East Asian Math. J.
Vol. 39 (2023), No. 3, pp. 305–317
http://dx.doi.org/10.7858/eamj.2023.023

EXISTENCE OF GREEN FUNCTIONS FOR STOKES SYSTEMS WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. We establish the existence and uniqueness of Green functions in Lipschitz domains for stationary Stokes systems with Neumann boundary conditions. For the uniqueness, we impose a different normalization condition from that in Choi et al. (*J. Math. Fluid Mech.*, 20(4):1745–1769, 2018).

1. Introduction and main result

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, where $d \geq 3$, and ν the outward unit normal to $\partial\Omega$. We consider a Neumann problem for the stationary Stokes system with variable coefficients

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega, \\ \mathcal{L}u + \nabla p = g & \text{in } \Omega, \\ \mathcal{B}u + p\nu = h & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathcal{L}$ is a 2nd order elliptic operator in divergence form

$$\mathcal{L}u = D_\alpha(A^{\alpha\beta}D_\beta u)$$

acting on column vector-valued functions $u = (u_1, \ldots, u_d)^\top$ and $\mathcal{B}u$ is the conormal derivative of u. The Neumann problem (1) arises from the variational principle and occurs when modeling a channel flow in which the output velocity dependence is a prior unknown. See [2, 10] and references therein.

In this paper, we are concerned with the Green functions of the Neumann problem. By the Green function, we mean a pair $(G, \Pi) = (G(x, y), \Pi(x, y))$

Received January 18, 2023; Accepted March 8, 2023.
2010 *Mathematics Subject Classification.* 35J08, 35J57, 76N10, 76D07.
Key words and phrases. Green function, stationary Stokes system, Neumann problem.
This work was supported by a 2-Year Research Grant of Pusan National University.
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(pISSN 1226-6973, eISSN 2287-2833)

satisfying

$$\begin{cases} \operatorname{div} G(\cdot, y) = 0 & \text{in } \Omega, \\ \mathcal{L}G(\cdot, y) + \nabla\Pi(\cdot, y) = -\delta_y I & \text{in } \Omega, \\ \mathcal{B}G(\cdot, y) + \nu\Pi(\cdot, y) = -\dfrac{1}{|\partial\Omega|} I & \text{on } \partial\Omega, \end{cases}$$

where δ_y is the Dirac delta function concentrated at y and I is the $d \times d$ identity matrix. See Definition 1.1 for a precise definition of the Green function. We prove that the Green function exists and satisfies the pointwise bound

$$|G(x, y)| \leq C|x - y|^{2-d}$$

away from the boundary of the domain under an interior continuity assumption of weak solutions to the system. In particular, for the uniqueness of Green functions, we impose the normalization condition

$$\int_{\partial\Omega} G(x, y)\, dx = 0. \tag{2}$$

Notice that if Ω is irregular so that neither the outer normal nor the trace of a Sobolev function on the boundary are defined, one may use the normalization condition

$$\int_{\Omega} G(x, y)\, dx = 0 \tag{3}$$

instead of (2), which enables to construct the Green functions in domains that are more general than Lipschitz. We refer the reader to [4] for the Green function with the condition (3). It is worth noting that in establishing the representation formula of the solution u to (1), the condition (2) requires the data g and h to satisfy only a compatibility condition (see Remark 1.5), whereas the condition (3) requires those to be of the form

$$g = \tilde{g} + D_\alpha \tilde{h}_\alpha, \quad h = \tilde{h}_\alpha \nu_\alpha$$

so that no boundary term appears in the weak formulation (see [4, Remark 2.3]). If Ω is unbounded, such normalization conditions are not needed. We refer the reader to [11] for Green functions in three dimensional half-spaces for the classical Stokes system with the Laplace operator. See also [6] for Green functions in bounded Lipschitz domains for elliptic systems with the normalization condition (2) and [5] for those in both bounded and unbounded domains for elliptic and parabolic systems with the normalization condition (3) on bounded domains.

The remainder of the paper is organized as follows. In the rest of this section, we state our main result along with some definitions and assumptions. In Section 2, we provide the proof of the main theorem.

Throughout this paper, we denote by Ω a bounded Lipschitz domain in the Euclidean space $\mathbb{R}^d$, where $d \geq 3$. Given $x \in \Omega$ and $r > 0$, we write

$$\Omega_r(x) = \Omega \cap B_r(x),$$

where $B_r(x)$ is the usual Euclidean ball or radius r centered at x. We also denote $d_x = \operatorname{dist}(x, \partial\Omega)$. For $q \in [1, \infty]$, we define

$$\tilde{L}_q(\Omega) = \left\{ u \in L_q(\Omega) : \int_{\partial\Omega} u \, d\sigma = 0 \right\}$$

and

$$\tilde{W}^1_q(\Omega) = \left\{ u \in W^1_q(\Omega) : \int_{\partial\Omega} u \, d\sigma = 0 \right\},$$

where $W^1_q(\Omega)$ is the usual Sobolev space. For a function u on Ω, we use $(u)_\Omega$ to denote the average of u in Ω, that is

$$(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx = \fint_\Omega u \, dx,$$

where $|\Omega|$ is the d-dimensional Legesgue measure of Ω.

Let $\mathcal{L}$ be an elliptic operator in divergence form

$$\mathcal{L}u = D_\alpha(A^{\alpha\beta} D_\beta u),$$

where the coefficients $A^{\alpha\beta} = A^{\alpha\beta}(x)$ are $d \times d$ matrix-valued functions on $\mathbb{R}^d$ satisfying the strong ellipticity condition, that is, there is a constant $\lambda \in (0, 1]$ such that for any $x \in \mathbb{R}^d$ and $\xi^\alpha \in \mathbb{R}^d$, $\alpha \in \{1, \ldots, d\}$, we have

$$|A^{\alpha\beta}(x)| \le \lambda^{-1}, \quad \sum_{\alpha,\beta=1}^{d} A^{\alpha\beta}(x) \xi^\beta \cdot \xi^\alpha \ge \lambda \sum_{\alpha=1}^{d} |\xi^\alpha|^2.$$

We denote by $\mathcal{B}u = A^{\alpha\beta} D_\beta u \nu_\alpha$ the conormal derivative of u on $\partial\Omega$ associated with $\mathcal{L}$. The adjoint operator $\mathcal{L}^*$ and the conormal derivative operator $\mathcal{B}^*$ associated with $\mathcal{L}^*$ are defined by

$$\mathcal{L}^* u = D_\alpha((A^{\beta\alpha})^\top D_\beta u), \quad \mathcal{B}^* u = (A^{\beta\alpha})^\top D_\beta u \nu_\alpha.$$

Let $g \in L_{2d/(d+2)}(\Omega)^d$ and $h \in L_2(\partial\Omega)^d$ satisfy the *compatibility condition*

$$\int_\Omega g \, dx = \int_{\partial\Omega} h \, d\sigma.$$

We say that $(u, p) \in W^1_2(\Omega)^d \times L_2(\Omega)$ is a weak solution of

$$\begin{cases} \mathcal{L}u + \nabla p = g & \text{in } \Omega, \\ \mathcal{B}u + p\nu = h & \text{on } \partial\Omega \end{cases}$$

if

$$\int_\Omega A^{\alpha\beta} D_\beta u \cdot D_\alpha \phi \, dx + \int_\Omega p \operatorname{div} \phi \, dx = -\int_\Omega g \cdot \phi \, dx + \int_{\partial\Omega} h \cdot \phi \, d\sigma$$

holds for any $\phi \in W^1_2(\Omega)^d$. A weak solution of the adjoint problem

$$\begin{cases} \mathcal{L}^* u + \nabla p = g & \text{in } \Omega, \\ \mathcal{B}^* u + p\nu = h & \text{on } \partial\Omega \end{cases}$$

is defined similarly. For $u \in W_1^1(\Omega)^d$ and $f \in L_1(\Omega)$, by $\operatorname{div} u = f$ in Ω, we mean the equation holds in the almost everywhere sense.

In the definition below, $G = G(x, y)$ is a $d \times d$ matrix-valued function with the entries $G_{ij} : \Omega \times \Omega \to [-\infty, \infty]$, and $\Pi = \Pi(x, y)$ is a $1 \times d$ vector-valued function with the entries $\Pi_i : \Omega \times \Omega \to [-\infty, \infty]$.

Definition 1.1 (Green function)**.** We say that (G, Π) is a Green function for $\mathcal{L}$ in a bounded Lipschitz domain Ω if it satisfies the following properties:
(a) For any $y \in \Omega$ and $r > 0$,

$$G(\cdot, y) \in \tilde{W}_1^1(\Omega)^{d\times d} \cap W_2^1(\Omega \setminus B_r(y))^{d\times d},$$
$$\Pi(\cdot, y) \in L_1(\Omega)^d \cap L_2(\Omega \setminus B_r(y))^d.$$

(b) For any $y \in \Omega$, $(G(\cdot, y), \Pi(\cdot, y))$ satisfies

$$\begin{cases} \operatorname{div} G(\cdot, y) = 0 & \text{in } \Omega, \\ \mathcal{L}G(\cdot, y) + \nabla\Pi(\cdot, y) = -\delta_y I & \text{in } \Omega, \\ \mathcal{B}G(\cdot, y) + \nu\Pi(\cdot, y) = -\dfrac{1}{|\partial\Omega|} I & \text{on } \partial\Omega, \end{cases}$$

in the sense that, for $k \in \{1, \ldots, d\}$ and $\phi \in W_\infty^1(\Omega)^d \cap C(\Omega)^d$, we have

$$\operatorname{div} G_{\cdot k}(\cdot, y) = 0 \quad \text{in } \Omega$$

and

$$\int_\Omega A^{\alpha\beta} D_\beta G_{\cdot k}(\cdot, y) \cdot D_\alpha \phi \, dx + \int_\Omega \Pi_k(\cdot, y) \operatorname{div} \phi \, dx = \phi_k(y) - \fint_{\partial\Omega} \phi_k \, dx,$$

where $G_{\cdot k}(\cdot, y)$ is the kth column of $G(\cdot, y)$.
(c) If $(u, p) \in \tilde{W}_2^1(\Omega)^d \times L_2(\Omega)$ is a weak solution of the adjoint problem

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega, \\ \mathcal{L}^* u + \nabla p = g & \text{in } \Omega, \\ \mathcal{B}^* u + p\nu = \dfrac{1}{|\partial\Omega|} \displaystyle\int_\Omega g \, dx & \text{on } \partial\Omega, \end{cases}$$

where $f \in L_\infty(\Omega)$ and $g \in L_\infty(\Omega)^d$, then for a.e. $y \in \Omega$, we have

$$u(y) = -\int_\Omega G(x, y)^\top g(x) \, dx + \int_\Omega \Pi(x, y)^\top f(x) \, dx.$$

where $G(x, y)^\top$ and $\Pi(x, y)^\top$ are the transposes of $G(x, y)$ and $\Pi(x, y)$.

Remark 1.2. By the solvability result in Lemma 2.2, we get the uniqueness of a Green function in the sense that if $(\tilde{G}, \tilde{\Pi})$ is another Green function satisfying the properties in Definition 1.1, then for any $\phi \in C_0^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega)^d$, we have

$$\int_\Omega \big(G(x, y)^\top - \tilde{G}(x, y)^\top\big)\varphi(x) \, dx = \int_\Omega \big(\Pi(x, y)^\top - \tilde{\Pi}(x, y)^\top\big)\phi(x) \, dx = 0$$

for a.e. $y \in \Omega$.

To construct the Green function, we impose the following assumption. Note that the assumption holds when the coefficients $A^{\alpha\beta}$ are variably partially BMO; see [4, Theorem 6.2].

Assumption 1.3. There exist constants $R_0 \in (0,1]$ and $A_0 > 0$ such that the following holds: Let $x_0 \in \Omega$ and $0 < R < \min\{R_0, d_{x_0}\}$. If $(u,p) \in W^1_2(B_R(x_0))^d \times L_2(B_R(x_0))$ satisfies

$$\begin{cases} \operatorname{div} u = 0 & \text{in } B_R(x_0), \\ \mathcal{L}u + \nabla p = g & \text{in } B_R(x_0), \end{cases}$$

where $g \in L_\infty(B_R(x_0))^d$, then we have $u \in C(\overline{B_{R/2}(x_0)})^d$ (in fact, a version of u belongs to $C(\overline{B_{R/2}(x_0)})^d$) with the estimate

$$\|u\|_{L_\infty(B_{R/2}(x_0))} \le A_0\big(R^{-d/2}\|u\|_{L_2(B_R(x_0))} + R^2\|g\|_{L_\infty(B_R(x_0))}\big).$$

The same statement holds true when $\mathcal{L}$ is replaced by $\mathcal{L}^*$.

Theorem 1.4. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, where $d \ge 3$. Then under Assumption 1.3, there exist Green functions (G, Π) and (G^*, Π^*) for $\mathcal{L}$ and $\mathcal{L}^*$, respectively, such that for given $y \in \Omega$, we have*

$$G(\cdot, y),\ G^*(\cdot, y) \in C(\Omega \setminus \{y\})^{d\times d},$$

and there exists a measure zero set $N_y \subset \Omega$ such that

$$G(x,y) = G^*(y,x)^\top, \quad G(y,x) = G^*(x,y)^\top \quad \text{for all } x \in \Omega \setminus N_y. \tag{4}$$

Moreover, for any $x, y \in \Omega$ satisfying

$$0 < |x-y| < \frac{1}{2}\min\{R_0, d_y\},$$

we have

$$|G(x,y)| \le C|x-y|^{2-d}. \tag{5}$$

Furthermore, the following estimates hold for all $y \in \Omega$ and $0 < R < \min\{R_0, d_y\}$:
(i) $\|G(\cdot,y)\|_{L_{2d/(d-2)}(\Omega\setminus B_R(y))} + \|DG(\cdot,y)\|_{L_2(\Omega\setminus B_R(y))} \le CR^{(2-d)/2}$.
(ii) $\|\Pi(\cdot,y)\|_{L_2(\Omega\setminus B_R(y))} \le CR^{(2-d)/2}$.
(iii) $\big|\{x \in \Omega : |G(x,y)| > t\}\big| \le Ct^{-d/(d-2)}$ *for all* $t > \min\{R_0, d_y\}^{2-d}$.
(iv) $\big|\{x \in \Omega : |D_xG(x,y)| > t\}\big| \le Ct^{-d/(d-1)}$ *for all* $t > \min\{R_0, d_y\}^{1-d}$.
(v) $\big|\{x \in \Omega : |\Pi(x,y)| > t\}\big| \le Ct^{-d/(d-1)}$ *for all* $t > \min\{R_0, d_y\}^{1-d}$.
(vi) $\|G(\cdot,y)\|_{L_q(B_R(y))} \le C_qR^{2-d+d/q}$, *where* $q \in [1, d/(d-2))$.
(vii) $\|DG(\cdot,y)\|_{L_q(B_R(y))} \le C_qR^{1-d+d/q}$, *where* $q \in [1, d/(d-1))$.
(viii) $\|\Pi(\cdot,y)\|_{L_q(B_R(y))} \le C_qR^{1-d+d/q}$, *where* $q \in [1, d/(d-1))$.
In the above, the constant C depends only on d, λ, K, and A_0, and C_q depends also on q, where the constant K is such that

$$\|\phi\|_{L_{2d/(d-2)}(\Omega)} + \|\phi\|_{L_2(\partial\Omega)} \le K\|D\phi\|_{L_2(\Omega)} \quad \text{for all } \phi \in \tilde{W}^1_2(\Omega)^d. \tag{6}$$

We finish this section with the following remark on the representation formula.

Remark 1.5. Let (G,Π) and (G^*,Π^*) be the Green functions for $\mathcal{L}$ and $\mathcal{L}^*$, respectively, derived in Theorem 1.4. If $(u,p) \in \tilde{W}^1_2(\Omega)^d \times L_2(\Omega)$ is a weak solution of

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega, \\ \mathcal{L}u + \nabla p = g & \text{in } \Omega, \\ \mathcal{B}u + p\nu = h & \text{on } \partial\Omega, \end{cases}$$

where $f \in L_\infty(\Omega)$, $g \in L_\infty(\Omega)^d$, and $h \in L_\infty(\partial\Omega)^d$ satisfying

$$\int_\Omega g\,dx = \int_{\partial\Omega} h\,d\sigma,$$

then for a.e. $y \in \Omega$, we have

$$u(y) = -\int_\Omega G^*(x,y)^\top g(x)\,dx + \int_{\partial\Omega} G^*(x,y)^\top h(x)\,d\sigma_x + \int_\Omega \Pi^*(x,y)^\top f(x)\,dx.$$

Hence by (4) we see that

$$u(y) = -\int_\Omega G(y,x)g(x)\,dx + \int_{\partial\Omega} G(y,x)h(x)\,d\sigma_x + \int_\Omega \Pi^*(x,y)^\top f(x)\,dx.$$

2. Proof of the main theorem

Throughout this paper, we mean by K the constant from (6). We also use the following notation.

Notation 2.1. For nonnegative (variable) quantities A and B, we denote $A \lesssim B$ if there exists a generic positive constant C such that $A \le CB$. We add subscript letters like $A \lesssim_{a,b} B$ to indicate the dependence of the implicit constant C on the parameters a and b.

Notation 2.2. For a given function f, if there is a continuous version of f, that is, there is a continuous function $\tilde{f}$ such that $\tilde{f} = f$ in the almost everywhere sense, then we replace f with $\tilde{f}$ and denote the version again by f.

2.1. W^1_2-solvability

In this subsection, we prove the solvability of the Stokes system in $\tilde{W}^1_2(\Omega)^d \times L_2(\Omega)$. To this end, we shall use the following lemma, in which we show that the divergence equation is solvable in $\tilde{W}^1_2(\Omega)^d$.

Lemma 2.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$ and $f \in L_2(\Omega)$. Then there exists $u \in \tilde{W}^1_2(\Omega)^d$ such that*

$$\operatorname{div} u = f \ \text{ in } \Omega, \quad \|Du\|_{L_2(\Omega)} \lesssim_d \|f\|_{L_2(\Omega)}.$$

Proof. By [3, Lemma 3.1], there exists $v \in W^1_2(\Omega)^d$ such that

$$\int_\Omega v\,dx = 0, \quad \operatorname{div} v = f \ \text{ in } \Omega,$$

and

$$\|Dv\|_{L_2(\Omega)} \lesssim_d \|f\|_{L_2(\Omega)}.$$

Then the lemma follows by setting $u = v - \fint_{\partial\Omega} v \, d\sigma$. □

In the lemma below, we do not impose any regularity assumptions on the coefficients $A^{\alpha\beta}$ of the operator $\mathcal{L}$.

Lemma 2.2. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$, where $d \geq 3$. Then for any $f \in L_2(\Omega)$, $g \in L_{2d/(d+2)}(\Omega)^d$, and $h \in L_2(\partial\Omega)^d$ satisfying*

$$\int_\Omega g \, dx = \int_{\partial\Omega} h \, d\sigma,$$

there exists a unique $(u, p) \in \tilde{W}_2^1(\Omega)^d \times L_2(\Omega)$ satisfying

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega, \\ \mathcal{L}u + \nabla p = g & \text{in } \Omega, \\ \mathcal{B}u + p\nu = h & \text{on } \partial\Omega. \end{cases}$$

Moreover, we have

$$\|Du\|_{L_2(\Omega)} + \|p\|_{L_2(\Omega)} \lesssim_{d,\lambda,K} \|f\|_{L_2(\Omega)} + \|g\|_{L_{2d/(d+2)}(\Omega)} + \|h\|_{L_2(\partial\Omega)}. \tag{7}$$

Proof. Due to Lemma 2.1, $\tilde{W}_2^1(\Omega)^d$ can be understood as a Hilbert space with inner product

$$\langle v, w \rangle = \int_\Omega Dv \cdot Dw \, dx.$$

Using this fact and following the proof of [7, Lemma 3.2], we see that the lemma holds. □

2.2. Approximated Green functions

In this subsection, we assume that the hypotheses in Theorem 1.4 hold. Under the hypotheses, we shall construct an approximated Green function and derive its various estimates. For this, we adapt the arguments in [9, 4].

Let $y \in \Omega$, $\varepsilon \in (0, 1]$, and $k \in \{1, \ldots, d\}$. Set

$$\Phi^{\varepsilon,y} = -\frac{1}{|\Omega_\varepsilon(y)|} \chi_{\Omega_\varepsilon(y)},$$

where $\chi_{\Omega_\varepsilon(y)}$ is the characteristic function. By Lemma 2.2, there exists a unique $(v, \pi) = (v^{\varepsilon,y,k}, \pi^{\varepsilon,y,k}) \in \tilde{W}_2^1(\Omega)^d \times L_2(\Omega)$ satisfying

$$\begin{cases} \operatorname{div} v = 0 & \text{in } \Omega, \\ \mathcal{L}v + \nabla\pi = \Phi^{\varepsilon,y} e_k & \text{in } \Omega, \\ \mathcal{B}v + p\nu = -|\partial\Omega|^{-1} e_k & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where e_k is the kth unit vector in $\mathbb{R}^d$. We define a pair

$$(G^\varepsilon(\cdot, y), \Pi^\varepsilon(\cdot, y)) \in \tilde{W}_2^1(\Omega)^{d\times d} \times L_2(\Omega)^d$$

by

$$G^\varepsilon_{jk}(\cdot, y) = v^j = v_j^{\varepsilon,y,k} \quad \text{and} \quad \Pi^\varepsilon_k(\cdot, y) = \pi = \pi^{\varepsilon,y,k},$$

and call it the *approximated Green function for $\mathcal{L}$ in Ω with a pole at y*. Notice from Assumption 1.3 (combined with Notation 2.2) that $G^{\varepsilon}(\cdot,y)$ is continuous in Ω.

The following lemma is about a L_2-estimate for the approximated Green functions. Note that the lemma is independent of $|\partial\Omega|$. Thus it does not follow directly from the L_2-estimate (7) applied to (8).

Lemma 2.3. *Let $y \in \Omega$ and $\varepsilon \in (0,1]$. Then we have*

$$\|DG^{\varepsilon}(\cdot,y)\|_{L_2(\Omega)} + \|\Pi^{\varepsilon}(\cdot,y)\|_{L_2(\Omega)} \lesssim_{d,\lambda,K} |\Omega_{\varepsilon}(y)|^{(2-d)/(2d)}.$$

Proof. For $k \in \{1,\dots,d\}$, we denote

$$(v,\pi) = (G^{\varepsilon}_{\cdot k}(\cdot,y), \Pi^{\varepsilon}_k(\cdot,y)), \tag{9}$$

where $G^{\varepsilon}_{\cdot k}(\cdot,y)$ is the kth column of $G^{\varepsilon}(\cdot,y)$. Notice that

$$\int_{\Omega} A^{\alpha\beta} D_{\beta} v \cdot D_{\alpha}\phi\,dx + \int_{\Omega} \pi \operatorname{div}\phi\,dx = \fint_{\Omega_{\varepsilon}(y)} \phi_k\,dx - \fint_{\partial\Omega} \phi_k\,d\sigma$$

for all $\phi \in W^1_2(\Omega)^d$. In particular, by the definition of $\tilde{W}^1_2(\Omega)$, it holds that

$$\int_{\Omega} A^{\alpha\beta} D_{\beta} v \cdot D_{\alpha}\phi\,dx + \int_{\Omega} \pi \operatorname{div}\phi\,dx = \fint_{\Omega_{\varepsilon}(y)} \phi_k\,dx \tag{10}$$

for all $\phi \in \tilde{W}^1_2(\Omega)^d$. By taking $\phi = v$ in (10), and then using Hölder's inequality and (6), we have

$$\begin{aligned}\|Dv\|^2_{L_2(\Omega)} &\lesssim |\Omega_{\varepsilon}(y)|^{(2-d)/(2d)} \|v\|_{L_{2d/(d-2)}(\Omega)} \\ &\lesssim_{d,\lambda,K} |\Omega_{\varepsilon}(y)|^{(2-d)/(2d)} \|Dv\|_{L_2(\Omega)},\end{aligned}$$

and thus, we get

$$\|Dv\|_{L_2(\Omega)} \lesssim |\Omega_{\varepsilon}(y)|^{(2-d)/(2d)}. \tag{11}$$

Similarly, by taking $\phi = u$ in (10), where $u \in \tilde{W}^1_2(\Omega)^d$ is such that (see Lemma 2.1)

$$\operatorname{div} u = \pi \ \text{ in } \ \Omega, \quad \|Du\|_{L_2(\Omega)} \lesssim_d \|\pi\|_{L_2(\Omega)},$$

we have

$$\|\pi\|^2_{L_2(\Omega)} \lesssim \big(\|Dv\|_{L_2(\Omega)} + |\Omega_{\varepsilon}(y)|^{(2-d)/(2d)}\big)\|\pi\|_{L_2(\Omega)},$$

from which together with (11), we get

$$\|\pi\|_{L_2(\Omega)} \lesssim |\Omega_{\varepsilon}(y)|^{(2-d)/(2d)}.$$

The lemma is proved. □

In the next lemma, we obtain a pointwise bound for $G^{\varepsilon}(\cdot,y)$.

Lemma 2.4. *Let $x,y \in \Omega$ and $\varepsilon \in (0,1]$ satisfying*

$$0 < 2\varepsilon < \frac{|x-y|}{2} < \min\{R_0, d_y/3\}.$$

Then we have

$$|G^{\varepsilon}(x,y)| \lesssim_{d,\lambda,K,A_0} |x-y|^{2-d}.$$

Proof. Recall the notation (9). We first claim that

$$\|v\|_{L_1(\Omega_R(x))} \lesssim_{d,\lambda,K,A_0} R^2 \tag{12}$$

for any R with $2\varepsilon < R < \min\{R_0, d_y\}$. Set $g = \chi_{\Omega_R(x)}(\operatorname{sgn} v^1, \dots, \operatorname{sgn} v^d)^\top$. By Lemma 2.2, there exists a unique $(u,p) \in \tilde{W}^1_2(\Omega)^d \times L_2(\Omega)$ satisfying that

$$\begin{cases} \operatorname{div} u = 0 & \text{in } \Omega, \\ \mathcal{L}^* u + \nabla p = -g + (g)_\Omega & \text{in } \Omega, \\ \mathcal{B} v + p\nu = 0 & \text{on } \partial\Omega, \end{cases}$$

and that

$$\|Du\|_{L_2(\Omega)} \le C\|g - (g)_\Omega\|_{L_{2d/(d+2)}(\Omega)} \le C R^{1+d/2},$$

where $C = C(d,\lambda,K)$, but independent of $|\Omega|$. Hence, from Assumption 1.3, Hölder's inequality, and (6), it follows that

$$\begin{aligned} \|u\|_{L_\infty(B_{R/2}(y))} &\lesssim R^{1-d/2}\|u\|_{L_{2d/(d-2)}(B_R(y))} + R^2 \\ &\lesssim R^{1-d/2}\|Du\|_{L_2(\Omega)} + R^2 \\ &\lesssim R^2. \end{aligned} \tag{13}$$

Since we have

$$\int_\Omega g \cdot v \, dz = \int_\Omega A^{\alpha\beta} D_\beta v \cdot D_\alpha u \, dx = \fint_{B_\varepsilon(y)} u_k \, dz,$$

by (13) and the fact that $\varepsilon < R/2$, we obtain

$$\|v\|_{L_1(\Omega_R(x))} \lesssim \|u\|_{L_\infty(B_{R/2}(y))} \lesssim R^2.$$

Thus we get the claim (12).

Now, let

$$0 < 2\varepsilon < R := \frac{|x-y|}{2} < \min\{R_0, d_y/3\}.$$

Note that $B_R(x) \subset \Omega$ and $B_R(x) \cap B_\varepsilon(y) = \emptyset$, which along with (8) shows

$$\begin{cases} \operatorname{div} v = 0 & \text{in } B_R(x), \\ \mathcal{L} v + \nabla \pi = 0 & \text{in } B_R(x). \end{cases}$$

Then by Assumption 1.3 and a well known argument (see [8, pp. 80–82]), we have

$$\|v\|_{L_\infty(B_{R/2}(x))} \lesssim R^{-d}\|v\|_{L_1(B_R(x))}.$$

This together with (12) and the continuity of v yields the desired inequality. □

Lemma 2.5. *Let $y \in \Omega$, $0 < R < \min\{R_0, 4d_y/5\}$, and $0 < \varepsilon < R/4$. Set*

$$\tilde{\Pi}^\varepsilon(\cdot, y) = \Pi^\varepsilon(\cdot, y) - \big(\Pi^\varepsilon(\cdot, y)\big)_{B_R(y)\setminus B_{R/2}(y))}.$$

Then for $k \in \{1, \dots, d\}$, we have

$$\|\tilde{\Pi}^\varepsilon_k(\cdot, y)\|_{L_2(B_R(y)\setminus B_{R/2}(y))} \lesssim_{d,\lambda} R^{-1}\|G^\varepsilon_k(\cdot, y)\|_{L_2(B_{5R/4}(y)\setminus B_{R/4}(y))}.$$

Proof. Recall the notation (9), and set

$$\tilde{\pi} = \pi - (\pi)_{B_R(y)\setminus B_{R/2}(y)}.$$

Since $(\tilde{\pi})_{B_R(y)\setminus B_{R/2}(y)} = 0$, by the existence of solutions to the divergence equation (see, for instance, [1]), there exists a function $\phi \in \mathring{W}^1_2(B_R(y)\setminus \overline{B_{R/2}(y)})^d$ such that

$$\operatorname{div}\phi = \tilde{\pi} \quad \text{in } \ B_R(y)\setminus \overline{B_{R/2}(y)}$$

and

$$\|D\phi\|_{L_2(B_R(y)\setminus B_{R/2}(y))} \le C\|\tilde{\pi}\|_{L_2(B_R(y)\setminus B_{R/2}(y))}, \tag{14}$$

where by a scaling argument, it is easily seen that the constant C in the above inequality depends only on d. We extend ϕ by zero in $\mathbb{R}^2 \setminus \big(B_R(y)\setminus \overline{B_{R/2}(y)}\big)$. By testing (8) with ϕ and using $\phi \equiv 0$ on $\partial\Omega$, we obtain that

$$\int_{B_R(y)\setminus B_{R/2}(y)} |\tilde{\pi}|^2\,dx = \int_{B_R(y)\setminus B_{R/2}(y)} \pi\tilde{\pi}\,dx = -\int_\Omega A^{\alpha\beta}D_\beta v\cdot D_\alpha\phi\,dx,$$

from which together with (14) we get

$$\|\tilde{\pi}\|_{L_2(B_R(y)\setminus B_{R/2}(y))} \lesssim_{d,\lambda} \|Dv\|_{L_2(B_R(y)\setminus B_{R/2}(y))}. \tag{15}$$

Let $z \in B_R(y)\setminus B_{R/2}(y)$, and observe that (using $\varepsilon < R/4$)

$$\begin{cases} \operatorname{div} v = 0 & \text{in } \ B_{R/4}(z),\\ \mathcal{L}v + \nabla\pi = 0 & \text{in } \ B_{R/4}(z). \end{cases}$$

By the well known Caccioppoli-type inequality (see, for instance, [4, Lemma 3.3]) with the fact that $B_{R/4}(z) \subset \big(B_{5R/4}(y)\setminus B_{R/4}(y)\big)$, we have

$$\|Dv\|_{L_2(B_{R/8}(z))} \lesssim_{d,\lambda} R^{-1}\|v\|_{L_2(B_{5R/4}(y)\setminus B_{R/4}(y))}.$$

Since the above inequality holds for any $z \in B_R(y)\setminus B_{R/2}(y)$, we obtain

$$\|Dv\|_{L_2(B_R(y)\setminus B_{R/2}(y))} \lesssim R^{-1}\|v\|_{L_2(B_{5R/4}(y)\setminus B_{R/4}(y))},$$

and thus, from (15) we get the desired inequality. □

Based on Lemmas 2.3 – 2.5, we establish the following estimates uniformly in $\varepsilon \in (0,1)$.

Lemma 2.6. *Let $y \in \Omega$, $0 < R < \min\{R_0, d_y\}$, and $0 < \varepsilon \le 1$. Then we have*

$$\|G^\varepsilon(\cdot,y)\|_{L_{2d/(d-2)}(\Omega\setminus B_R(y))} + \|DG^\varepsilon(\cdot,y)\|_{L_2(\Omega\setminus B_R(y))} \lesssim_{d,\lambda,K,A_0} R^{(2-d)/2} \tag{16}$$

and

$$\|\Pi^\varepsilon(\cdot,y)\|_{L_2(\Omega\setminus B_R(y))} \lesssim_{d,\lambda,K,A_0} R^{(2-d)/2}. \tag{17}$$

Proof. Certainly, we may assume that $0 < R < d_y/3$. If $R/16 \le \varepsilon \le 1$, then the lemma follows immediately from Lemma 2.3 combined with (6). Thus we only need to consider the case of $0 < \varepsilon < R/16$.

We first prove the estimate (16). Recall the notation (9), and let η be an infinitely differentiable function in $\mathbb{R}^d$ satisfying

$$0 \le \eta \le 1, \quad \eta \equiv 1 \text{ in } B_{R/2}(y), \quad \operatorname{supp} \eta \subset B_R(y), \quad |\nabla \eta| \lesssim_d R^{-1}. \tag{18}$$

By applying $(1-\eta)^2 v$ as a test function to (8), we have

$$\|(1-\eta)Dv\|^2_{L_2(\Omega)} \lesssim R^{-2}\|v\|^2_{L_2(B_R(y)\setminus B_{R/2}(y))} + I + J,$$

where

$$I = \left| \int_\Omega \pi \operatorname{div}\big((1-\eta)^2 v\big)\, dx \right|, \quad J = \left| \fint_{\partial\Omega} (1-\eta)^2 v \, d\sigma \right|.$$

Notice that (using $\operatorname{div} v = 0$)

$$I = \left| \int_\Omega \pi \operatorname{div}\big((\eta^2 - 2\eta)v\big)\, dx \right| = \left| \int_{B_R(y)\setminus B_{R/2}(y)} \tilde{\pi} \nabla(\eta^2 - 2\eta) \cdot v \, dx \right|,$$

where $\tilde{\pi} = \pi - (\pi)_{B_R(y)\setminus B_{R/2}(y)}$. Then by Hölder's inequality and Lemma 2.5, we have

$$I \lesssim R^{-2}\|v\|^2_{L_2(B_{5R/4}(y)\setminus B_{R/4}(y))}.$$

Since $(1-\eta)^2 \equiv 1$ on $\partial\Omega$, we also obtain that

$$J = \left| \fint_{\partial\Omega} v \, d\sigma \right| = 0.$$

Combining these together, and then using Lemma 2.4 with the fact that

$$0 < 2\varepsilon < \frac{R}{8} \le \frac{|x-y|}{2} < \frac{5R}{8} < \min\{R_0, d_y/3\}$$

for all $x \in B_{5R/4}(y) \setminus B_{R/4}(y)$, we have

$$\|(1-\eta)Dv\|_{L_2(\Omega)} \lesssim R^{-1}\|v\|_{L_2(B_{5R/4}(y)\setminus B_{R/4}(y))} \lesssim R^{(2-d)/2}. \tag{19}$$

Hence by (6),

$$\begin{aligned}
\|(1-\eta)v\|_{L_{2d/(d-2)}(\Omega)} &\lesssim_K \|D((1-\eta)v)\|_{L_2(\Omega)} \\
&\lesssim \|(1-\eta)Dv\|_{L_2(\Omega)} + R^{-1}\|v\|_{L_2(B_R(y)\setminus B_{R/2}(y))} \\
&\lesssim R^{(2-d)/2}.
\end{aligned}$$

From (19) and the inequality above, we get (16).

We next prove that the estimate (17) holds when $0 < R < d_y/3$ and $0 < \varepsilon < R/16$. By Lemma 2.1 combined with (6), there exists $\phi \in \tilde{W}^1_2(\Omega)^d$ such that

$$\operatorname{div} \phi = \pi \chi_{\Omega \setminus B_R(y)} \text{ in } \Omega$$

and

$$\|\phi\|_{L_{2d/(d-2)}(\Omega)} + \|D\phi\|_{L_2(\Omega)} \lesssim_{d,K} \|\pi\|_{L_2(\Omega\setminus B_R(y))}. \tag{20}$$

We apply $(1-\eta)\phi$ as a test function to (8), where η is as in (18), to get

$$\int_\Omega \pi \operatorname{div}((1-\eta)\phi)\, dx = -\int_\Omega A^{\alpha\beta} D_\beta v \cdot D_\alpha((1-\eta)\phi)\, dx - \fint_{\partial\Omega} (1-\eta)\phi_k \, d\sigma.$$

Since it holds that

$$\int_\Omega \pi \operatorname{div}((1-\eta)\phi)\,dx = \int_\Omega \pi \operatorname{div}\phi\,dx - \int_\Omega \pi \operatorname{div}(\eta\phi)\,dx$$
$$= \int_{\Omega\setminus B_R(y)} |\pi|^2\,dx - \int_{B_R(y)\setminus B_{R/2}(y)} \tilde{\pi}\nabla\eta\cdot\phi\,dx,$$

where $\tilde{\pi} = \pi - (\pi)_{B_R(y)\setminus B_{R/2}(y))}$, and that

$$⨍_{\partial\Omega}(1-\eta)\phi_k\,d\sigma = ⨍_{\partial\Omega}\phi_k\,d\sigma = 0,$$

we have

$$\int_{\Omega\setminus B_R(y)} |\pi|^2\,dx = -\int_\Omega A^{\alpha\beta}D_\beta v\cdot D_\alpha((1-\eta)\phi)\,dx$$
$$+ \int_{B_R(y)\setminus B_{R/2}(y)} \tilde{\pi}\nabla\eta\cdot\phi\,dx.$$

By Hölder's inequality,

$$\|\pi\|^2_{L_2(\Omega\setminus B_R(y))} \lesssim \|Dv\|_{L_2(\Omega\setminus B_{R/2}(y))}\big(\|\phi\|_{L_{2d/(d-2)}(\Omega)} + \|D\phi\|_{L_2(\Omega)}\big)$$
$$+ \|\tilde{\pi}\|_{L_2(B_R(y)\setminus B_{R/2}(y))}\|\phi\|_{L_{2d/(d-2)}(\Omega)},$$

and thus we get from (20) that

$$\|\pi\|_{L_2(\Omega\setminus B_R(y))} \lesssim \|Dv\|_{L_2(\Omega\setminus B_{R/2}(y))} + \|\tilde{\pi}\|_{L_2(B_R(y)\setminus B_{R/2}(y))}.$$

This together with Lemma 2.5 and (16) yields (17). The lemma is proved. □

We finish this subsection with the following lemma on uniform weak type and L_q-estimates, the proof of which proceeds in a standard manner. See the proofs of [4, Lemmas 4.4 and 4.5] for the details.

Lemma 2.7. *Let $y \in \Omega$ and $\varepsilon \in (0,1]$. Then we have*

$$\big|\{x\in\Omega : |G^\varepsilon(x,y)| > t\}\big| \le Ct^{-d/(d-2)}, \quad \forall t > \min\{R_0, d_y\}^{2-d},$$
$$\big|\{x\in\Omega : |D_x G^\varepsilon(x,y)| > t\}\big| \le Ct^{-d/(d-1)}, \quad \forall t > \min\{R_0, d_y\}^{1-d},$$
$$\big|\{x\in\Omega : |\Pi^\varepsilon(x,y)| > t\}\big| \le Ct^{-d/(d-1)}, \quad \forall t > \min\{R_0, d_y\}^{1-d}.$$

Moreover, for any R with $0 < R < \min\{R_0, d_y\}$, we have

$$\|G^\varepsilon(\cdot,y)\|_{L_q(B_R(y))} \le C_q R^{2-d+d/q}, \quad q \in [1, d/(d-2)),$$
$$\|DG^\varepsilon(\cdot,y)\|_{L_q(B_R(y))} \le C_q R^{1-d+d/q}, \quad q \in [1, d/(d-1)),$$
$$\|\Pi^\varepsilon(\cdot,y)\|_{L_q(B_R(y))} \le C_q R^{1-d+d/q}, \quad q \in [1, d/(d-1)).$$

In the above, $C = C(d,\lambda,K,A_0)$ and C_q depends also on q.

2.3. Proof of Theorem 1.4

The existence of the Green function (G, Π) for $\mathcal{L}$ in Ω satisfying the estimates (i) – $(viii)$ follows from the weak compactness theorem combined with the uniform estimates in Lemmas 2.6 and 2.7. Then due to Assumption 1.3 and the estimate (i), we obtain the pointwise bound (5). By the same reasoning, we verify the existence of the Green function (G^*, Π^*) for the adjoint operator $\mathcal{L}^*$ in Ω satisfying the natural counterparts of the properties of (G, Π). Then it is easily seen that G and G^* satisfy the symmetry (4). For details, we refer the reader to the proof of [4, Theorem 2.4]. □

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