gamma+u^m) - (1-u^m)(\Gamma+d^m)|\\
&\quad + |(1-d^m)(\Gamma+t^m) - (1-t^m)(\Gamma+d^m)|\Big]^2\\
&\leq \frac{2(\Gamma-1)^2}{\Gamma^4}(|d^m - u^m| + |d^m - t^m|)^2\\
&\leq \frac{2m^2(\Gamma-1)^2}{\Gamma^4}(|d-u| + |d-t|)^2\\
&\leq \left(\frac{\Gamma-1}{\Gamma}\right)^2(|d-fd| + |d-t|)^2\\
&= \delta\Omega_b(d, fd, t), \ where\ \delta = \left(\frac{\Gamma-1}{\Gamma}\right)^2.
\end{aligned}$$

Now,

$$s\Omega_b(fd, f^2d, ft) \leq \delta\Omega_b(d, fd, t) \leq \delta W(d, t, l),$$

then

$$e^{s\Omega_b(fd, f^2d, ft)} \leq e^{\delta W(d,t,l)}.$$

Therefore, we have

$$1 \leq \frac{e^{\delta W(d,t,l)}}{e^{s\Omega_b(fd,f^2d,ft)}}.$$

It means that

$$1 \leq 1 + \ln \frac{e^{\delta W(d,t,l)}}{e^{s\Omega_b(fd,f^2d,ft)}},$$

it implies

$$1 \leq H\bigg(\theta s\Omega_b(fd, f^2d, ft), \theta\delta W(d,t,l)\bigg).$$

Hence, f satisfy the conditions of $\Omega_b(\mathcal{H}, \theta, s)$-contraction. Theorem 3.5 ensures that Φ_f has only one element.

4. Application

To Show the novelty of our work, we introduce this application.
By utilizing our results the following equation:

$$d^{m+1} + d^m + \Gamma d = 1, \text{ where } m, \Gamma \in \mathbb{R} \text{ and } m > \frac{1}{\sqrt{2}}, \ \Gamma \geq \sqrt{2}\, m, \tag{4.1}$$

has not only a solution in the unit interval $[0,1]$ as intermediate value theorem say, but also, the solution is unique.

To prove this, it is similar to show that Φ_f of the following mapping has only one element in the unit interval $[0,1]$.

$$f(d) = \frac{1 - d^m}{\Gamma + d^m}, \text{ where } m, \Gamma \in \mathbb{R} \text{ and } m > \frac{1}{\sqrt{2}}, \ \Gamma \geq \sqrt{2}\, m. \tag{4.2}$$

Example 3.12 ensures that Φ_f has only one element and so the equation (4.1) has a unique solution.

Example 4.1. If $m = 100$, $\Gamma = 150$ in Equation (4.2). By employing MATLAB simulation we find out that the fixed point of f is $d \approx 0.0666666666665433$ and so, it is the unique solution of the Equation (4.1).

5. Conclusion

In this study, We employed our new contraction namely $\Omega_b(\mathcal{H}, \theta, s)$-contraction to discuss and unify some new fixed point results in the literature, this contraction established by using the notion of G_b-metric spaces which equipped with generalized Ω-distance mappings (Ω_b-distance mappings) and the notion of $\mathcal{H}$-simulation functions and the class of Θ functions. Finally, we show the importance of our work by setting up some interesting examples and application.

References

[1] K. Abodayeh, A. Bataihah and W. Shatanawi, *Generalized Ω-distance mappings and some fixed point theorems,* U.P.B. Sci. Bull. Series A, **79**(2) (2017), 223–232.

[2] I. Abu-Irwaq, I. Nuseir and A. Bataihah, *Common fixed point theorems in G-metric spaces with Ω-distance,* J. Math. Anal., **8**(1) (2017), 120-129.

[3] I. Abu-Irwaq, W. Shatanawi, A. Bataihah and I. Nuseir, *Fixed point results for nonlinear contractions with generalized Ω-distance mappings,* U.P.B. Sci. Bull. Series A., **81**(1) (2019), 57-64.

[4] A. Aghajani, M. Abbas and J.R. Roshan, *Common fixed point of generalized weak contractive mappings in partially ordered G_b-metric spaces,* Filomat, **28**(6) (2014), 1087–1101.

[5] I.A. Bakhtin, *The contraction mapping principle in almost metric spaces,* Funct. Anal., Gos. Ped. Inst., Unianowsk, **30** (1989), 26–37.

[6] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux quations intégrales,* Fund. Math, (1922), 133-181.

[7] A. Bataihah, T. Qawasmeh and M. Shatnawi, *Discussion on b-metric spaces and related results in metric and G-metric spaces,* Nonlinear Funct. Anal. Appl., **27**(2) (2022), 233-247.

[8] A. Bataihah, A. Tallafha and W. Shatanawi, *Fixed point results with Ω-distance by utilizing simulation functions,* Italian J. Pure Appl. Math., **43** (2020), 185-196.

[9] A. Bataihah, W. Shatanawi and A. Tallafha, *Fixed point results with simulation functions,* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 13-23.

[10] A. Bataihah, W. Shatanawi, T. Qawasmeh and R. Hatamleh, *On H-Simulation Functions and Fixed Point Results in the Setting of ωt-Distance Mappings with Application on Matrix Equations,* Mathematics, **8**(5) (2020), 837.

[11] Seong-Hoon Cho, *Fixed point for L-contraction in generalized metric spaces,* Abst. Appl. Anal., **2018**, Article ID 1327691.

[12] R. Hatamleh, *On the form of correlation function for a class of nonstationary field with a zero spectrum,* Rocky Mountain Journal of Mathematics, **33**(1) (2003), 159-173.

[13] R. Hatamleh, A. Qazza and H. Migdadi, *Stationary Connected Curves in Hilbert Spaces,* J. Math. Stat., **10**(2) (2014), 262-266.

[14] R. Hatamleh and V.A. Zolotarev, *On model representations of non-self-adjoint operators with infinitely dimensional imaginary components,* J. Math. Phy. Anal. Geom., **11**(2) (2015), 174186.

[15] R. Hatamleh and V.A. Zolotarev, *On Two-Dimensional Model Representations of One Class of Commuting Operators,* Ukrainian Math. J., **66**(1) (2014), 122-144.

[16] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle,* J. Ineq. Appl., **2014**(38) (2014).

[17] T. Kamran, M. Samreen and Q. UL Ain, *A generalization of b-metric space and some fixed point theorems,* Mathematics, **19**(5) (2017).

[18] F. Khojasteh, S. Shukla and S. Radenovic, *A new approach to the study of fixed point theory for simulation functions,* Filomat, **29** (2015), 1189–1194.

[19] Z. Mustafa and B. Sims, *A new approach to generalized metric spaces,* J. Nonlinear Convex Anal., **7**(2) (2006), 289-297.

[20] T. Qawasmeh and R. Hatamleh, *A new contraction based on $\mathcal{H}$-simulation functions in the frame of extended b-metric spaces and application,* International Journal of Electrical and Computer Engineering, **13**(4), (2023), 4212–4221. DOI: 10.11591/ijece.v13i4.pp4212-4221.

[21] T. Qawasmeh, W. Shatanawi, A. Bataihah and A. Tallafha, *Fixed point results and* (α, β)*- triangular admissibility in the frame of complete extended b-metric spaces and application,* U.P.B. Sci. Bull., Series A, **83**(1) (2021), 113-124.

[22] T. Qawasmeh, W. Shatanawi, A. Bataihah and A. Tallafha, *Common Fixed Point Results for Rational* $(\alpha, \beta)_\varphi - m\omega$ *Contractions in Complete Quasi Metric Spaces,* Mathematics, **7**(392) (2019).

[23] T. Qawasmeh, W. Shatanawi and A. Tallafha, *Fixed point theorems through modified* ω*-distance and application to nontrivial equations,* Axioms, **8**(5) (2019).

[24] M. Shakhatreh and T. Qawasmeh, *Associavity of Max-Min Composition of Three Fuzzy Relations,* Italian J. Pure Appl. Math., **2020**(44) (2020), 224228.

[25] M. Shakhatreh and T. Qawasmeh, *General Method to Generate Fuzzy Equivalence Relations in Matrix Form,* Jordan J. Math. Stat., **13**(3) (2020), 401-420.

[26] W. Shatanawi, A. Bataihah and A. Pitea, *Fixed and common fixed point results for cyclic mappings of* Ω*-distance,* j. Nonlinear Sci. Appl., **9**(3) (2016), 727–735.

[27] W. Shatanawi, A. Bataihah and A. Tallafha, *Four-Step Iteration Scheme to Approximate Fixed Point for Weak Contractions,* Comput. Materials & Continu, **64**(3) (2020), 1491-1504.

[28] W. Shatanawi and M. Postolache, *Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces,* Fixed Point Theory Appl., **2013**(60) (2013).

[29] W. Shatanawi, T. Qawasmeh, A. Bataihah and A. Tallafha, *New contraction and some fixed point results with application based on extended quasi b-metric spaces,* U.P.B. Sci. Bull., Series A, **83**(2) (2021), 39-48.

[30] M. Younis, D. Singh, M. Asadi and Vishal Joshi, *Results on contractions of Reich type in graphical b-metric spaces with applications,* FILOMAT, **33**(17) (2019), 5723–5735.

[31] D. Zheng, Z. Cai and P. Wang, *New fixed point theorems for* $\theta - \phi$ *contraction in complete metric spaces,* J. Nonlinear Sci. Appl., **10**(5) (2017), 2662–2670.