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A NOTE ON THREE DERIVATIONS OF GREGORY-LEIBNIZ SERIES FOR π VIA A HYPERGEOMETRIC SERIES APPROACH

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Abstract. The aim of this note is to provide three derivations of the well-known Gregory-Leibniz series for π via a hypergeometric series approach.

1. Introduction

We first recall the definition of generalized hypergeometric series with p numerator and q denominator parameters [1, 2, 4, 6] viz.

$$ {}_pF_q\left[\begin{array}{cccc} a_1, & a_2, & \ldots, & a_p \\ & & & \\ b_1, & b_2, & \ldots, & b_q \end{array} ; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n\ldots(a_p)_n}{(b_1)_n(b_2)_n\ldots(b_q)_n}\frac{z^n}{n!}, \tag{1}$$

where $(a)_n$ denotes the well-known Pochhammer symbols (or the shifted factorial or the raised factorial, since $(1)_n = n!$), which is defined as

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\cdots(a+n-1), & n \in \mathbb{N} \\ 1, & n = 0, \end{cases} \tag{2}$$

where $\Gamma(z)$ is the well-known gamma function.

For details about the convergence (including its absolutely convergence) of the generalized hypergeometric series, we refer the standard books [1, 2, 4, 6].

It should be remarked here that whenever a generalized hypergeometric series reduces to the gamma function, the results are very important from the applications point of view. Thus the classical summation theorems for the series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$ and others listed, for example, in the standard book of Slater [6] play

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an important role. However, in our present investigation, we shall require the following classical summation theorems recorded in [3, 6] viz.

$$ {}_2F_1\left[\begin{matrix} a, \quad b \\ 1+a-b \end{matrix}\;;-1\right] = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1+a-b\right)}{\Gamma\left(1+a\right)\Gamma\left(1+\frac{1}{2}a-b\right)}, \tag{3} $$

$$ \begin{aligned} &{}_3F_2\left[\begin{matrix} a, \quad b, \quad c \\ 1+a-b, \quad 1+a-c \end{matrix}\;;1\right] \\ &\quad = \frac{\Gamma\left(1+\frac{1}{2}a\right)\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)\Gamma\left(1+\frac{1}{2}a-b-c\right)}{\Gamma\left(1+a\right)\Gamma\left(1+\frac{1}{2}a-b\right)\Gamma\left(1+\frac{1}{2}a-c\right)\Gamma\left(1+a-b-c\right)}, \end{aligned} \tag{4} $$

provided $\Re(a-2b-2c) > -2$,
and

$$ {}_4F_3\left[\begin{matrix} a, \quad 1+\frac{1}{2}a, \quad b, \quad c \\ \frac{1}{2}a, \quad 1+a-b, \quad 1+a-c \end{matrix}\;;-1\right] = \frac{\Gamma\left(1+a-b\right)\Gamma\left(1+a-c\right)}{\Gamma\left(1+a\right)\Gamma\left(1+a-b-c\right)}. \tag{5} $$

In literature (3) is known as the classical Kummer summation theorem while (4) is known as the classical Dixon summation theorem.

Next, we mention the following well-known and very useful Gregory-Leibniz π-series viz.

$$ 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots=\frac{\pi}{4}. \tag{6} $$

Our aim of this note is to provide three derivations of (6) via a hypergeometric series approach, the same will be given in the next section.

2. Three derivations of the Gregory-Leibniz series

In this section, we shall derive the Gregory-Leibniz series by three methods via a hypergeometric series approach.

2.1. First method

Let us denote the left-hand side of Gregory-Leibniz series for π by s, we have

$$ \begin{aligned} s &= 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\cdots+\frac{(-1)^n}{2n+1}+\cdots \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+\frac{1}{2}}. \end{aligned} $$

Using the identity (2), it is easy to see that

$$n+\frac{1}{2}=\frac{1}{2}\frac{\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_n}.$$

Thus

$$s=\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{3}{2}\right)_n}{\left(\frac{1}{2}\right)_n}$$

since $(1)_n=n!$. Thus

$$s=\sum_{n=0}^{\infty}(-1)^n\frac{(1)_n\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n n!}.$$

Summing up the series, we have

$$s={}_2F_1\left[\begin{matrix}1, & \frac{1}{2} \\ & \frac{3}{2}\end{matrix};-1\right].$$

We now observe that the ${}_2F_1$ can be evaluated with the help of classical Kummer summation theorem (3) by letting $a=1$ and $b=\frac{1}{2}$ and after some calculation, we easily arrive at the right-hand side of (6). This completes the first derivation of the π-series.

2.2. Second method

In this case, consider the Gregory-Leibniz series for π in the form

$$\begin{aligned}s&=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots+\left(\frac{1}{4n+1}-\frac{1}{4n+3}\right)+\cdots\\&=2\left(\frac{1}{1\cdot 3}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{(4n+1)(4n+3)}+\cdots\right)\\&=2\sum_{n=0}^{\infty}\frac{1}{(4n+1)(4n+3)}\\&=\frac{1}{8}\sum_{n=0}^{\infty}\frac{1}{\left(n+\frac{1}{4}\right)\left(n+\frac{3}{4}\right)}.\end{aligned}$$

Proceeding as in case of first method, we see that

$$s=\frac{2}{3}\;{}_3F_2\left[\begin{matrix}1, & \frac{1}{4}, & \frac{3}{4} \\ & \frac{5}{4}, & \frac{7}{4}\end{matrix};1\right]$$

We now observe that the ${}_3F_2$ can be evaluated with the help of the classical Dixon summation theorem (4) by letting $a=1, b=\frac{1}{4}$ and $c=\frac{3}{4}$ and we easily arrive at the right-hand side of (6). This completes the second derivation of the π-series.

2.3. Third method

In this case, consider the Gregory-Leibniz series for π in the form

$$
\begin{aligned}
s &= \left(1-\frac{1}{3}+\frac{1}{5}\right)-\left(\frac{1}{7}-\frac{1}{9}+\frac{1}{11}\right)+\cdots+(-1)^n\left(\frac{1}{6n+1}-\frac{1}{6n+3}+\frac{1}{6n+5}\right)+\cdots \\
&= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6n+1}-\frac{1}{6n+3}+\frac{1}{6n+5}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{6n+1}+\frac{1}{6n+5}\right)-\sum_{n=0}^{\infty}\frac{(-1)^n}{6n+3} \\
&= \sum_{n=0}^{\infty}(-1)^n\left(\frac{12n+6}{(6n+1)(6n+5)}\right)-\sum_{n=0}^{\infty}\frac{(-1)^n}{6n+3}.
\end{aligned}
$$

$$
s = A - B. \tag{7}
$$

Now proceeding as in case of second derivation, it is not difficult to see that

$$
A = \frac{4}{3}\ {}_4F_3\left[\begin{matrix} 1, & \frac{3}{2}, & \frac{1}{6}, & \frac{5}{6} \\ & \frac{1}{2}, & \frac{7}{6}, & \frac{11}{6} \end{matrix}\ ; -1\right] = \frac{\pi}{3},
$$

by using (5), by letting $a = 1, b = \frac{1}{6}$ and $c = \frac{5}{6}$ and

$$
B = \frac{1}{3}{}_2F_1\left[\begin{matrix} 1, & \frac{1}{2} \\ & \frac{3}{2} \end{matrix}\ ; -1\right] = \frac{\pi}{12},
$$

by using (3), by letting $a = 1$ and $b = \frac{1}{2}$.

Thus by (7), we have

$$
s = \frac{\pi}{3} - \frac{\pi}{12} = \frac{\pi}{4}.
$$

Remark 1. For an interesting generalization of the Gregory-Leibniz π-series, we refer a very recent paper by Rathie and Paris [5].

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