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EXISTENCE OF MULTIPLE POSITIVE SOLUTIONS FOR A SCHRÖDINGER-TYPE SINGULAR FALLING ZERO PROBLEM

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Abstract. Extending [14], we establish the existence of multiple positive solutions for a Schrödinger-type singular elliptic equation:

$$\begin{cases} -\Delta u + V(x)u &= \lambda \frac{f(u)}{u^\beta}, \quad x \in \Omega, \\ u &= 0, \quad\quad\; x \in \partial\Omega, \end{cases}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1$, with a smooth boundary $\partial\Omega$, $\beta \in [0,1)$, $f \in C[0,\infty)$, $V : \Omega \to \mathbb{R}$ is a bounded function and λ is a positive parameter. In particular, when $f(s) > 0$ on $[0,\sigma)$ and $f(s) < 0$ for $s > \sigma$, we establish the existence of at least three positive solutions for a certain range of λ by using the method of sub and supersolutions.

1. Introduction

We consider a Schrödinger- type singluar problem on $\mathbb{R}^N$

$$(P_\lambda) \qquad \begin{cases} -\Delta u + V(x)u &= \lambda \frac{f(u)}{u^\beta}, \quad x \in \Omega, \\ \\ u &= 0, \quad\quad\; x \in \partial\Omega, \end{cases}$$

where $0 \in \Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1, 0 \leq \beta < 1$, $V \in L^\infty(\Omega)$ and λ is a positive parameter. We assume that $f : [0,\infty) \to \mathbb{R}$ is a continuous function satisfying

($H1$) There exists $\sigma > 0$ such that $f(s) > 0$ for all $0 \leq s < \sigma$ and $f(s) < 0$ for all $s > \sigma$.

We further assume that $V \in L^\infty(\Omega)$ satisfies the condition:

($H2$) There exists a constant $c_V > 0$ such that $V(x) \geq -c_V$ for $x \in \Omega$ and $1 - c_V c_1 > 0$, where $c_1 > 0$ is a constant such that $\int_\Omega |u|^2 dx \leq$

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$c_1 \int_\Omega |\nabla u|^2 dx \ \forall u \in W_0^{1,2}(\Omega)$.

The first equation in (P_λ) is derived from the nonlinear Schrödinger equation (details in [21]). Nonlinear Schrödinger equations have been studied widely to demonstrate the existence of solutions which act on V in the whole space $\mathbb{R}^N$ (see [2], [13], [19]) or on bounded domains (see [10]). In the case when $V \equiv 0$, the existence of multiple positive solutions of (P_λ) with falling zero nonlinearity has been widely investigated for a long time (see [3],[4], [5], [18] and [22] and references therein). In this paper, when $V \in L^\infty(\Omega)$ satisfies $(H2)$, by using the method of sub and supersolutions, we establish the existence of a positive solution of (P_λ) for all $\lambda > 0$ and the existence of multiple positive solutions of (P_λ) for a certain range of λ.

We first state the existence result:

Theorem 1.1. *Assume $(H1)$ and $(H2)$. Then (P_λ) has a positive solution $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ for all $\lambda > 0$.*

Next, to state the multiplicity result, we let

$$A = \frac{(N+1)^{N+1}}{N^N} \text{ and } B = \frac{R^2}{AN + \|V\|_\infty R^2},$$

where R is the radius of the largest inscribed ball B_R in Ω. We define $f^*(s) := \max_{t \in [0,s]} f(t)$ and for any $0 < a < d < b$,

$$Q(a,d,b) := \frac{\frac{d^{\beta+1}}{f(d)} \frac{1}{B}}{\min\left\{ \frac{a^{\beta+1}}{f^*(a)\|w\|_\infty^{\beta+1}}, \frac{2d^\beta}{f(d)AB} b \right\}}.$$

Theorem 1.2. *Assume $(H1)$ and $(H2)$. If there exist a, b and d with $0 < a < d < b$ such that $Q(a,d,b) < 1$,*

$$\tilde{f}(s) := \frac{f(s)}{s^\beta} - \frac{f(d)}{d^{\beta+1}} B \|V\|_\infty s > 0, \ \forall s \in [0,b]$$

and $\tilde{f}(s)$ is nondecreasing on $[a,b]$, then the problem (P_λ) has at least three positive solutions $u_\lambda \in C^2(\Omega) \cap C(\bar{\Omega})$ for all $\lambda_* < \lambda < \lambda^*$, where

$$\lambda_* = \frac{d^{\beta+1}}{f(d)} \frac{1}{B} \text{ and } \lambda^* = \min\left\{ \frac{a^{\beta+1}}{f^*(a)\|w\|_\infty^{\beta+1}}, \frac{2d^\beta}{f(d)AB} b \right\}.$$

In order to obtain at least three positive solutions for a certain range of λ using the method of sub and supersolution, it is important to construct two pairs of sub and supersolutions $(\psi_1, Z_1), (\psi_2, Z_2)$ of (P_λ) with the property that $\psi_1 \le \psi_2 \le Z_1$, $\psi_1 \le Z_2 \le Z_1$ such that $\psi_2 \not\le Z_2$ so that three solution results in [1] can be applied. However, the term $V(x)$ acting on u gives a nontrivial difficulty in the construction of the second pair of sub and supersolution (ψ_2, Z_2) satisfying $\psi_2 \not\le Z_2$. We overcome the difficulty from the singularity by the arguments used in [15] and [22] and by combining $V(x)u$ and $\frac{f(u)}{u^\beta}$ with a suitable

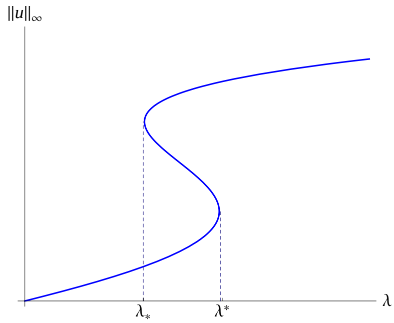

FIGURE 1. S-shaped bifurcation diagram showing the existence of multiple positive solutions for (P_λ)

way, that is $\tilde{f}(s) = \frac{f(s)}{s^\beta} - \frac{f(d)}{d^{\beta+1}} B\|V\|_\infty s$, we enable to construct the second pair of sub and supersolution (ψ_2, Z_2) satisfying $\psi_2 \not\leq Z_2$.

This paper is organized as follows: In Section 2, we analyze the phosphorous cycling model which is applicable to our results. Moreover, we provide S-shaped bifurcation curves verifying Theorem 1.1 and Theorem 1.2 obtained numerically via Quadrature method. In Section 3, we recall a method of sub and supersolutions for (P_λ) and a three solution theorem for singular problem (P_λ). Section 4 is devoted to the proofs of Theorem 1.1 and Theorem 1.2.

2. Example and numerical results

In this section, we introduce a model which is applicable in our main results. A simple model reads as

$$\begin{cases} -\Delta u + V(x)u &= \frac{\lambda}{u^\beta}\left(\tau - u + \frac{cu^4}{1+u^4}\right), \quad x \in \Omega, \\ u &= 0, \qquad\qquad\qquad\qquad\quad x \in \partial\Omega. \end{cases} \tag{1}$$

This model describes phosphorous cycling in stratified lake and the colonization of barren soils in drylands by vegetation. In particular, it illustrates the decrease in the amount of phosphorous in the eqilimnion (upper layer) and the rapid recycling that occurs when the hypolimnion (lower layer) is depleted of oxygen. It also describes the colonization of barren soils in drylands by vegetation (more details in [5] and [6]).

Denote $g(s) := \frac{f(s)}{s^\beta}$ for $s > 0$. Let $0 < \beta < 1$ be arbitrary fixed. Now we provide the necessary conditions for the value of $\tau > 0$ and $c > 0$ in order to obtain the existence and multiplicity result Theorem 1.1 and Theorem 1.2.

Proposition 2.1. *If* $c > \frac{16\beta}{(4-\beta)^2}\tau + \frac{16(1-\beta)}{(4-\beta)^2}\sqrt[4]{\frac{4-\beta}{4+\beta}} =: c(\tau)$ *for each* $\tau > 0$, *then there exist* $0 < m < M < \infty$ *such that* $g'(m) = g'(M) = 0$.

Proof. As $g'(s) = \frac{d}{ds}\left[\frac{f(s)}{s^\beta}\right] = \frac{sf'(s)-\beta f(s)}{s^{\beta+1}}$, we evaluate

$$\begin{aligned} sf'(s) - \beta f(s) &= -s + \frac{4cs^4}{(1+s^4)^2} - \tau\beta + \beta s - \frac{\beta c s^4}{1+s^4} \\ &= \frac{cs^4(4-\beta-\beta s^4)}{(1+s^4)^2} - \tau\beta - (1-\beta)s =: j(s) - i(s) \end{aligned}$$

where $j(s) := \frac{cs^4(4-\beta-\beta s^4)}{(1+s^4)^2}$ and $i(s) := \tau\beta + (1-\beta)s$ for $s > 0$. We claim that for each $\tau > 0$ if $c > c(\tau)$, then $j(s) - i(s) = 0$ has two positive solutions. This implies that $g'(s)$ has exactly two zeros in $(0, \infty)$. Indeed, as $j'(s) = \frac{4cs^3(4-\beta-(4+\beta)s^4)}{(1+s^4)^3}$, it follows that $j(s)$ achieves exactly one local maximum $\frac{c(4-\beta)^2}{16}$ at $s = \sqrt[4]{\frac{4-\beta}{4+\beta}}$ in $(0,\infty)$. Hence, if $j\left(\sqrt[4]{\frac{4-\beta}{4+\beta}}\right) > i\left(\sqrt[4]{\frac{4-\beta}{4+\beta}}\right)$, then the linear line $i(s)$ will cut $j(s)$ at exactly two different points on $(0,\infty)$. This implies that there exist exactly two positive critical points $0 < m < \sqrt[4]{\frac{4-\beta}{4+\beta}} < M < \infty$ such that $g'(m) = g'(M) = 0$ if $c > \frac{16\beta}{(4-\beta)^2}\tau + \frac{16(1-\beta)}{(4-\beta)^2}\sqrt[4]{\frac{4-\beta}{4+\beta}}$. □

Proposition 2.2. *If* $\tau > \frac{3}{4}\sqrt[4]{\frac{3}{5}} - \frac{1}{4}\left(\sqrt[4]{\frac{3}{5}}\right)^5 =: \tau_0$*, then there exists a unique* $\sigma > 0$ *such that* $g(\sigma) = 0$.

Proof. Since g has the local minimum at $s = m$, we can see that if $g(m) > 0$, then g has a unique zero in $(0,\infty)$. Now it is enough to show $f(m) > 0$ as $g(m) = \frac{f(m)}{m^\beta}$. Note that m is the solution of $j(s) = i(s)$ at the previous lemma. Hence, m satisfies

$$\frac{cm^4}{(1+m^4)^2} = \frac{\tau\beta + (1-\beta)m}{4-\beta-\beta m^4}. \tag{2}$$

Hence, using (2), we evaluate

$$f(m) = \tau - m + \frac{cm^4}{1+m^4} = \tau - m + \frac{(\tau\beta + (1-\beta)m)(1+m^4)}{4-\beta-\beta m^4},$$

which is simplified by

$$f(m) = \frac{4\tau + m^5 - 3m}{4-\beta-\beta m^4}.$$

Now we note $4 - \beta - \beta m^4 > 0$ as $0 < m^4 < \frac{4-\beta}{4+\beta}$. Hence, $f(m) > 0$ if $\tau > \frac{1}{4}(3m - m^5)$. Observing that $\sup\limits_{(0,\sqrt[4]{\frac{4-\beta}{4+\beta}})} \dfrac{1}{4}(3m - m^5)$ is achieved at $m = \sqrt[4]{\frac{3}{5}}$ and noting $\sqrt[4]{\frac{3}{5}} < \sqrt[4]{\frac{4-\beta}{4+\beta}}$ as $0 < \beta < 1$, we obtain $f(m) > 0$ if $\tau > \frac{3}{4}\sqrt[4]{\frac{3}{5}} - \frac{1}{4}\left(\sqrt[4]{\frac{3}{5}}\right)^5$. This implies that g has an unique zero $\sigma > 0$ if $\tau > \tau_0$. □

Hence, for all $\tau > \tau_0$ and $c > \frac{16\beta}{(4-\beta)^2}\tau + \frac{16(1-\beta)}{(4-\beta)^2}\sqrt[4]{\frac{4-\beta}{4+\beta}}$, the function $g(s) = \frac{f(s)}{s^\beta}$ satisfies the condition $(H1)$ and the graph of g is given in Figure 2.

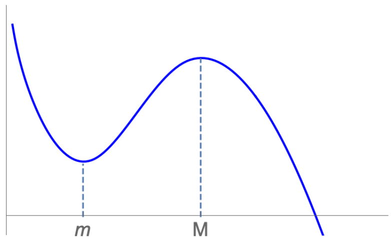

FIGURE 2. Graph of g satisfies $(H1)$ for $c > c(\tau)$ and $\tau > \tau_0$.

Now we provide the bifurcation curves of models (1) obtained numerically via Quadrature method when τ and c are satisfied with Proposition 2.1 and Proposition 2.2 and $V(x)$ is a constant with $\|V\|_\infty \approx 0$. Consider the following the one dimensional problem of (1) taking $V(x)$ as a constant $\mu \in \mathbb{R}$

$$\begin{cases} -u'' + \mu u &= \lambda \frac{f(u)}{u^\beta}, \quad x \in (0,1), \\ u(0) = 0 &= u(1), \end{cases} \tag{3}$$

where $f(u) = \tau - u + \frac{cu^4}{1+u^4}$. It is well-known that if u is a positive solution of (3), then u is symmetric about $x = \frac{1}{2}$, u is increasing on $(0, \frac{1}{2})$ and decreasing on $(\frac{1}{2}, 1)$ and $\|u\|_\infty = u(\frac{1}{2})$. By integrating (3) over $(0, \frac{1}{2})$ and using the above propeties, we deduce

$$\frac{1}{\sqrt{2}} = \int_0^\rho \frac{ds}{\sqrt{\lambda[F(\rho) - F(s)] - \frac{\mu}{2}(\rho^2 - s^2)}} := G[\rho, \lambda], \tag{4}$$

where $F(u) = \int_0^u \frac{f(s)}{s^\beta} ds$ and $\rho = \|u\|_\infty$. Conversely, by the modification of Quadrature method in [3], it can be shown that if for each $\rho > 0$ there exists $\lambda > 0$ satisfying (4), then (3) has a positive solution u_λ with $\|u\|_\infty = \rho$. Here we provide the S-shaped bifurcation curve for (3) with the specific value of τ and c satisfying the above lemma and the various values of μ in Figure 3.

3. Preliminary

In this section, we recall a method of obtaining sub and supersolutions and a three solution theorem for the singular problem (P_λ). We also recall results on the principal eigenvalue and eigenfunction of an eigenvalue problem. Finally, we prove that the singular problem (8) has a positive solution.

3.1. Method of sub- and supersolutions

A subsolution of (P_λ) is defined as a function $\psi : \bar{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta\psi + V(x)\psi \leq \lambda \frac{f(\psi)}{\psi^\beta}, & x \in \Omega, \\ \psi > 0, & x \in \Omega, \\ \psi = 0, & x \in \partial\Omega, \end{cases} \tag{5}$$

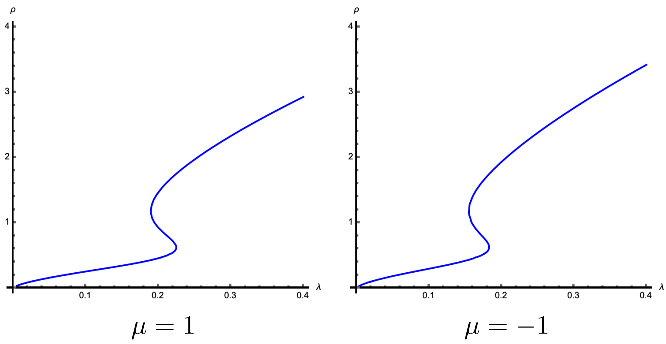

FIGURE 3. S-shaped bifurcation Diagram of (3) for the value of $\tau = 10$ and $c = 100$ via Mathematica

while a supersolution of $(P_\lambda)$is defined as a function $Z : \bar{\Omega} \to \mathbb{R}$ satisfying

$$\begin{cases} -\Delta Z + V(x)Z \geq \lambda \frac{f(Z)}{Z^\beta}, & x \in \Omega, \\ Z > 0, & x \in \Omega, \\ Z = 0, & x \in \partial\Omega. \end{cases} \tag{6}$$

The following Lemmas hold.

Lemma 3.1. *([8]) If a subsolution ψ and a supersolution Z of (P_λ) exist such that $\psi \leq Z$ on $\bar{\Omega}$, then (P_λ) has at least one solution $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $\psi \leq u \leq Z$ on $\bar{\Omega}$.*

Lemma 3.2. *([9]) Suppose there exist two pairs of ordered sub-supersolutions (ψ_1, Z_1) and (ψ_2, Z_2) of (P_λ) with the property that $\psi_1 \leq \psi_2 \leq Z_1$, $\psi_1 \leq Z_2 \leq Z_1$ and $\psi_2 \not\leq Z_2$. Additionally, assume that ψ_2 and Z_2 are not solutions of (P_λ). Then there exist at least three solutions $u_i, i = 1, 2, 3$ for $(P_\lambda)$where $u_1 \in [\psi_1, Z_2], u_2 \in [\psi_2, Z_1]$ and $u_3 \in [\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$.*

3.2. An eigenvalue problem

Let λ_1 be the principal eigenvalue and ϕ_1 be a corresponding eigenfunction of

$$\begin{cases} -\Delta\phi + V(x)\phi = \lambda\phi, & x \in \Omega, \\ \phi = 0, & x \in \partial\Omega, \end{cases} \tag{7}$$

If $V \in L^\infty(\Omega)$, then we can choose $\phi_1 \in C^2(\Omega) \cap C^1(\bar{\Omega})$ such that $\phi_1 > 0$ in Ω. (See [7],[11] and [20]).

3.3. A singular problem

Lemma 3.3. (*see* [17]) *Assume* $(H2)$. *Then, the following singular problem*

$$\begin{cases} -\Delta w + V(x)w = \frac{1}{w^\beta}, & in\ \Omega, \\ w = 0, & on\ \partial\Omega. \end{cases} \tag{8}$$

has a solution $w \in C^2(\Omega) \cap C(\bar{\Omega})$ *such that* $w(x) > 0$ *for* $x \in \Omega$ *and* $\frac{\partial w}{\partial \eta} < 0$ *on* $\partial\Omega$ *where* η *is the outward unit normal to* $\partial\Omega$.

Proof. It has been proved in [17]. For the reader's convenience, we recall it. First, we consider an energy functional on $W_0^{1,2}(\Omega)$ corresponding to (8) given by

$$E(w) = \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{1}{1-\beta}\int_\Omega (w^+)^{1-\beta} dx + \frac{1}{2}\int_\Omega V(x) w^2 dx, \ w \in W_0^{1,2}(\Omega).$$

Since this functional satisfies

$$\begin{aligned} E(w) &= \frac{1}{2}\int_\Omega |\nabla w|^2 dx - \frac{1}{1-\beta}\int_\Omega (w^+)^{1-\beta} dx + \frac{1}{2}\int_\Omega V(x) w^2 dx, \\ &\geq \frac{1}{2}\|\nabla w\|_2^2 - \frac{1}{1-\beta}\|w\|_2^{1-\beta} - \frac{c_V}{2}\|w\|_2^2 \\ &= \frac{1}{2}(1 - c_V c_1)\|\nabla w\|_2^2 - \frac{1}{1-\beta}\|w\|_2^{1-\beta} \end{aligned}$$

and since $0 < 1-\beta < 1$, $E(w)$ is coercive and weakly lower semicontinuous on $W_0^{1,2}(\Omega)$. It follows that E possesses a global minimizer $w \in W_0^{1,2}(\Omega)$. Moreover, note that $w \not\equiv 0$ in Ω since $E(0) = 0 > E(\epsilon\phi_1)$ for small enough $\epsilon > 0$.

Next we consider the polar decomposition $w = w^+ - w^-$ for $w \in W_0^{1,2}(\Omega)$ which yields $\nabla w = \nabla w^+ - \nabla w^-$. Hence, if w is a global minimizer of E, then $|w|$ is also global minimizer of E, which implies that $E(|w|) \leq E(w)$. Since w is a global minimizer of E, $E(|w|) = E(w)$ holds if and only if $w \geq 0$ a.e. in Ω. Then, any global minimizer of E must satisfy $w \geq 0$ a.e. in Ω.

Further, we recall that by the standard elliptic regularity theory a global minimizer $w \in W_0^{1,2}(\Omega)$ of E belongs to $C^2(\Omega) \cap C(\bar{\Omega})$ (see [12]).

Now, we will show that

$$w(x) \geq e(x)\|e\|_\infty^{-\frac{\beta}{1+\beta}} \text{ in } \Omega,$$

where e is the solution of :

$$\begin{cases} -\Delta e + \|V\|_\infty e = 1, & x \in \Omega, \\ e = 0, & x \in \partial\Omega. \end{cases}$$

Suppose $\tilde{\Omega} := \{x \in \Omega : w(x) < e(x)\|e\|_\infty^{-\frac{\beta}{1+\beta}}\} \neq \emptyset$. Then we obtain that

$$\begin{aligned} -\Delta(w - e\|e\|_\infty^{-\frac{\beta}{1+\beta}}) &= -V(x)w + \|V\|_\infty e\|e\|_\infty^{-\frac{\beta}{1+\beta}} + \frac{1}{w^\beta} - \|e\|_\infty^{-\frac{\beta}{1+\beta}} \\ &= (\|V\|_\infty - V(x))w + \|V\|_\infty(e\|e\|_\infty^{-\frac{\beta}{1+\beta}} - w) + \frac{1}{w^\beta} - \|e\|_\infty^{-\frac{\beta}{1+\beta}} \\ &\geq \frac{1}{w^\beta} - \|e\|_\infty^{-\frac{\beta}{1+\beta}} \\ &> e(x)^{-\beta}\|e\|_\infty^{\frac{\beta^2}{1+\beta}} - \|e\|_\infty^{-\frac{\beta}{1+\beta}} \\ &\geq \|e\|_\infty^{-\beta}\|e\|_\infty^{\frac{\beta^2}{1+\beta}} - \|e\|_\infty^{-\frac{\beta}{1+\beta}} = 0 \end{aligned}$$

in $\tilde{\Omega}$ and $w - e\|e\|_\infty^{-\frac{\beta}{1+\beta}} = 0$ on $\partial\tilde{\Omega}$, which implies that $w - e\|e\|_\infty^{-\frac{\beta}{1+\beta}} \geq 0$ in $\tilde{\Omega}$ by the Maximum principle. This contradicts to the definition of $\tilde{\Omega}$. Hence, we have $w(x) \geq e(x)\|e\|_\infty^{-\frac{\beta}{1+\beta}}$ on Ω. Since $e > 0$ in Ω and $\frac{\partial e}{\partial \eta} < 0$ on $\partial\Omega$, it follows that $w(x) > 0$ for $x \in \Omega$ and $\frac{\partial w}{\partial \eta} < 0$ on $\partial\Omega$. $\square$

4. Proof of the Main Theorems

4.1. Proof of Theorem 1.1

Proof. First, we construct a positive supersolution Z_1 for all $\lambda > 0$. Define $f^*(s) := \max_{t\in[0,s]} f(t)$. Then, clearly $f(s) \leq f^*(s)$ and $f^*(s)$ is nondecreasing for all $s \in [0, \infty)$. Then there exists $M_\lambda \gg 1$ such that for each $\lambda > 0$

$$M_\lambda^{1-\beta} \geq \lambda f^*(M_\lambda \|w\|_\infty). \tag{9}$$

Let $Z_1 = M_\lambda w$. Then, by using (9) and the definition of f^*, we have

$$\begin{aligned} -\Delta Z_1 + V(x)Z_1 &= \frac{M_\lambda}{w^\beta} \\ &\geq \lambda \frac{f^*(M_\lambda\|w\|_\infty)}{M_\lambda^\beta w^\beta} \geq \lambda \frac{f^*(M_\lambda w)}{(M_\lambda w)^\beta} \geq \lambda \frac{f(M_\lambda w)}{(M_\lambda w)^\beta} = \lambda \frac{f(Z_1)}{Z_1^\beta} \end{aligned}$$

in Ω and $Z_1 = 0$ on $\partial\Omega$, which implies that Z_1 is a supersolution of (P_λ) for all $\lambda > 0$.

Next, we construct a positive subsolution ψ_1 satisfying $\psi_1 \leq Z_1$ for all λ. Since $\lim_{u\to 0^+} \frac{f(u)}{u^\beta} = \infty$, there exists a sufficiently small $m_\lambda > 0$ such that

$$\lambda_1 m_\lambda \phi_1 \leq \lambda \frac{f(m_\lambda \phi_1)}{(m_\lambda \phi_1)^\beta}.$$

Let $\psi_1 := m_\lambda \phi_1$. Then we obtain

$$-\Delta\psi_1 + V(x)\psi_1 = \lambda_1 m_\lambda \phi_1 \leq \lambda \frac{f(m_\lambda\phi_1)}{(m_\lambda\phi_1)^\beta} = \lambda \frac{f(\psi_1)}{\psi_1^\beta}$$

in Ω and $\psi_1 = 0$ on $\partial\Omega$. Hence, ψ_1 is a positive subsolution of (P_λ) for all $\lambda > 0$. Now, it is possible to choose $m_\lambda > 0$ small enough so that $\psi_1(x) \leq Z_1(x)$ since $\psi(x) > 0$ and $Z_1(x) > 0$ in Ω and $\frac{\partial Z_1}{\partial \eta} < 0$ on $\partial\Omega$. Therefore, by Lemma 3.1, there exists a solution u_λ such that $\psi_1 \leq u_\lambda \leq Z_1$ for all $\lambda > 0$. $\square$

4.2. Proof of Theorem 1.2

Proof. We first construct a positive supersolution of (P_λ) for $\lambda < \frac{a^{\beta+1}}{f^*(a)\|w\|_\infty^{\beta+1}}$. Let $Z_2 = a\frac{w}{\|w\|_\infty}$. Then it follows

$$\begin{aligned} -\Delta Z_2 + V(x)Z_2 &= \frac{a}{\|w\|_\infty w^\beta} \\ &> \lambda \frac{f^*(a)}{(a\frac{w}{\|w\|_\infty})^\beta} \geq \lambda \frac{f^*(a\frac{w}{\|w\|_\infty})}{(a\frac{w}{\|w\|_\infty})^\beta} \geq \frac{f(a\frac{w}{\|w\|_\infty})}{(a\frac{w}{\|w\|_\infty})^\beta} = \lambda \frac{f(Z_2)}{Z_2^\beta} \end{aligned}$$

in Ω when $\lambda < \frac{a^{\beta+1}}{f^*(a)\|w\|_\infty^{\beta+1}}$. Clearly, $Z_2 = 0$ on $\partial\Omega$. Hence, Z_2 is a positive supersolution of (P_λ) for $\lambda < \frac{a^{\beta+1}}{f^*(a)\|w\|_\infty^{\beta+1}}$.

Now we construct a positive subsolution ψ_2 of the following problem

$$\begin{cases} -\Delta u + \|V\|_\infty u = \lambda \frac{f(u)}{u^\beta}, \text{ in } \Omega. \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{10}$$

Then, ψ_2 is a positive subsolution of (P_λ) since

$$-\Delta\psi_2 + V(x)\psi_2 \le -\Delta\psi_2 + \|V\|_\infty\psi_2 \le \lambda\frac{f(\psi_2)}{\psi_2^\beta}.$$

We recall $\tilde{f}(u) = \frac{f(u)}{u^\beta} - \frac{f(d)}{d^\beta}B\|V\|_\infty u$. Note that $\tilde{f}(u)$ is nondecreasing on $[a,b]$. Let $a^* \in (0,a]$ be such that $\tilde{f}(a^*) = \min_{0<x\le a}\tilde{f}(x)$ and define a nonsingular function $h \in C([0,\infty))$ such that

$$h(u) = \begin{cases} \tilde{f}(a^*), & u \le a^*, \\ \tilde{f}(u) & u \ge a, \end{cases}$$

so that h is nondecreasing on $(0,a]$ and $h(u) \le \tilde{f}(u)$ for all $u > 0$.

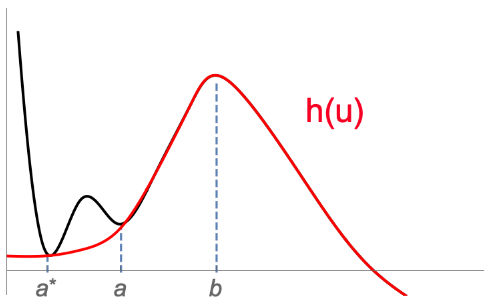

FIGURE 4. Graph of h

Now we consider the following nonsingular problem:

$$\begin{cases} -\Delta u = \lambda h(u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{11}$$

For $0 < \epsilon < R$ and $\delta, \mu > 1$ let us define $\rho : [0,R] \to [0,1]$ by

$$\rho(r) = \begin{cases} 1, \ 0 \le r \le \epsilon, \\ 1 - (1 - (\frac{R-r}{R-\epsilon})^\mu)^\delta, \ \epsilon < r \le R. \end{cases}$$

Then we find

$$\rho'(r) = \begin{cases} 0, \ 0 \le r \le \epsilon, \\ -\frac{\delta\mu}{R-\epsilon}(1 - (\frac{R-r}{R-\epsilon})^\mu)^{\delta-1}(\frac{R-r}{R-\epsilon})^{\mu-1}, \ \epsilon < r \le R. \end{cases}$$

Let $v(r) = d\rho(r)$. Notice that $|v'(r)| \le d\frac{\delta\mu}{R-\epsilon}$. Define ψ as the radially symmetric solution of

$$\begin{cases} -\Delta\psi = \lambda h(v(|x|)), \text{ in } B_R(0), \\ \psi = 0, \text{ on } \partial B_R(0). \end{cases}$$

Then ψ satisfies

$$\begin{cases} -(r^{N-1}(\psi'(r)))' = \lambda r^{N-1} h(v(r)), \\ \psi'(0) = 0, \ \psi(R) = 0. \end{cases} \tag{12}$$

Integrating the first equation of (12) for $0 < r < R$, we have

$$-\psi'(r) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} h(v(s))\, ds. \tag{13}$$

Here we claim that

$$\psi(r) \ge v(r), \ \forall\, 0 \le r \le R \tag{14}$$

and

$$\|\psi\|_\infty \le b \tag{15}$$

when $\dfrac{d^{\beta+1}}{f(d)}\dfrac{1}{B} < \lambda < \dfrac{2d^\beta}{f(d)AB} b$. In order to prove (14), it is enough to show that

$$-\psi'(r) \ge -v'(r) \ \forall\, 0 \le r \le R \tag{16}$$

as $\psi(R) = 0 = v(R)$. Notice that for $0 \le r \le \epsilon$, $\psi'(r) \le 0 = v'(r)$. Hence, for $r > \epsilon$ we obtain from (13)

$$\begin{aligned} -\psi'(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} h(v(s))\, ds \\ &> \frac{\lambda}{R^{N-1}} \int_0^\epsilon s^{N-1} h(v(s))\, ds \\ &= \frac{\lambda}{R^{N-1}} \frac{\epsilon^N}{N} h(d) = \frac{\lambda}{R^{N-1}} \frac{\epsilon^N}{N} \tilde{f}(d). \end{aligned}$$

If $\lambda > \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu$, then we can show (16) as $|v'(r)| \le d\frac{\delta\mu}{R-\epsilon}$. Note that

$$\inf_\epsilon \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu = \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}} \delta\mu$$

and is achieved at $\epsilon = \frac{NR}{N+1}$. Hence, if $\lambda > \frac{d}{\tilde{f}(d)} \frac{(N+1)^{N+1}}{R^2 N^{N-1}}$, then in the definition of ρ we can choose $\epsilon = \frac{NR}{N+1}$ and the values of δ and μ so that $\lambda \ge \frac{d}{\tilde{f}(d)} \frac{NR^{N-1}}{(R-\epsilon)\epsilon^N} \delta\mu$. Thus, we obtain (16) when $\lambda > \dfrac{d}{\tilde{f}(d)} \dfrac{(N+1)^{N+1}}{R^2 N^{N-1}} = \dfrac{d^{\beta+1}}{f(d)} \dfrac{1}{B}$, observing that

$$\tilde{f}(d) = (1 - B\|V\|_\infty) \frac{f(d)}{d^\beta}. \tag{17}$$

Next, integrating (13) from t to R, we obtain

$$\begin{aligned}\psi(t) &= \int_t^R \frac{\lambda}{r^{N-1}}\left(\int_0^r s^{N-1}h(v(s))\,ds\right)\,dr \\ &\leq \int_t^R \frac{\lambda}{r^{N-1}}h(d)\left(\int_0^r s^{N-1}\,ds\right)\,dr \\ &\leq \lambda\frac{h(d)}{N}\int_0^R r\,dr = \lambda\frac{\tilde{f}(d)}{2N}R^2.\end{aligned}$$

Hence, if $\lambda < \dfrac{b}{\tilde{f}(d)}\dfrac{2N}{R^2} = \dfrac{2d^\beta}{f(d)AB}b$ (using (17) again), then it follows $\|\psi\|_\infty \leq b$. Finally, we have

$$v(r) \leq \psi(r) \leq b, \forall\ 0 \leq r \leq R$$

when $\dfrac{d}{f(d)}\dfrac{1}{B} < \lambda < \dfrac{2d^\beta}{f(d)AB}b$. Now, from the fact $v(r) \leq \psi(r) \leq b, \forall\ 0 \leq r \leq R$ and h is nondecreasing on $[0,b]$, we can see

$$-\Delta\psi = \lambda h(v) \leq \lambda h(\psi), \quad \text{in } B_R(0) \text{ and } \psi = 0 \text{ on } \partial B_R(0).$$

Let us define $\xi(x) = \psi(x)$ if $x \in B_R(0)$ and $\xi(x) = 0$ if $x \in \Omega \setminus B_R(0)$. Then, ξ is a nonnegative subsolution of (11) for $\frac{d^{\beta+1}}{f(d)}\frac{1}{B} < \lambda < \frac{2b}{f(d)AB}$. However, ξ is not strictly positive in Ω. We iterate this subsolution ξ in a suitable manner, we obtain a positive subsolution ψ_2 of (11) such that $\psi_2 > 0$ in Ω (see details in [15]).

Finally, since $\lambda > \frac{d^{\beta+1}}{f(d)}\frac{1}{B}$, we obtain

$$\begin{aligned}-\Delta\psi_2 \leq \lambda h(\psi_2) \leq \lambda\tilde{f}(\psi_2) &= \lambda\left(\frac{f(\psi_2)}{\psi_2^\beta} - \frac{f(d)}{d^{\beta+1}}B\|V\|_\infty\psi_2\right) \\ &< \lambda\left(\frac{f(\psi_2)}{\psi_2^\beta} - \frac{1}{\lambda}\|V\|_\infty\psi_2\right) \\ &= \lambda\frac{f(\psi_2)}{\psi_2^\beta} - \|V\|_\infty\psi_2,\end{aligned}$$

which implies that ψ_2 is a positive subsolution of (10).

In the proof of Theorem 1.1 we have a sufficiently small subsolution $\psi_1 = m_\lambda\phi_1$ such that $\psi_1 \leq Z_2$ and a sufficiently large supersolution $Z_1 = M_\lambda w$ such that $\psi_2 \leq Z_1$. Hence, there exist a positive solutions u_1 and u_2 of (P_λ) such that $\psi_1 \leq u_1 \leq Z_2$ and $\psi_2 \leq u_2 \leq Z_1$. Note that $u_1 \neq u_2$ since $\psi_2 \not\leq Z_2$. Therefore by Lemma 3.2, there exists a positive solution u_3 such that $u_3 \in [\psi_1, Z_1] \setminus ([\psi_1, Z_2] \cup [\psi_2, Z_1])$. □

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