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FIXED POINT THEOREMS IN QUASI-METRIC SPACES

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Abstract. Fixed point theory is the center of focus for many mathematicians from last few decades. A lot of generalizations of the Banach contraction principle have been established. In this paper, we introduce the concepts of θ-contraction and θ-φ-contraction in quasi-metric spaces to study the existence of the fixed point for them.

1. Introduction

The problem of the existence of the solution of many mathematical models is equivalent to the existence of a fixed point problem for a certain map. The study of fixed points, therefore, has a central role in many disciplines of applied sciences. The most essential and key part of the theory of fixed points is the existence of the solution of operator equations satisfying certain conditions, for example, Fredholm integral equations, Voltera integral equations, two point boundary value problems in differential equations as well as some eigenvalue problems [2, 6, 7]. A beautiful blend of analysis, topology and geometry has laid down the foundation of the theory of fixed points.

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The most celebrated result of the theory of metric fixed points is the Banach contraction principle [1]. Due to its importance, several authors have obtained many interesting extensions and generalizations [3, 8, 10, 11, 16, 19, 22, 27, 28].

In 1931, for the first time, quasi-metric spaces were introduced by Wilson [25], in such a way that without the requirement that the (asymmetric) metric d has to satisfy $d(x, y) = d(y, x)$. As such, any metric space is a quasi-metric space but the converse is not true.

Quasi-metric spaces have numerous recent applications both in pure and applied mathematics, for example, in rate-independent models for plasticity [12], shape-memory alloys [13], models for material failure [17] and the questions of existence and uniqueness of Hamilton-Jacobi equations [14].

Various fixed point results were established on such spaces, see [4, 15, 20, 21, 23, 24, 26] and references therein. In quasi-metric spaces some notions such as convergence, compactness and completeness are different from these in metric space case. Collins and Zimer [5] discussed these notions in a quasi-metric space.

Recently, Samet *et al.* [9] introduced a new concept of θ-contraction and established some fixed point results for such mappings in complete generalized metric spaces and generalize the results of Banach in such spaces.

Very recently, Zheng *et al.* [29] introduced a new concept of θ-ϕ-contraction and established some fixed point results for such mappings in complete metric spaces and generalized the results of Brower and Kannan.

In this paper, aspired by the notion of Samet *et al.* [9] and the notion introduced by Zheng *et al.* [29], we present a new notion of generalized θ-contraction and θ-ϕ-contraction and establish various fixed point theorems for such mappings in complete quasi-metric spaces. The results presented in the paper improve and extend the corresponding results of Kannan [10] and Reich [16].

2. Preliminaries

Definition 2.1. ([18]) Let X be a nonempty set and $d : X \times X \to \mathbb{R}^+$ be a function such that for all $x, y, z \in X$,

(i) $d(x, y) = d(y, x) = 0$ if and only if $x = y$;
(ii) $d(x, y) \leq d(x, z) + d(z, y)$ (triangular inequality).

Then (X, d) is called a quasi-metric space.

Definition 2.2. ([5]) Let (X, d) be a quasi-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$.

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is forward (resp. backward) convergent to x if

$$\lim_{n\to+\infty} d(x,x_n)=0 \qquad (resp. \lim_{n\to+\infty} d(x_n,x)=0).$$

(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is forward-Cauchy if for every $\varepsilon>0$ there exists a positive integer $N=N(\varepsilon)$ such that $d(x_n,x_m)<\varepsilon$ for all $m\geq n\geq N$.

(iii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is backward-Cauchy if for every $\varepsilon>0$ there exists a positive integer $N=N(\varepsilon)$ such that $d(x_m,x_n)<\varepsilon$ for all $m\geq n\geq N$.

Lemma 2.3. ([5]) *Let (X,d) be a quasi-metric space and $\{x_n\}_n$ be a sequence in X. If $\{x_n\}_{n\in\mathbb{N}}$ is forward convergent to $x\in X$ and is backward convergent to $y\in X$, then $x=y$.*

Definition 2.4. ([5]) Let (X,d) be a quasi-metric space. Then X is said to be forward (resp. backward) complete if every forward-(resp. backward-) Cauchy sequence $\{x_n\}_n$ in X is forward (resp. backward) convergent to $x\in X$.

Definition 2.5. ([5]) Let (X,d) be a quasi-metric space. Then X is said to be complete if X is forward and backward complete.

The following definition was given by Samet *et al.* in [9].

Definition 2.6. ([9]) Let Θ_C be the family of all functions $\theta:\ ]0,+\infty[\ \to\ ]1,+\infty[$ such that

(θ_1) θ is increasing, that is, for all $x,y\in\mathbb{R}^+$ such that $x<y$, $\theta(x)<\theta(y)$;
(θ_2) for each sequence $\{x_n\}$ in $]0,+\infty[$,

$$\lim_{n\to\infty} x_n=0 \quad \text{if and only if} \quad \lim_{n\to\infty}\theta(x_n)=1;$$

(θ_3) θ is continuous.

Definition 2.7. ([9]) Let Θ_G be the family of all functions $\theta:\ ]0,+\infty[\ \to\ ]1,+\infty[$ such that

(θ_1) θ is increasing, that is, for all $x,y\in\mathbb{R}^+$ such that $x<y$, $\theta(x)<\theta(y)$;
(θ_2) for each sequence $\{x_n\}$ in $]0,+\infty[$,

$$\lim_{n\to\infty} x_n=0 \quad \text{if and only if} \quad \lim_{n\to\infty}\theta(x_n)=1;$$

(θ_3) there exist $r\in\ ]0,1[$ and $l>0$ such that $\lim_{n\to\infty}\frac{\theta(t)-1}{t^r}=l$;
(θ_4) θ is continuous.

In [29], Zheng presented the concept of θ-ϕ-contraction in metric spaces and proved the following nice result.

Definition 2.8. ([29]) Let Φ be the family of all functions ϕ: $[1,+\infty[\ \to [1,+\infty[$ such that

(ϕ_1) ϕ is increasing;
(ϕ_2) for each $t \in]1, +\infty[$, $\lim_{n\to\infty} \phi^n(t) = 1$;
(ϕ_3) ϕ is continuous.

Lemma 2.9. ([29]) *If $\phi \in \Phi$, then $\phi(1) = 1$ and $\phi(t) < t$ for all $t \in]1 + \infty[$.*

Definition 2.10. ([29]) Let (X, d) be a metric space and $T : X \to X$ be a mapping. T is said to be a θ-ϕ-contraction if there exist $\theta \in \Theta$ and $\phi \in \Phi$ such that for any $x, y \in X$,

$$d(Tx, Ty) > 0 \Rightarrow \theta[d(Tx, Ty)] \leq \phi[\theta(N(x, y))],$$

where

$$N(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}.$$

Theorem 2.11. ([29]) *Let (X, d) be a complete metric space and $T : X \to X$ be a θ-ϕ-contraction. Then T has a unique fixed point.*

3. Main results

In the following, we present the concepts of θ-contraction and θ-ϕ-contraction in quasi-metric spaces and we prove some fixed point results in such spaces. Also, we derive some useful corollaries of this result.

Theorem 3.1. *Let (X, d) be a quasi-metric space and $T : X \to X$ be a mapping. If there exist $\theta \in \Theta_G$ and $r \in]0, 1[$ such that for all $x, y \in X$*

$$\max\{d(Tx, Ty), d(Ty, Tx)\} > 0 \quad \Rightarrow \quad \theta[d(Tx, Ty)] \leq [\theta(M(x, y))]^r, \quad (3.1)$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$$

and

$$d(y, x) \leq d\left(T^2 y, x\right).$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. We define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$, for all $n \in \mathbb{N}$.

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ and $d(x_{n_0+1}, x_{n_0}) = 0$, then x_{n_0} is a fixed point of T. Then we assume that $d(x_n, x_{n+1}) > 0$ or $d(x_{n+1}, x_n) > 0$.

Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.1), we obtain

$$\theta(d(x_n, x_{n+1})) = \theta(d(Tx_{n-1}, Tx_n)) \leq [\theta(M(x_n, x_{n-1}))]^r,$$

where

$$\begin{aligned} M\left(x_{n-1}, x_n\right) &= \max\{d\left(x_{n-1}, x_n\right), d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\} \\ &= \max\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\}. \end{aligned}$$

Suppose that $d\left(x_{n-1}, x_n\right) \leq d\left(x_n, x_{n+1}\right)$ for some positive integer n. Then we have

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) \leq \left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right]^r < \theta\left(d\left(x_n, x_{n+1}\right)\right),$$

which is a contradiction. Hence

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) \leq \left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right]^r \leq ... \leq \left[\theta\left(d\left(x_0, x_1\right)\right)\right]^{r^n}. \tag{3.2}$$

Since $r \in]0, 1[$, we obtain

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) < \theta\left(d\left(x_{n-1}, x_n\right)\right).$$

By (θ_1), we have

$$d\left(x_n, x_{n+1}\right) < d\left(x_{n-1}, x_n\right). \tag{3.3}$$

Letting $x = x_n$ and $y = x_{n-1}$ in (3.1), we obtain

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) = \theta\left(d\left(Tx_n, Tx_{n-1}\right)\right) \leq \left[\theta\left(M\left(x_n, x_{n-1}\right)\right)\right]^r,$$

where

$$\begin{aligned} M\left(x_n, x_{n-1}\right) &= \max\{d\left(x_n, x_{n-1}\right), d\left(x_n, x_{n+1}\right), d\left(x_{n-1}, x_n\right)\} \\ &= \max\left(d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n-1}\right)\right\}. \end{aligned}$$

Suppose that $d\left(x_n, x_{n-1}\right) \leq d\left(x_{n+1}, x_n\right)$ for some $n \in \mathbb{N}$.

Case 1: $d\left(x_n, x_{n-1}\right) \geq d\left(x_{n-1}, x_n\right)$. We get

$$\theta\left(d\left(x_n, x_{n-1}\right)\right) \leq \theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \left[\theta\left(d\left(x_n, x_{n-1}\right)\right)\right]^r < \theta\left(d\left(x_n, x_{n-1}\right)\right),$$

which is a contradiction.

Case 2: $d\left(x_n, x_{n-1}\right) < d\left(x_{n-1}, x_n\right)$. We get

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right]^r.$$

Since $d\left(y, x\right) \leq d\left(T^2 y, x\right)$, $d\left(x_{n-1}, x_n\right) \leq d\left(x_{n+1}, x_n\right)$, which implies that

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right]^r \leq \left[\theta\left(d\left(x_{n+1}, x_n\right)\right)\right]^r < \theta\left(d\left(x_{n+1}, x_n\right)\right),$$

which is a contradiction. Hence

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \left[\theta\left(d\left(x_n, x_{n-1}\right)\right)\right]^r \leq ... \leq \left[\theta\left(d\left(x_1, x_0\right)\right)\right]^{r^n}. \tag{3.4}$$

Since $r \in]0, 1[$ and using (θ_1), we conclude that

$$d\left(x_{n+1}, x_n\right) < d\left(x_n, x_{n-1}\right). \tag{3.5}$$

From (3.3), the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is monotone nonincreasing. So there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \alpha.$$

Assume that $\alpha > 0$. By property of θ and using (3.2), we obtain

$$1 < \theta(\alpha) \leq \theta(d(x_n, x_{n+1})) \leq [\theta(d(x_0, x_1))]^{r^n}. \tag{3.6}$$

Taking the limit as $n \to \infty$ in (3.6) and using (θ_2), we get

$$1 < \theta(\alpha) \leq \lim_{n\to+\infty} [\theta(d(x_0, x_1))]^{r^n}.$$

Therefore,

$$1 < \theta(\alpha) \leq 1,$$

which is a contradiction. Thus $\alpha = 0$ and

$$\lim_{n\to\infty} d(x_{n,} x_{n+1}) = 0.$$

From (3.5), the sequence $\{d(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ is monotone nonincreasing. So there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lambda.$$

Assume that $\lambda > 0$. By property of θ and using (3.4), we obtain

$$1 < \theta(\lambda) \leq \theta(d(x_{n+1}, x_n)) \leq [\theta(d(x_1, x_0))]^{r^n}. \tag{3.7}$$

Taking the limit as $n \to \infty$ in (3.7) and using (θ_2), we get

$$1 < \theta(\lambda) \leq \lim_{n\to+\infty} [\theta(d(x_1, x_0))]^{r^n}.$$

Therefore,

$$1 < \theta(\alpha) \leq 1,$$

which is a contradiction. Thus $\lambda = 0$ and

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Step 2. We prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Firstly, we are going to show that $\{x_n\}_{n\in\mathbb{N}}$ is a forward-Cauchy sequence, that is, $\lim_{n,m\to\infty} d(x_n, x_{n+m}) = 0$.

From (θ_3), there exist $k \in]0,1[$ and $l > 0$ such that

$$\lim_{n\to\infty} \frac{\theta[d(x_n, x_{n+1})] - 1}{d(x_n, x_{n+1})^k} = l.$$

Suppose that $l < \infty$. In this case, let $A = \frac{l}{2}$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k} - l \right| \leq A$$

for all $n \geq n_0$. This implies that

$$\frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k} \geq A$$

for all $n \geq n_0$. Then

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq Bn\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right]$$

for all $n \geq n_0$, where $A = \frac{1}{B}$.

Now, suppose that $l = \infty$. Let $B > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta\left[d\left(x_n, x_{n+1}\right)\right] - 1}{d\left(x_n, x_{n+1}\right)^k} \right| \geq B$$

for all $n \geq n_0$. This implies that

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq An\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right]$$

for all $n \geq n_0$, where $A = \frac{1}{B}$. Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq An\left[\theta\left(d\left(x_n, x_{n+1}\right)\right) - 1\right]$$

for all $n \geq n_0$. By continuing this process, we have

$$n\left[d\left(x_n, x_{n+1}\right)^k\right] \leq An\left[\left(\theta\left(d\left(x_0, x_1\right)\right)\right)^{r^n} - 1\right] \tag{3.8}$$

for all $n \geq n_0$. Letting the limit as $n \to \infty$ in (3.8), we obtain

$$\lim_{n\to\infty} n\left[d\left(x_n, x_{n+1}\right)^k\right] = 0.$$

Thus there exists $n_1 \in \mathbb{N}$ such that

$$d\left(x_n, x_{n+1}\right) \leq \frac{1}{n^{\frac{1}{k}}} \tag{3.9}$$

for all $n \geq n_1$.

Now, by the triangular inequality and using (3.9), for all $m > n \geq n_1$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + ... + d(x_{m-1}, x_m) \\ &\leq \frac{1}{n^{\frac{1}{k}}} + \frac{1}{(n+1)^{\frac{1}{k}}} + ... + \frac{1}{(m-1)^{\frac{1}{k}}} \\ &\leq \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

From the convergence of the series $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a forward-Cauchy sequence in (X, d).

Secondly, we are going to show $\{x_n\}_{n\in\mathbb{N}}$ is a backward-Cauchy sequence, that is, $\lim_{m,n\to\infty} d(x_{n+m}, x_n) = 0$.

Let $x = x_n$ and $y = x_{n-1}$ in (3.1). From (θ_3), there exist $k \in]0, 1[$ and $l > 0$ such that

$$\lim_{n\to\infty} \frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} = l.$$

Suppose that $l < \infty$. In this case, let $H = \frac{l}{2}$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} - l \right| \leq H$$

for all $n \geq n_0$. This implies that

$$\frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} \geq H$$

for all $n \geq n_0$. Then

$$n\left[d(x_{n+1}, x_n)^k\right] \leq Mn\left[\theta(d(x_{n+1}, x_n)) - 1\right]$$

for all $n \geq n_0$, where $H = \frac{1}{M}$.

Suppose that $l = \infty$. Let $M > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\theta[d(x_{n+1}, x_n)] - 1}{d(x_{n+1}, x_n)^k} \right| \geq M$$

for all $n \geq n_0$. This implies that

$$n\left[d(x_{n+1}, x_n)^k\right] \leq Hn\left[\theta(d(x_{n+1}, x_n)) - 1\right]$$

for all $n \geq n_0$, where $H = \frac{1}{M}$. Thus, in all cases, there exist $H > 0$ and $n \in \mathbb{N}$ such that

$$n\left[d\left(x_{n+1}, x_n\right)^k\right] \leq An\left[\theta\left(d\left(x_{n+1}, x_n\right)\right) - 1\right]$$

for all $n \geq n_0$. By continuing this process, we have

$$n\left[d\left(x_{n+1}, x_n\right)^k\right] \leq Hn\left[\left(\theta\left(d\left(x_1, x_0\right)\right)\right)^{r^n} - 1\right] \tag{3.10}$$

for all $n \geq n_0$. Letting the limit as $n \to \infty$ in (3.10), we obtain

$$\lim_{n\to\infty} n\left[d\left(x_{n+1}, x_n\right)^k\right] = 0.$$

Thus there exists $n_2 \in \mathbb{N}$ such that

$$d\left(x_{n+1}, x_n\right) \leq \frac{1}{n^{\frac{1}{k}}} \tag{3.11}$$

for all $n \geq n_2$.

Now, by the triangular inequality and using (3.11), we get

$$\begin{aligned} d\left(x_m, x_n\right) &\leq d\left(x_m, x_{m+1}\right) + d\left(x_{m+1}, x_{m+2}\right) + ... + d\left(x_{n-1}, x_n\right) \\ &\leq \frac{1}{m^{\frac{1}{k}}} + \frac{1}{(m+1)^{\frac{1}{k}}} + ... + \frac{1}{(n-1)^{\frac{1}{k}}} \\ &\leq \sum_{i=m}^{\infty} \frac{1}{i^{\frac{1}{k}}} \end{aligned}$$

for all $n > m \geq n_1$. From the convergence of the series $\sum_{i=m}^{\infty} \frac{1}{i^{\frac{1}{k}}}$, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a backward-Cauchy sequence in (X, d).

Finally, we deduce that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete quasi-metric space (X, d). By completeness of (X, d), there exist z, $w \in X$ such that

$$\lim_{n\to\infty} d\left(x_n, z\right) = 0, \qquad \lim_{n\to\infty} d\left(w, x_n\right) = 0.$$

By Lemma 2.3, we get $z = w$.

Step 3. We prove that $z = Tz$, that is, $d\left(Tz, z\right) = 0$ and $d\left(z, Tz\right) = 0$.

Arguing by contradiction, we assume that $d\left(Tz, z\right) > 0$ or $d\left(z, Tz\right) > 0$.

First, assume that $d\left(z, Tz\right) > 0$. By the triangular inequality, we get

$$d\left(Tx_n, Tz\right) \leq d\left(Tx_n, z\right) + d\left(z, Tz\right) \tag{3.12}$$

and

$$d\left(z, Tz\right) \leq d\left(z, Tx_n\right) + d\left(Tx_n, Tz\right). \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\lim_{n\to+\infty} d\left(Tx_n, Tz\right) = d\left(z, Tz\right).$$

So there exists $n_0 \in \mathbb{N}$ such that

$$d\left(Tx_n, Tz\right) \geq d\left(z, Tz\right) > 0$$

for all $n \geq n_0$. Hence

$$\max\{d\left(Tx_n, Tz\right), d\left(Tz, Tx_n\right)\} > 0.$$

Letting $x = x_n$ and $y = z$ in (3.1), we obtain

$$\theta\left(d\left(Tx_n, Tz\right)\right) \leq \left[\theta\left(M\left(x_n, z\right)\right)\right]^r, \tag{3.14}$$

where

$$M\left(x_n, z\right) = \max\left\{d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, z\right)\right\}$$

and

$$\lim_{n\to+\infty} M\left(x_n, z\right) = d\left(z, Tz\right). \tag{3.15}$$

Taking the limit as $n \to \infty$ in (3.14), using (3.15) and the properties of θ, we obtain

$$\begin{aligned}\lim_{n\to+\infty} \theta\left(d\left(Tx_n, Tz\right)\right) = \theta\left(\lim_{n\to+\infty} d\left(Tx_n, Tz\right)\right) &= \theta\left(d\left(z, Tz\right)\right) \\ &\leq \left[\theta\left(\lim_{n\to+\infty} M\left(x_n, z\right)\right)\right]^r = \left[\theta\left(d\left(z, Tz\right)\right)\right]^r \\ &< \theta\left(d\left(z, Tz\right)\right),\end{aligned}$$

which is a contradiction.

If $d\left(Tz, z\right) > 0$, by a similar method, we get a contradiction. Therefore, $d\left(z, Tz\right) = d\left(Tz, z\right) = 0$. Hence $z = Tz$.

Step 4. Uniqueness.

Suppose that there are two distinct points $z, u \in X$ such that $Tz = z$ and $Tu = u$. Then $d(z, u) = d(Tz, Tu) > 0$ or $d(u, z) = d(Tu, Tz) > 0$.

Letting $x = z$ and $y = u$ in (3.1), we obtain

$$\theta\left(d(z, u)\right) \leq \left[\theta\left(M(z, u)\right)\right]^r,$$

where

$$M(z, u) = \max\left\{d\left(z, u\right), d\left(z, Tz\right), d\left(u, Tu\right)\right\} = d(z, u)$$

which implies that $\theta\left(d(z, u)\right) < \theta\left(d(z, u)\right)$. This is a contradiction. Thus $z = u$. □

Example 3.2. Let $X = [1, +\infty[$. Define $d : X \times X \to [0, +\infty[$ by

$$d(x, y) = \max\{y - x, 0\}$$

for all $x, y \in X$. Then (X, d) is a complete quasi-metric space.

Define a mapping $T : X \to X$ by

$$T(x) = \sqrt{x}.$$

Then $T(x) \in [1, +\infty[$. Let $\theta(t) = e^{\sqrt{t}}$ and $r = \frac{1}{2}$. It is obvious that $\theta \in \Theta$ and $r \in]0, 1[$.

Let $x, y \in [1, +\infty[$. Then we have

$$d(y, x) = \max\{x - y, 0\}, \qquad d(T^2 y, x) = \max\{x - y^{\frac{1}{4}}, 0\}.$$

So

$$\max\{x - y, 0\} \leq \max\{y - y^{\frac{1}{4}}, 0\},$$

which implies that

$$d(y, x) \leq d(T^2 y, x)$$

for all $x, y \in X$.

On the other hand,

$$d(Tx, Ty) = d\left(\sqrt{x}, \sqrt{y}\right) = \max\{\sqrt{y} - \sqrt{x}, 0\}$$

and

$$\begin{aligned} M(x, y) &= \max\{d(x, y), d(x, Tx), d(y, Ty)\} \\ &= \max\left\{\max\{y - x, 0\}, \max\{\sqrt{x} - x, 0\}, \max\{\sqrt{y} - y, 0\}\right\}. \end{aligned}$$

First observe that $\max\{d(Tx, Ty), d(Ty, Tx)\} > 0$ if and only if $y > x$. Hence

$$d(Tx, Ty) = \sqrt{y} - \sqrt{x}, \;\; \theta(d(Tx, Ty)) = e^{\sqrt{\sqrt{y} - \sqrt{x}}}$$

and

$$M(x, y) = \max\left\{y - x, \sqrt{x} - x, \sqrt{y} - y\right\} = y - x.$$

Then we have

$$[\theta(d(x, y))]^{\frac{1}{2}} = \left[e^{\sqrt{y - x}}\right]^{\frac{1}{2}} = e^{\sqrt{\sqrt{y - x}}}.$$

On the other hand,

$$\theta(d(Tx, Ty)) - [\theta(d(x, y))]^{\frac{1}{2}} = e^{\sqrt{\sqrt{y} - \sqrt{x}}} - e^{\sqrt{\sqrt{y - x}}}.$$

Since $x, y \in [1, +\infty[$,

$$\sqrt{y} - \sqrt{x} \leq \sqrt{y - x}.$$

Since $e^{\sqrt{x}}$ is increasing,

$$e^{\sqrt{\sqrt{y} - \sqrt{x}}} \leq e^{\sqrt{\sqrt{y - x}}},$$

which implies that

$$\begin{aligned}\theta(d(Tx,Ty)) &\leq [\theta(d(x,y))]^{\frac{1}{2}}\\ &\leq [\theta(\max\{d(x,y),d(x,Tx),d(y,Ty)\}]^{\frac{1}{2}}.\end{aligned}$$

Hence the condition (3.1) is satisfied. Therefore, T has a unique fixed point $z = 1$.

If we remove our condition $d(y,x) \leq d(Ty^2, x)$ for all $x, y \in X$, then it may be that T does not admit a fixed point.

Example 3.3. Let $X = \left[\frac{1}{4}, \frac{3}{5}\right]$. Define $d : X \times X \to [0, +\infty[$ by

$$d(x,y) = \max\{y - x, 0\}$$

for all $x, y \in X$. Then (X, d) is a complete quasi-metric space.

Define a mapping $T : X \to X$ by

$$T(x) = \frac{\sqrt{x}+1}{4}.$$

Then $T(x) \in \left[\frac{1}{4}, \frac{3}{5}\right]$. Let $\theta(t) = e^{\sqrt{t}}$ and $r = \frac{1}{2}$. It is obvious that $\theta \in \Theta$ and $r \in \left]0, 1\right[$.

Let $x, y \in \left[\frac{1}{4}, \frac{3}{5}\right]$. Then we have

$$d(y,x) = \max\{x-y,0\}, \quad d(T^2y,x) = \max\left\{x - \frac{1}{4}\left[\sqrt{\frac{\sqrt{y}+1}{4}}+1\right], 0\right\}.$$

If $x > y$ and $y = \frac{1}{4}$, then

$$\max\{x-y,0\} = x - \frac{1}{4} > \max\left\{x - \frac{1}{4}\left[\sqrt{\frac{\sqrt{y}+1}{4}}+1\right], 0\right\},$$

which implies that

$$d(y,x) > d(T^2y,x).$$

On the other hand,

$$d(Tx,Ty) = d\left(\frac{\sqrt{x}+1}{4}, \frac{\sqrt{y}+1}{4}\right) = \max\left\{\frac{\sqrt{y}-\sqrt{x}}{4}, 0\right\}$$

and

$$\begin{aligned}M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty)\}\\ &= \max\left\{\max\{y-x,0\}, \max\{\frac{\sqrt{x}+1}{4} - x, 0\}, \max\{\frac{\sqrt{y}+1}{4} - y, 0\}\right\}.\end{aligned}$$

First observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0$ if and only if $y > x$. Hence

$$d(Tx,Ty) = \frac{\sqrt{y}-\sqrt{x}}{4}, \quad \theta(d(Tx,Ty)) = e^{\frac{\sqrt{\sqrt{y}-\sqrt{x}}}{2}}$$

and

$$M(x,y) = \max\left\{y-x, \frac{\sqrt{x}+1}{4}-x, \frac{\sqrt{y}+1}{4}-y\right\}$$
$$\geq y-x.$$

Then we have

$$[\theta(d(x,y))] = e^{\sqrt{\sqrt{y-x}}}.$$

On the other hand,

$$\theta(d(Tx,Ty)) - \sqrt{[\theta(d(x,y))]} = e^{\frac{\sqrt{\sqrt{y}-\sqrt{x}}}{2}} - e^{\sqrt{\sqrt{y-x}}}.$$

Since $x, y \in \left[\frac{1}{4}, \frac{3}{5}\right]$ and the function e^t is increasing,

$$e^{\frac{\sqrt{\sqrt{y}-\sqrt{x}}}{2}} \leq e^{\sqrt{\sqrt{y-x}}},$$

which implies that

$$\theta(d(Tx,Ty)) \leq [\theta(d(x,y))]^{\frac{1}{2}}$$
$$\leq [\theta(\max\{d(x,y), d(x,Tx), d(y,Ty)\}, d(y,Tx))]^{\frac{1}{2}}.$$

Hence T has no fixed point.

Theorem 3.4. *Let (X,d) be a quasi-metric space and $T : X \to X$ be a mapping. If there exist $\phi \in \phi$ and $\theta \in \Theta$ such that for all $x, y \in X$,*

$$\max\{d(Tx,Ty), d(Ty,Tx)\} > 0 \;\Rightarrow\; \theta[d(Tx,Ty)] \leq \phi[\theta(M(x,y))], \quad (3.16)$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}$$

and

$$d(y,x) \leq d\left(T^2y, x\right).$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Then we define a sequence $\{x_n\}_{n\in\mathbb{N}}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) = 0$ and $d(x_{n_0+1}, x_{n_0}) = 0$, then x_{n_0} is a fixed point of T. Then we assume that $d(x_n, x_{n+1}) > 0$ or $d(x_{n+1}, x_n) > 0$. So,

$$\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_n)\} > 0.$$

Step 1. We claim that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Letting $x = x_{n-1}$ and $y = x_n$ in (3.16), we obtain

$$\theta(d(x_n, x_{n+1})) = \theta(d(Tx_{n-1}, Tx_n)) \leq \phi[\theta(M(x_{n-1}, x_n))],$$

where

$$\begin{aligned} M\left(x_{n-1}, x_n\right) &= \max\{d\left(x_{n-1}, x_n\right), d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\} \\ &= \max\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right)\}. \end{aligned}$$

Suppose that $d\left(x_{n-1}, x_n\right) \leq d\left(x_n, x_{n+1}\right)$ for some positive integer n. Then we have

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) \leq \phi\left[\theta\left(d\left(x_n, x_{n+1}\right)\right)\right].$$

By Lemma 2.9, we obtain

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) < \theta\left(d\left(x_n, x_{n+1}\right)\right),$$

which is a contradiction. So

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) \leq \phi\left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right] \leq ... \leq \phi^n\left[\theta\left(d\left(x_0, x_1\right)\right)\right]. \tag{3.17}$$

By Lemma 2.9, we obtain

$$\theta\left(d\left(x_n, x_{n+1}\right)\right) < \theta\left(d\left(x_{n-1}, x_n\right)\right).$$

By (θ_1), we have

$$d\left(x_n, x_{n+1}\right) < d\left(x_{n-1}, x_n\right). \tag{3.18}$$

Letting $x = x_n$ and $y = x_{n-1}$ in (3.16), we obtain

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) = \theta\left(d\left(Tx_n, Tx_{n-1}\right)\right) \leq \phi\left[\theta\left(M\left(x_n, x_{n-1}\right)\right)\right],$$

where

$$\begin{aligned} M\left(x_n, x_{n-1}\right) &= \max\{d\left(x_n, x_{n-1}\right), d\left(x_n, x_{n+1}\right), d\left(x_{n-1}, x_n\right)\} \\ &= \max\{d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n-1}\right)\}. \end{aligned}$$

Suppose that $d\left(x_n, x_{n-1}\right) \leq d\left(x_{n+1}, x_n\right)$ for some $n \in \mathbb{N}$.

Case 1: $d\left(x_n, x_{n-1}\right) \geq d\left(x_{n-1}, x_n\right)$. We get

$$\theta\left(d\left(x_n, x_{n-1}\right)\right) \leq \theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \phi\left[\theta\left(d\left(x_n, x_{n-1}\right)\right)\right] < \theta\left(d\left(x_n, x_{n-1}\right)\right),$$

which is a contradiction.

Case 2: $d\left(x_n, x_{n-1}\right) < d\left(x_{n-1}, x_n\right)$. We get

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \phi\left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right].$$

Since $d\left(y, x\right) \leq d\left(T^2 y, x\right)$, $d\left(x_{n-1}, x_n\right) \leq d\left(x_{n+1}, x_n\right)$, which implies that

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \phi\left[\theta\left(d\left(x_{n-1}, x_n\right)\right)\right] \leq \phi\left[\theta\left(d\left(x_{n+1}, x_n\right)\right)\right] < \theta\left(d\left(x_{n+1}, x_n\right)\right).$$

This is a contradiction. Hence

$$\theta\left(d\left(x_{n+1}, x_n\right)\right) \leq \phi\left[\theta\left(d\left(x_n, x_{n-1}\right)\right)\right] \leq ... \leq \phi^n\left[\theta\left(d\left(x_1, x_0\right)\right)\right]. \tag{3.19}$$

By Lemma 2.9 and using (θ_1), we conclude that

$$d\left(x_{n+1}, x_n\right) < d\left(x_n, x_{n-1}\right). \tag{3.20}$$

From (3.18), the sequence $\{d(x_n, x_{n+1})\}_{n\in\mathbb{N}}$ is monotone nonincreasing. So there exists $\alpha \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \alpha.$$

Taking the limit as $n \to \infty$ in (3.17), using (ϕ_2) and (θ_3), we obtain

$$1 \leq \lim_{n\to+\infty} \theta(d(x_n, x_{n+1})) \leq \lim_{n\to+\infty} \phi^n [\theta(d(x_{n-1}, x_n))].$$

Thus $\lim_{n\to+\infty} \theta(d(x_n, x_{n+1})) = 1$. By (θ_2),

$$\lim_{n\to\infty} d(x_{n,} x_{n+1}) = 0. \tag{3.21}$$

From (3.18), the sequence $\{d(x_{n+1}, x_n)\}_{n\in\mathbb{N}}$ is monotone nonincreasing. So there exists $\lambda \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lambda.$$

Taking the limit as $n \to \infty$ in (3.19), using (ϕ_2) and (θ_3), we have

$$1 \leq \lim_{n\to+\infty} \theta(d(x_{n+1}, x_n)) \leq \lim_{n\to+\infty} \phi^n [\theta(d(x_n, x_{n-1}))].$$

Thus $\lim_{n\to+\infty} \theta(d(x_{n+1}, x_n)) = 1$. By (θ_2),

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0.$$

Step 2. We prove that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Firstly, we show that $\{x_n\}_{n\in\mathbb{N}}$ is a forward-Cauchy sequence. If otherwise there exist an $\varepsilon > 0$ and sequences $\{n_{(k)}\}_k$ and $\{m_{(k)}\}_k$ such that, for all positive integers k, $n_{(k)} > m_{(k)} > k$,

$$d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) < \varepsilon$$

and

$$d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) < \varepsilon.$$

By the triangular inequality, we obtain

$$\begin{aligned} \varepsilon \leq d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq d\left(x_{m_{(k)}}, x_{n_{(k)-1}}\right) + d\left(x_{n_{(k)-1}}, x_{n_{(k)}}\right) \\ &< \varepsilon + d\left(x_{n_{(k)-1}}, x_{n_{(k)}}\right). \end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{3.22}$$

Now, by the triangular inequality, we have

$$\begin{aligned} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) &\leq d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right) + d\left(x_{m_{(k)}}, x_{n_{(k)+1}}\right) \\ &\leq d\left(x_{m_{(k)+1}}, x_{m(k)}\right) + d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), \end{aligned}$$

$$\begin{aligned} d\left(x_{m_{(k)}}, x_{n_{(k)}}\right) &\leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)}}\right) \\ &\leq d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right) + d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right). \end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{3.23}$$

By (3.23), let $B = \frac{\varepsilon}{2} > 0$, from the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that

$$\left| d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) - \varepsilon \right| \leq B$$

for all $n \geq n_0$. This implies that

$$d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) \geq B > 0$$

for all $n \geq n_0$. Letting $x = x_{m_{(k)}}$ and $y = x_{m_{(k)}}$ in (3.16), we have

$$\theta\left(d\left(x_{m_{(k)+1}}, x_{m_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(M\left(x_{m_{(k)}}, x_{n_{(k)}}\right)\right)\right], \tag{3.24}$$

where

$$M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \max\left\{d\left(x_{m_{(k)}}, x_{n_{(k)}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right)\right\}.$$

Therefore, by (3.22) and (3.21), we get that

$$\lim_{k\to+\infty} M\left(x_{m_{(k)}}, x_{n_{(k)}}\right) = \varepsilon. \tag{3.25}$$

Taking the limit as $k \to \infty$ in (3.24), using (3.25), (ϕ_3), (θ_3) and Lemma 2.9, we obtain

$$\theta(\varepsilon) \leq \phi\left[\theta(\varepsilon)\right] < \theta(\varepsilon).$$

This is a contradiction. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a forward-Cauchy sequence in (X, d).

Secondly, we prove that $\{x_n\}_n \in \mathbb{N}$ is a backward-Cauchy sequence. If otherwise there exist an $\varepsilon > 0$ and sequences $\{n_{(k)}\}_k$ and $\{m_{(k)}\}_k$ such that, for all positive integers k, $n_{(k)} > m_{(k)} > k$,

$$d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) \leq \varepsilon$$

and

$$d\left(x_{n_{(k)-1}}, xm_{(k)}\right) < \varepsilon.$$

By the triangular inequality, we obtain

$$\begin{aligned}\varepsilon \leq d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) &\leq d\left(x_{n_{(k)}}, x_{n_{(k)-1}}\right) + d\left(x_{n(k)-1}, x_{m_{(k)}}\right) \\ &< d\left(x_{n_{(k)}}, x_{n_{(k)-1}}\right) + \varepsilon.\end{aligned}$$

Taking the limit as $k \to \infty$, we obtain

$$\lim_{k\to\infty} d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) = \varepsilon. \tag{3.26}$$

Now, by the triangular inequality, we have

$$\begin{aligned}d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) &\leq d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{m_{(k)+1}}\right) \\ &\leq d\left(x_{n_{(k)+1}}, x_{n_{(k)}}\right) + d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) + d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\end{aligned}$$

and

$$\begin{aligned}d\left(x_{n_{(k)}}, x_{m_{(k)}}\right) &\leq d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{m_{(k)}}\right) \\ &\leq d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right) + d\left(x_{n_{(k)+1}}, x_{m(k)+1}\right) + d\left(x_{m_{(k)+1}}, x_{m_{(k)}}\right).\end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequalities, we obtain

$$\lim_{k\to\infty} d\left(x_{m_{(k)+1}}, x_{n_{(k)+1}}\right) = \varepsilon. \tag{3.27}$$

By (3.27), let $A = \frac{\varepsilon}{2} > 0$, from the definition of the limit, there exists $n_1 \in \mathbb{N}$ such that

$$\left| d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) - \varepsilon \right| \leq A$$

for all $n \geq n_1$. This implies that

$$d\left(x_{n_{(k)+1}}, x_{m_{(k)+1}}\right) \geq A > 0$$

for all $n \geq n_1$. Letting $x = x_{n_{(k)}}$ and $y = x_{m_{(k)}}$ in (3.16), we have

$$\theta\left(d\left(x_{n_{(k)+1}}, x_{n_{(k)+1}}\right)\right) \leq \phi\left[\theta\left(M\left(x_{n_{(k)}}, x_{m_{(k)}}\right)\right)\right], \tag{3.28}$$

where

$$M\left(x_{n_{(k)}}, x_{m_{(k)}}\right) = \max\left\{d\left(x_{n_{(k)}}, x_{m_{(k)}}\right), d\left(x_{n_{(k)}}, x_{n_{(k)+1}}\right), d\left(x_{m_{(k)}}, x_{m_{(k)+1}}\right)\right\}.$$

Therefore by (3.21) and (3.26), we get that

$$\lim_{k\to+\infty} M\left(x_{n_{(k)}}, x_{m_{(k)}}\right) = \varepsilon.$$

Taking the limit as $k \to \infty$ in (3.28), using (3.27) and Lemma 2.9, we obtain

$$\theta(\varepsilon) \leq \phi\left[\theta(\varepsilon)\right] < \theta(\varepsilon).$$

This is a contradiction. Consequently, $\{x_n\}_{n\in\mathbb{N}}$ is a backward-Cauchy sequence in (X, d). Hence, by completeness of (X, d), there exist $z, u \in X$ such that

$$\lim_{n\to+\infty} d\left(x_n, z\right) = \lim_{n\to+\infty} d\left(u, x_n\right) = 0.$$

So from Lemma 2.3, we get $z = u$ and hence

$$\lim_{n\to+\infty} d\left(x_n, z\right) = \lim_{n\to+\infty} d\left(z, x_n\right) = 0.$$

Step 3. We prove that $z = Tz$, i.e., $d\left(Tz, z\right) = 0$ and $d\left(z, Tz\right) = 0$.

Arguing by contradiction, we assume that $d\left(Tz, z\right) > 0$ or $d\left(z, Tz\right) > 0$.

First, assume that $d\left(z, Tz\right) > 0$. As in the proof of Theorem 3.1, we have

$$\lim_{n\to+\infty} d\left(Tx_n, Tz\right) = d\left(z, Tz\right).$$

So there exists $n_0 \in \mathbb{N}$ such that

$$d\left(Tx_n, Tz\right) \geq d\left(z, Tz\right) > 0$$

for all $n \geq n_0$. Letting $x = x_n$ and $y = z$ in (3.16), we obtain

$$\theta\left(d\left(Tx_n, Tz\right)\right) \leq \phi\left[\theta\left(M\left(x_n, z\right)\right)\right], \tag{3.29}$$

where

$$M\left(x_n, z\right) = \max\left\{d\left(x_n, Tx_n\right), d\left(z, Tz\right), d\left(x_n, z\right)\right\}.$$

Since $\lim_{n\to+\infty} d\left(x_n, x_{n+1}\right) = \lim_{n\to+\infty} d\left(x_n, z\right) = 0$, we obtain

$$\lim_{n\to+\infty} M\left(x_n, z\right) = d\left(z, Tz\right).$$

Taking the limit as $n \to \infty$ in (3.29), using the properties of ϕ and θ, we obtain

$$\begin{aligned}\lim_{n\to+\infty} \theta\left(d\left(Tx_n, Tz\right)\right) &= \theta\left(\lim_{n\to+\infty} d\left(Tx_n, Tz\right)\right) = \theta\left(d\left(z, Tz\right)\right) \\ &\leq \phi\left[\theta\left(\lim_{n\to+\infty} M\left(x_n, z\right)\right)\right] = \phi\left[\theta\left(d\left(z, Tz\right)\right)\right] \\ &< \theta\left(d\left(z, Tz\right)\right).\end{aligned}$$

This is a contradiction.

If $d\left(Tz, z\right) > 0$, by a similar method, we get a contradiction. Therefore, $d\left(z, Tz\right) = d\left(Tz, z\right) = 0$. Hence $z = Tz$.

Step 4. Uniqueness.

Suppose that there are two distinct points $z, u \in X$ such that $Tz = z$ and $Tu = u$. Then $d(z,u) = d(Tz,Tu) > 0$ or $d(u,z) = d(Tu,Tz) > 0$.

Letting $x = z$ and $y = u$ in (3.16), we obtain

$$\theta\left(d(z,u)\right) \leq \phi\left[\theta\left(M(z,u)\right)\right],$$

where

$$M(z,u) = \max\left\{d\left(z,u\right), d\left(z,Tz\right), d\left(u,Tu\right)\right\} = d(z,u).$$

This implies that $\theta\left(d(z,u)\right) < \theta\left(d(z,u)\right)$, which is a contradiction. Thus $z = u$. □

Corollary 3.5. *Let (X,d) be a quasi-metric space and $T : X \to X$ be a mapping. If there exist $\theta \in \Theta_C$ and $r \in \left]0,1\right[$ such that for all $x, y \in X$,*

$$\max\{d\left(Tx,Ty\right), d\left(Ty,Tx\right)\} > 0 \;\;\Rightarrow\;\; \theta\left[d\left(Tx,Ty\right)\right] \leq \left[\theta\left(M\left(x,y\right)\right)\right]^r,$$

where

$$M\left(x,y\right) = \max\left\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\right\}$$

and

$$d\left(y,x\right) \leq d\left(T^2y,x\right).$$

Then T has a unique fixed point.

Proof. Let $\phi\left(t\right) = t^k$ for all $t \in \left[1,+\infty\right[$. It is obvious that $\phi \in \Phi$ and we have

$$\max\{d\left(Tx,Ty\right), d\left(Ty,Tx\right)\} > 0 \Rightarrow \theta\left[d\left(Tx,Ty\right)\right] \leq \phi\left[\theta\left(M\left(x,y\right)\right)\right].$$

Hence T satisfies the assumption of Theorem 3.4 and there is a unique fixed point of T. □

Corollary 3.6. *Let (X,d) be a complete quasi-metric space. Assume that there exists $\alpha \in \left]0,\frac{1}{2}\right[$ such that for all $x, y \in X$ with*

$$\max\{d\left(Tx,Ty\right), d\left(Ty,Tx\right)\} > 0,$$

we have

$$d\left(Tx,Ty\right) \leq \alpha\left[d\left(Tx,x\right) + d\left(y,Ty\right)\right].$$

Then T has a unique fixed point.

Proof. Let $\theta(t) = e^t$ for all $t \in \left]0,+\infty\right[$ and $\phi\left(t\right) = t^{2\alpha}$ for all $t \in \left[1,+\infty\right[$.

It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. So

$$\begin{aligned}
\theta\left(d\left(Tx,Ty\right)\right) &= e^{d\left(Tx,Ty\right)} \\
&\leq e^{\alpha\left(d\left(Tx,x\right)+d\left(y,Ty\right)\right)} \\
&= e^{2\alpha\left(\frac{d\left(Tx,x\right)+d\left(y,Ty\right)}{2}\right)} \\
&= \left[e^{\left(\frac{d\left(Tx,x\right)+d\left(y,Ty\right)}{2}\right)}\right]^{2\alpha} \\
&= \phi\left[\theta\left(\frac{d\left(Tx,x\right)+d\left(y,Ty\right)}{2}\right)\right] \\
&\leq \phi\left[\theta\left(\max\{d\left(x,y\right),d\left(Tx,x\right),d\left(y,Ty\right)\}\right)\right].
\end{aligned}$$

Therefore, from Theorem 3.4, T has a unique fixed point. □

Corollary 3.7. *Let (X,d) be a complete quasi-metric space. Assume that there exists $\lambda \in \left]0,\frac{1}{3}\right[$ such that for all $x,y \in X$ with*

$$\max\{d\left(Tx,Ty\right),d\left(Ty,Tx\right)\} > 0,$$

we have

$$d\left(Tx,Ty\right) \leq \alpha\left[d\left(x,y\right)+d\left(Tx,x\right)+d\left(y,Ty\right)\right].$$

Then T has a unique fixed point.

Proof. Let $\theta(t) = e^t$ for all t $\in \left]0,+\infty\right[$ and $\phi\left(t\right) = t^{3\lambda}$ for all t $\in [1,+\infty[$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$. So

$$\begin{aligned}
\theta\left(d\left(Tx,Ty\right)\right) &= e^{d\left(Tx,Ty\right)} \\
&\leq e^{3\lambda\frac{\left(d\left(x,y\right)+d\left(Tx,x\right)+d\left(y,Ty\right)\right)}{3}} \\
&= \left[e^{\frac{\left(d\left(x,y\right)+d\left(Tx,x\right)+d\left(y,Ty\right)\right)}{3}}\right]^{3\lambda} \\
&= \phi\left[\theta\left(\frac{\left(d\left(x,y\right)+d\left(Tx,x\right)+d\left(y,Ty\right)\right)}{3}\right)\right] \\
&\leq \phi\left[\theta\left(\max\{d\left(x,y\right),d\left(Tx,x\right),d\left(y,Ty\right)\}\right)\right].
\end{aligned}$$

Therefore, from Theorem 3.4, T has a unique fixed point. □

Example 3.8. Let $X = [1,+\infty[$. Define $d : X \times X \to [0,+\infty[$ by

$$d(x,y) = \max\{y-x,0\}$$

for all $x, y \in X$. Then (X, d) is a complete quasi-metric space.

Define a mapping $T : X \to X$ by

$$T(x) = \frac{\sqrt{x}+1}{2}.$$

Then $T(x) \in [1, +\infty[$. Let $\theta(t) = \sqrt{t}+1$ and $\phi(t) = \frac{t+1}{2}$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$.

Let $x, y \in [1, +\infty[$. Then we have

$$d(y,x) = \max\{x-y, 0\}, \quad d(T^2y, x) = \max\left\{x - \sqrt{\frac{\sqrt{y}+1}{8}} - \frac{1}{2}, 0\right\}.$$

So

$$\max\{x-y,0\} \le \max\left\{x - \sqrt{\frac{\sqrt{y}+1}{8}} - \frac{1}{2}, 0\right\},$$

which implies that

$$d(y,x) \le d(T^2y, x)$$

for all $x, y \in X$. On the other hand,

$$d(Tx,Ty) = d\left(\frac{\sqrt{x}+1}{2}, \frac{\sqrt{y}+1}{2}\right) = \max\left\{\frac{\sqrt{y}-\sqrt{x}}{2}, 0\right\}$$

and

$$\begin{aligned} M(x,y) &= \max\{d(x,y), d(x,Tx), d(y,Ty)\} \\ &= \max\left\{\max\{y-x,0\}, \max\{\frac{\sqrt{x}+1}{2} - x, 0\}, \max\{\frac{\sqrt{y}+1}{2} - y, 0\}\right\}. \end{aligned}$$

First observe that $\max\{d(Tx,Ty), d(Ty,Tx)\} > 0$ if and only if $y > x$. Hence

$$d(Tx,Ty) = \frac{\sqrt{y}-\sqrt{x}}{2}, \ \theta(d(Tx,Ty)) = \sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} + 1$$

and

$$M(x,y) = y - x.$$

Then we have

$$\phi\left[\theta(d(x,y))\right] = \frac{\sqrt{y-x}}{2} + 1.$$

On the other hand,

$$\begin{aligned} \theta(d(Tx,Ty)) - \phi\left[\theta(d(x,y))\right] &= \sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} + 1 - \frac{\sqrt{y-x}}{2} + 1 \\ &= \sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} - \frac{\sqrt{y-x}}{2}. \end{aligned}$$

Since $x, y \in [1, +\infty[$,

$$\sqrt{\frac{\sqrt{y}-\sqrt{x}}{2}} - \frac{\sqrt{y-x}}{2} \leq 0.$$

This implies that

$$\begin{aligned}\theta(d(Tx,Ty)) &\leq \phi\left[\theta(d(x,y))\right] \\ &\leq \phi\left[\theta(\max\left\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\right\}, d\left(y,Tx\right))\right].\end{aligned}$$

Hence the condition (3.16) is satisfied. Therefore, T has a unique fixed point $z = 1$.

If we remove our condition $d(y,x) \leq d(Ty^2, x)$ *for all* $x, y \in X$, then it may be that T does not admit a fixed point.

Example 3.9. Let $X = \left[\frac{1}{4}, \frac{1}{2}\right]$. Define $d : X \times X \to [0, +\infty[$ by

$$d(x,y) = \max\{y - x, 0\}$$

for all $x, y \in X$. Then (X, d) is a complete quasi-metric space.

Define a mapping $T : X \to X$ by

$$T(x) = \frac{\sqrt{x}+4}{16}.$$

Then $T(x) \in \left[\frac{1}{4}, \frac{1}{2}\right]$. Let $\theta(t) = \sqrt{t} + 1$ and $\phi(t) = \frac{t+1}{2}$. It is obvious that $\theta \in \Theta$ and $\phi \in \Phi$.

Let $x, y \in \left[\frac{1}{4}, \frac{1}{2}\right]$. Then we have

$$d(y,x) = \max\{x-y, 0\} \text{ and } d(T^2y, x) = \max\left\{x - \frac{1}{16}\left[\sqrt{\frac{\sqrt{y}+4}{16}} + 4\right], 0\right\}.$$

If $x > y$ *and* $y = \frac{1}{4}$, then

$$\max\{x-y, 0\} = x - \frac{1}{4} > \max\left\{x - \frac{1}{16}\left[\sqrt{\frac{\sqrt{y}+4}{16}} + 4\right], 0\right\}.$$

This implies that

$$d(y,x) > d(T^2y, x).$$

On the other hand,

$$d(Tx,Ty) = d\left(\frac{\sqrt{x}+4}{16}, \frac{\sqrt{y}+4}{16}\right) = \max\left\{\frac{\sqrt{y}-\sqrt{x}}{16}, 0\right\}$$

and

$$\begin{aligned}M(x,y) &= \max\{d\left(x,y\right), d\left(x,Tx\right), d\left(y,Ty\right)\} \\ &= \max\left\{\max\{y-x, 0\}, \max\{\frac{\sqrt{x}+4}{16} - x, 0\}, \max\{\frac{\sqrt{y}+4}{16} - y, 0\}\right\}.\end{aligned}$$

First observe that $\max\{d(Tx,Ty),d(Ty,Tx)\}>0$ if and only if $y>x$. Hence

$$d(Tx,Ty)=\frac{\sqrt{y}-\sqrt{x}}{16},\ \ \theta(d(Tx,Ty))=\sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}+1$$

and

$$\begin{aligned}M(x,y)&=\max\left\{y-x,\frac{\sqrt{x}+4}{16}-x,\frac{\sqrt{y}+4}{16}-y\right\}\\&\geq y-x.\end{aligned}$$

Then we have

$$\phi\left[\theta(d(x,y))\right]=\frac{\sqrt{y-x}}{2}+1.$$

On the other hand,

$$\begin{aligned}\theta(d(Tx,Ty))-\phi\left[\theta(d(x,y))\right]&=\sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}+1-\frac{\sqrt{y-x}}{16}-1\\&=\sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}-\frac{\sqrt{y-x}}{2}.\end{aligned}$$

Since $x,y\in\left[\frac{1}{4},\frac{1}{2}\right]$,

$$\sqrt{\frac{\sqrt{y}-\sqrt{x}}{16}}-\frac{\sqrt{y-x}}{2}\leq 0.$$

This implies that

$$\begin{aligned}\theta(d(Tx,Ty))&\leq\phi\left[\theta(d(x,y))\right]\\&\leq\phi\left[\theta(\max\left\{d\left(x,y\right),d\left(x,Tx\right),d\left(y,Ty\right)\right\},d\left(y,Tx\right))\right].\end{aligned}$$

Hence T has no fixed point.

4. Conclusion

In this paper, we introduced the concept of θ-contraction and θ-φ-contraction in quasi-metric spaces to study the existence of the fixed point for them. We plan to study some contractions in other metric spaces.

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