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# THE CLASSIFICATION OF $\omega$-LEFT-SYMMETRIC ALGEBRAS IN LOW DIMENSIONS

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Abstract. $\omega$-left-symmetric algebras contain left-symmetric algebras as a subclass and the commutator defines an $\omega$-Lie algebra. In this paper, we classify $\omega$-left-symmetric algebras in dimension 3 up to an isomorphism based on the classification of $\omega$-Lie algebras and the technique of Lie algebras.

## 1. Introduction

A vector space $L$ over $\mathbb{F}$ is called an $\omega$-Lie algebra if there is a bilinear map $[\cdot,\cdot]: L \times L \to L$ and a skew-symmetric bilinear form $\omega : L \times L \to \mathbb{F}$ such that

(1) $[x, y] = -[y, x]$,

(2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = \omega(x, y)z + \omega(y, z)x + \omega(z, x)y$,

hold for any $x, y, z \in L$, denote by $L_\omega$. The notation is given by Nurowski in [17], and there are a lot of studies in this field such as [7–9, 20, 21]. Clearly $\omega$-Lie algebras include Lie algebras as a subclass.

It is well-known that left-symmetric algebras are defined by the representation of Lie algebras. A natural question is to define $\omega$-left-symmetric algebras by the representation of $\omega$-Lie algebras, which is given in [19] as follows. Let $V_\omega$ be a vector space over $\mathbb{F}$ with a bilinear map $(x, y) \mapsto xy$. If there is a bilinear map $\omega : V_\omega \times V_\omega \to \mathbb{F}$ such that

$$(xy)z - x(yz) - (yx)z + y(xz) = \omega(x, y)z, \ \forall x, y, z \in V_\omega,$$

then $V_\omega$ is called an $\omega$-left-symmetric algebra. Left-symmetric algebras (or pre-Lie algebras, quasi-associative algebras, Vinberg algebras and so on) are $\omega$-left-symmetric algebras with $\omega = 0$, which are first introduced by A. Cayley in 1896 ([5]). They appear in many fields in mathematics and mathematical physics, for more details see [2–4, 6, 10–16, 18] and so on. Moreover $V_\omega$ is an

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$\omega$-Lie algebra under the commutator $[x, y] = xy - yx$, which is denoted by $(V_\omega, [\cdot, \cdot])$.

The classification of left-symmetric algebras in dimension 3 is given by Bai in [1]. This paper is to classify $\omega$-left-symmetric algebras with $\omega \neq 0$ in dimension 3 based on the classification of $\omega$-Lie algebras given by Nurowski in [17].

The paper is organized as follows. In Section 3, we recall some notations and results on $\omega$-Lie algebras. In particular, we list the classification of $\omega$-Lie algebras in dimension 3 given by Nurowski. Here we point out that there are two $\omega$-Lie algebras in Nurowski's list which are $\omega$-isomorphic. In Section 4, we obtain $\omega$-left-symmetric algebras in dimension 3 based on the classification of $\omega$-Lie algebras given by Nurowski, i.e., Theorem 3.3. In Section 5, we compute the automorphisms of $\omega$-left-symmetric algebras given in Theorem 3.3, and then give the classification up to an $\omega$-isomorphism.

## 2. $\omega$-Lie algebras

**Definition 2.1** ([17])**.** Let $L$ be a vector space over $\mathbb{F}$. If there is a bilinear map $[\cdot, \cdot] : L \times L \to L$ and a skew-symmetric bilinear form $\omega : L \times L \to \mathbb{F}$ such that

(1) $[x, y] = -[y, x]$,
(2) $[[x, y], z] + [[y, z], x] + [[z, x], y] = \omega(x, y)z + \omega(y, z)x + \omega(z, x)y$,

hold for any $x, y, z \in L$, then $L$ is called an $\omega$-Lie algebra, denote by $L_\omega$. The second identity is called the $\omega$-Jacobi identity, and $L_\omega$ is called simple if $L_\omega$ has no non-trivial ideal.

Clearly Lie algebras are $\omega$-Lie algebras with $\omega = 0$. Let $L_\omega$ be an $\omega$-Lie algebra in dimension 2 with $\omega \neq 0$. Then there exists a basis $\{e_1, e_2\}$ of $L_\omega$ such that

(1) $[e_1, e_2] = 0$, $\omega(e_1, e_2) = a$ for some $a \neq 0$, or
(2) $[e_1, e_2] = e_2$, $\omega(e_1, e_2) = a$ for some $a \neq 0$.

**Definition 2.2.** Let $L_\omega$ and $L_\Omega$ be $\omega$-Lie algebras over $\mathbb{F}$. If there is a linear isomorphism $\rho : L_\omega \to L_\Omega$ such that

$$\rho([x, y]) = [\rho(x), \rho(y)], \ \forall x, y \in L_\omega,$$

then $\rho$ is called an isomorphism from $L_\omega$ to $L_\Omega$. Furthermore, if $\omega(x, y) = \Omega(\rho(x), \rho(y))$, then $\rho$ is called an $\omega$-isomorphism.

Denote by $\mathrm{Isom}(L_\omega, L_\Omega)$ and $\mathrm{Isom}_{\omega,\Omega}(L_\omega, L_\Omega)$ the sets of isomorphisms and $\omega$-isomorphisms from $L_\omega$ to $L_\Omega$, respectively. Clearly

$$\mathrm{Isom}_{\omega,\Omega}(\mathrm{L}_\omega, \mathrm{L}_\Omega) \subseteq \mathrm{Isom}(\mathrm{L}_\omega, \mathrm{L}_\Omega).$$

Set $\mathrm{Aut}(L_\omega) = \mathrm{Isom}(L_\omega, L_\omega)$ and $\mathrm{Aut}_\omega(L_\omega) = \mathrm{Isom}_{\omega,\omega}(L_\omega, L_\omega)$.

**Example 2.3.** Let $L_\omega$ be an $\omega$-Lie algebra in dimension 2 with a basis $\{e_1, e_2\}$ satisfying

$$[e_1, e_2] = e_1, \ \omega(e_1, e_2) = 1.$$

It is easy to see that $\mathrm{Aut}(L_\omega) = \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ for $a \neq 0$. Furthermore, if $f \in \mathrm{Aut}_\omega(L_\omega)$, then

$$a = \omega(f(e_1), f(e_2)) = \omega(e_1, e_2) = 1.$$

That is, $\mathrm{Aut}_\omega(L_\omega) = \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$. For this case,

$$\mathrm{Aut}_\omega(L_\omega) \subset \mathrm{Aut}(L_\omega).$$

**Proposition 2.4.** *Let $L_\omega$ and $L_\Omega$ be $\omega$-Lie algebras with* $\dim L_\omega = \dim L_\Omega \geq 3$. *Then*

$$\mathrm{Isom}(L_\omega, L_\Omega) = \mathrm{Isom}_{\omega,\Omega}(L_\omega, L_\Omega).$$

*Proof.* It is enough to prove $\mathrm{Isom}(L_\omega, L_\Omega) \subseteq \mathrm{Isom}_{\omega,\Omega}(L_\omega, L_\Omega)$. For any $\rho \in \mathrm{Isom}(L_\omega, L_\Omega)$,

$$\begin{aligned}
&\omega(x,y)\rho(z) + \omega(y,z)\rho(x) + \omega(z,x)\rho(y) \\
&= \rho(\omega(x,y)z + \omega(y,z)x + \omega(z,x)y) \\
&= \rho([[x,y],z] + [[y,z],x] + [[z,x],y]) \\
&= [[\rho(x),\rho(y)],\rho(z)] + [[\rho(y),\rho(z)],\rho(x)] + [[\rho(z),\rho(x)],\rho(y)] \\
&= \Omega(\rho(x),\rho(y))\rho(z) + \Omega(\rho(y),\rho(z))\rho(x) + \Omega(\rho(z),\rho(x))\rho(y)
\end{aligned}$$

by the $\omega$-Jacobi identity and the definition of an isomorphism. For any $x, y \in L_\omega$, there exists $z \in L_\omega$ which does not belong to the subspace generated by $x$ and $y$. It means that $\rho(z)$ does not belong to the subspace in $L_\Omega$ generated by $\rho(x)$ and $\rho(y)$. Hence the above identity shows that $\Omega(\rho(x), \rho(y)) = \omega(x, y)$, i.e., $\rho \in \mathrm{Isom}_{\omega,\Omega}(L_\omega, L_\Omega)$. $\square$

**Theorem 2.5** ([17])**.** *Let $L_\omega$ be an $\omega$-Lie algebra of dimension 3 over $\mathbb{R}$ with $\omega \neq 0$. Then $L_\omega$ is one of the following types. That is, there exists a basis $\{e_1, e_2, e_3\}$ of $L_\omega$ such that*

(1) $[e_1, e_2] = e_2$, $[e_2, e_3] = e_1$, $[e_3, e_1] = -e_3$, $\omega(e_1, e_2) = 0$, $\omega(e_2, e_3) = 2$ *and* $\omega(e_3, e_1) = 0$. *It is type* $IV_T$.

(2) $[e_1, e_2] = -e_1$, $[e_2, e_3] = e_1 + e_3$, $[e_3, e_1] = -e_2$, $\omega(e_1, e_2) = 0$, $\omega(e_2, e_3) = 0$ *and* $\omega(e_3, e_1) = -2$. *It is type* $VI_S$.

(3) $[e_1, e_2] = e_2$, $[e_2, e_3] = e_1$, $[e_3, e_1] = -e_2 - e_3$, $\omega(e_1, e_2) = 0$, $\omega(e_2, e_3) = 2$ *and* $\omega(e_3, e_1) = 0$. *It is type* $VI_T$.

(4) $[e_1, e_2] = e_2 - e_1$, $[e_2, e_3] = e_1 + e_3$, $[e_3, e_1] = -e_2 - e_3$, $\omega(e_1, e_2) = 0$, $\omega(e_2, e_3) = 2$ *and* $\omega(e_3, e_1) = -2$. *It is type* $VI_N$.

(5) $[e_1, e_2] = e_2$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2 - e_3$, $\omega(e_1, e_2) = 0$, $\omega(e_2, e_3) = 2$ *and* $\omega(e_3, e_1) = 0$. *It is type* $VII_T$.

(6) $[e_1, e_2] = -e_3$, $[e_2, e_3] = e_1 - ae_2$, $[e_3, e_1] = e_2 + ae_1$, $\omega(e_1, e_2) = -2a$, $\omega(e_2, e_3) = 0$ *and* $\omega(e_3, e_1) = 0$. *It is type* $VIII_a$.

(7) $[e_1, e_2] = ae_2 - e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2 - ae_3$, $\omega(e_1, e_2) = 0$, $\omega(e_2, e_3) = 2a$ *and* $\omega(e_3, e_1) = 0$. *It is type* $VIII_{Ta}$.

(8) $[e_1, e_2] = ae_2 - e_3$, $[e_2, e_3] = e_1 - ae_2$, $[e_3, e_1] = ae_1 + e_2 - ae_3$, $\omega(e_1, e_2) = -2a$, $\omega(e_2, e_3) = 2a$ *and* $\omega(e_3, e_1) = 0$. *It is* $VIII_{Na}$.

(9) $[e_1, e_2] = e_3,\ [e_2, e_3] = e_1 - ae_2,\ [e_3, e_1] = e_2 + ae_1,\ \omega(e_1, e_2) = 2a,\ \omega(e_2, e_3) = 0$ *and* $\omega(e_3, e_1) = 0$. *It is type* $IX_a$.

*Here* $a > 0$ *is a real number.*

*Remark* 2.6. In the above classification, two $\omega$-Lie algebras of types $V_I S$ and $V_I T$ are isomorphic, and there is no isomorphism for the other types of $\omega$-Lie algebras. Assume that $L_\omega$ and $L_\Omega$ are $\omega$-Lie algebras of types $VI_S$ and $VI_T$, respectively. Let $\{e_1, e_2, e_3\}$ be the basis of $L_\omega$ satisfying

$$[e_1, e_2] = -e_1,\ [e_2, e_3] = e_1 + e_3,\ [e_3, e_1] = -e_2,$$
$$\omega(e_1, e_2) = \omega(e_2, e_3) = 0,\ \omega(e_3, e_1) = -2$$

and let $\{E_1, E_2, E_3\}$ be the basis of $L_\Omega$ satisfying

$$[E_1, E_2] = E_2,\ [E_2, E_3] = E_1,\ [E_3, E_1] = -E_2 - E_3,$$
$$\Omega(E_1, E_2) = \Omega(E_3, E_1) = 0,\ \Omega(E_2, E_3) = 2.$$

Define a linear map $f$ from $L_\omega$ to $L_\Omega$ by

$$f(e_1) = E_2,\ f(e_2) = E_1,\ f(e_3) = E_3.$$

It is easy to see that $f \in \mathrm{Isom}(L_\omega, L_\Omega) = \mathrm{Isom}_{\omega,\Omega}(L_\omega, L_\Omega)$.

**Definition 2.7** ([21])**.** Let $L_\omega$ be an $\omega$-Lie algebra and $M$ a vector space. If there is a linear map $\psi : L_\omega \to \mathrm{End}(M)$ such that

$$\psi([x, y])m = \psi(x)\psi(y)m - \psi(y)\psi(x)m + \omega(x, y)m,\ \forall x, y \in L_\omega, m \in M,$$

then $(\psi, M)$ or $\psi$ is called a representation of $L_\omega$.

## 3. $\omega$-left-symmetric algebras

**Definition 3.1** ([19])**.** Let $V_\omega$ be a vector space over $\mathbb{F}$ with a bilinear map $(x, y) \mapsto xy$. If there is a bilinear map $\omega : V_\omega \times V_\omega \to \mathbb{F}$ such that

$$(xy)z - x(yz) - (yx)z + y(xz) = \omega(x, y)z,\ \forall x, y, z \in V_\omega. \tag{3.1}$$

Then $V_\omega$ is called an $\omega$-left-symmetric algebra.

For an $\omega$-left-symmetric algebra $V_\omega$, it is easy to check that

(1) $\omega$ is skew-symmetric, and clearly $V_\omega$ is a left-symmetric algebra if $\omega = 0$.
(2) $V_\omega$ is an $\omega$-Lie algebra under the commutator $[x, y] = xy - yx$. Denote it by $(V_\omega, [\cdot, \cdot])$.
(3) Define a linear map $l : V_\omega \to \mathrm{End}(V_\omega)$ by $l(x)(y) = l_x(y) = xy$. Then $l$ is a representation of the $\omega$-Lie algebra $(V_\omega, [\cdot, \cdot])$.

That is, an $\omega$-left-symmetric algebra can be considered as an extension of a left symmetric algebra, and the relationship between $\omega$-left-symmetric algebra and $\omega$-Lie algebra is similar to that between Lie algebra and left-symmetric algebra. The following is to classify $\omega$-left-symmetric algebras in low dimensions which are not left-symmetric algebras, i.e., $\omega \neq 0$.

**Theorem 3.2** ([19]). *Let $V_\omega$ be an $\omega$-left-symmetric algebra in dimension 2 with $\omega \neq 0$. Then there is a basis $\{e_1, e_2\}$ of $V_\omega$ such that $\omega(e_1, e_2) = 1$, and*

(1) $e_1e_1 = e_1$, $e_1e_2 = e_2$, $e_2e_1 = -e_1 + e_2$, $e_2e_2 = ae_1 + be_2$, *or*
(2) $e_1e_1 = e_1 + ae_2$, $e_1e_2 = e_2$, $e_2e_1 = -e_1 + e_2$, $e_2e_2 = -2e_2$.

We will classify $\omega$-left-symmetric algebras of dimension 3 over $\mathbb{R}$ based on the classification of $\omega$-Lie algebras given by Nurowski. Assume that $V_\omega$ is an $\omega$-left-symmetric algebra of dimension 3 with the product $(x, y) \mapsto xy$. Then $V_\omega$ is an $\omega$-Lie algebra of dimension 3 under the commutator $[x, y] = xy - yx$. Suppose that there is a basis in $V_\omega$ such that

$$[e_1, e_2] = k^i e_i,\ [e_2, e_3] = l^i e_i,\ [e_3, e_1] = p^i e_i.$$
$$\omega(e_1, e_2) = c_{12},\ \omega(e_2, e_3) = c_{23},\ \omega(e_3, e_1) = c_{31}.$$

Then the product of the $\omega$-left-symmetric algebra is equivalent to that, for any $x \in V_\omega$,

$$\begin{aligned}(e_1e_2)x - e_1(e_2x) - (e_2e_1)x + e_2(e_1x) &= c_{12}x,\\ (e_2e_3)x - e_2(e_3x) - (e_3e_2)x + e_3(e_2x) &= c_{23}x,\\ (e_3e_1)x - e_3(e_1x) - (e_1e_3)x + e_1(e_3x) &= c_{31}x.\end{aligned}$$

Let $l_x$ denote the left multiplication on $V_\omega$, i.e., $l_x(y) = xy$, and denote by $A, B, C$ the matrices of $l_{e_1}$, $l_{e_2}$, $l_{e_3}$ under the basis $\{e_1, e_2, e_3\}$, respectively, i.e.,

$$\begin{aligned}l_{e_1}(e_1, e_2, e_3) &= (e_1, e_2, e_3)A,\\ l_{e_2}(e_1, e_2, e_3) &= (e_1, e_2, e_3)B,\\ l_{e_3}(e_1, e_2, e_3) &= (e_1, e_2, e_3)C.\end{aligned}$$

Then the above equations are equivalent to

$$\begin{cases} k^i l_{e_i} - AB + BA = c_{12},\\ l^i l_{e_i} - BC + CB = c_{23},\\ p^i l_{e_i} - CA + AC = c_{31}.\end{cases} \tag{3.2}$$

### 3.1. $(V_\omega, [\cdot,\cdot])$ is type $IV_T$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_2,\ [e_2, e_3] = e_1,\ [e_3, e_1] = -e_3,$$
$$\omega(e_1, e_2) = 0,\ \omega(e_2, e_3) = 2,\ \omega(e_3, e_1) = 0.$$

It is easy to see that $(V_\omega, [\cdot,\cdot])$ is simple as an $\omega$-Lie algebra. Moreover, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix},\ B = \begin{pmatrix} a_{12} & b_{12} & b_{13}\\ a_{22}-1 & b_{22} & b_{23}\\ a_{32} & b_{32} & b_{33}\end{pmatrix},\ C = \begin{pmatrix} a_{13} & b_{13}-1 & c_{13}\\ a_{23} & b_{23} & c_{23}\\ a_{33}-1 & b_{33} & c_{33}\end{pmatrix}.$$

By (3.2), we have

$$AB - BA - B = 0,\ BC - CB - A + 2I = 0,\ AC - CA - C = 0.$$

By the second one, we have $A = [B, C] + 2I$. Putting into the other two, we have

$$B = [[B, C], B],\ C = [[B, C], C].$$

It means that $\{B, C, [B, C]\}$ generates a Lie subalgebra of $\mathbb{R}^{3\times 3}$. Then

$$[[B, C], [B, C]] + [[C, [B, C]], B] + [[[B, C], B], C] = 0.$$

It follows that $2[B, C] = 0$. Then $B = C = [B, C] = 0$, which is impossible.

That is, there is no $\omega$-left-symmetric algebra $V_\omega$ such that $(V_\omega, [\cdot, \cdot])$ is type $IV_T$.

### 3.2. $(V_\omega, [\cdot, \cdot])$ is type $VI_T$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_2,\ [e_2, e_3] = e_1,\ [e_3, e_1] = -e_2 - e_3,$$
$$\omega(e_1, e_2) = 0,\ \omega(e_2, e_3) = 2,\ \omega(e_3, e_1) = 0.$$

Clearly, $(V_\omega, [\cdot, \cdot])$ is simple. Moreover, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} a_{12} & b_{12} & b_{13} \\ a_{22} - 1 & b_{22} & b_{23} \\ a_{32} & b_{32} & b_{33} \end{pmatrix}, C = \begin{pmatrix} a_{13} & b_{13} - 1 & c_{13} \\ a_{23} - 1 & b_{23} & c_{23} \\ a_{33} - 1 & b_{33} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$AB - BA - B = 0,\ BC - CB - A + 2I = 0,\ AC - CA - C - B = 0.$$

By the second one, we have $A = [B, C] + 2I$. Putting into the other two, we have

$$B = [[B, C], B],\ B + C = [[B, C], C].$$

That is, $\{B, C, [B, C]\}$ generates a Lie subalgebra of $\mathbb{R}^{3\times 3}$. Then

$$[[B, C], [B, C]] + [[C, [B, C]], B] + [[[B, C], B], C] = 0.$$

It follows that $[B, C] = 0$. Then $B = C = [B, C] = 0$, which is impossible.

That is, there is no $\omega$-left-symmetric algebra $V_\omega$ such that $(V_\omega, [\cdot, \cdot])$ is type $VI_T$.

### 3.3. $(V_\omega, [\cdot, \cdot])$ is type $VI_N$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_2 - e_1,\ [e_2, e_3] = e_1 + e_3,\ [e_3, e_1] = -e_3 - e_2,$$
$$\omega(e_1, e_2) = 0,\ \omega(e_2, e_3) = 2,\ \omega(e_3, e_1) = -2.$$

Clearly, $(V_\omega, [\cdot, \cdot])$ is not simple. Moreover, we have

$$A = \begin{pmatrix} a_{11} & a_{12} & c_{11} \\ a_{21} & a_{22} & c_{21} + 1 \\ a_{31} & a_{32} & c_{31} + 1 \end{pmatrix}, B = \begin{pmatrix} a_{12} + 1 & b_{12} & c_{12} + 1 \\ a_{22} - 1 & b_{22} & c_{22} \\ a_{32} & b_{32} & c_{32} + 1 \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$B-A-AB+BA=0,\ A+C-BC+CB=2I,\ -B-C-CA+AC=-2I.$$

That is,

$$B-A=[A,B],\ A+C=2I+[B,C],\ B+C=2I+[A,C].$$

Consider the dimension $k$ of the Lie algebra $L$ generated by $\{A,B,C,I\}$.

First we know $k\neq 1$.

If $k=4$, then $\{A,B,C,I\}$ is a basis of $L$. Moreover, under this basis, we have

$$\mathrm{ad}_A=\begin{pmatrix}0&-1&0&0\\0&1&1&0\\0&0&1&0\\0&0&-2&0\end{pmatrix},\ \mathrm{ad}_B=\begin{pmatrix}1&0&1&0\\-1&0&0&0\\0&0&1&0\\0&0&-2&0\end{pmatrix}.$$

It follows that

$$\mathrm{ad}_{B-A}=\begin{pmatrix}1&1&1&0\\-1&-1&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\ \mathrm{ad}_{[A,B]}=\begin{pmatrix}1&1&-1&0\\-1&-1&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

It contradicts to $B-A=[A,B]$.

If $k=3$, we divide into four cases:

(1) $\{A,B,C\}$ is a basis of $L$. Let $2I=xA+yB+zC$. Then

$$\begin{aligned}&B-A=[A,B],\quad (1-x)A-yB+(1-z)C=[B,C],\\&-xA+(1-y)B+(1-z)C=[A,C].\end{aligned}$$

It means that

$$\mathrm{ad}_A=\begin{pmatrix}0&-1&-x\\0&1&1-y\\0&0&1-z\end{pmatrix},\ \mathrm{ad}_B=\begin{pmatrix}1&0&1-x\\-1&0&y\\0&0&1-z\end{pmatrix},\ \mathrm{ad}_C=\begin{pmatrix}x&x-1&0\\y-1&y&0\\z-1&z-1&0\end{pmatrix}.$$

Since $2I\in L$ and $\mathrm{ad}_{2I}=0$, we have $x\mathrm{ad}_A+y\mathrm{ad}_B+z\mathrm{ad}_C=0$, i.e.,

$$0=\begin{pmatrix}y+xz&z(x-1)-x&-y(x-1)-x^2\\z(y-1)-y&x+yz&-x(y-1)-y^2\\z(z-1)&z(z-1)&-(x+y)(z-1)\end{pmatrix}.$$

By $z(z-1)=0$, we have $z=0$ or $z=1$. If $z=1$, then $z(y-1)-y=-1\neq 0$, which is a contradiction. Thus $z=0$. Thus $x=y=0$. It follows that $2I=xA+yB+cZ=0$, which is also a contradiction.

(2) $\{A,B,I\}$ is a basis of $L$. Let $C=xA+yB+zI$. Then we have

$$\begin{aligned}&B-A=[A,B],\quad (1+x)A+yB+(z-2)I=-x[A,B],\\&xA+(1+y)B+(z-2)I=y[A,B].\end{aligned}$$

Putting the first one into the second one, we have $A+(x+y)B+(z-2)I=0$, which is impossible.

(3) $\{A, C, I\}$ is a basis of $L$. Let $B = xA + yC + zI$. Then we have

$$(x-1)A + yC + zI = y[A, C], \quad A + C - 2I = x[A, C],$$
$$xA + (1+y)C + (z-2)I = [A, C].$$

Putting the third one into the first and second one, we have $y = z = 0$ and $x = 1$. That is, $A = B$ and $A + C = 2I$, which is impossible.

(4) $\{B, C, I\}$ is a basis of $L$. Let $A = xB + yC + zI$. Then we have

$$(1-x)B - yC - zI = -y[B, C],$$
$$xB + (1+y)C + (z-2)I = [B, C], \quad B + C - 2I = x[B, C].$$

Putting the second one into the first and third one, we have $y = z = 0$ and $x = 1$. That is, $A = B$ and $B + C = 2I$, which is impossible.

If $k = 2$, we will discuss the following three cases:

(1) If $\{A, I\}$, or $\{B, I\}$, or $\{C, I\}$ is a basis of $L$, then $[A, B] = [A, C] = [B, C] = 0$. Then $A = B$ and $A + C = 2I$. That is,

$$A = \begin{pmatrix} a_1 & a_1 - 1 & 2 - a_1 \\ a_2 & a_2 + 1 & 1 - a_2 \\ a_3 & a_3 & 1 - a_3 \end{pmatrix}, \ B = \begin{pmatrix} a_1 & a_1 - 1 & 2 - a_1 \\ a_2 & a_2 + 1 & 1 - a_2 \\ a_3 & a_3 & 1 - a_3 \end{pmatrix}, \ C = \begin{pmatrix} 2 - a_1 & 1 - a_1 & a_1 - 2 \\ -a_2 & 1 - a_2 & -1 + a_2 \\ -a_3 & -a_3 & 1 + a_3 \end{pmatrix}.$$

(2) If $\{A, B\}$ is a basis of $L$, then we have that $A - B = [B - A, C] = [[A, B], C] = 0$, i.e., $A = B$, which is impossible.

(3) If $\{A, C\}$ or $\{B, C\}$ is a basis of $L$, we have the same solution as (1).

That is, if $V_\omega$ is an $\omega$-left-symmetric algebra such that $(V_\omega, [\cdot, \cdot])$ is type $VI_N$, then there exists a basis $\{e_1, e_2, e_3\}$ of $V_\omega$ such that

$$\begin{cases} e_1e_1 = e_2e_1 = a_1e_1 + a_2e_2 + a_3e_3 = 2e_1 - e_3e_1, \\ e_1e_2 = e_2e_2 = (a_1 - 1)e_1 + (a_2 + 1)e_2 + a_3e_3 = 2e_2 - e_3e_2, \\ e_1e_3 = e_2e_3 = (2 - a_1)e_1 + (1 - a_2)e_2 + (1 - a_3)e_3 = 2e_3 - e_3e_3, \\ \omega(e_1, e_2) = 0, \ \omega(e_2, e_3) = 2, \ \omega(e_3, e_1) = -2. \end{cases}$$

### 3.4. $(V_\omega, [\cdot, \cdot])$ is type $VII_T$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = e_2, \ [e_2, e_3] = e_1, \ [e_3, e_1] = e_2 - e_3,$$
$$\omega(e_1, e_2) = 0, \ \omega(e_2, e_3) = 2, \ \omega(e_3, e_1) = 0.$$

Clearly, $(V_\omega, [\cdot, \cdot])$ is simple. Moreover we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ B = \begin{pmatrix} a_{12} & b_{12} & b_{13} \\ a_{22} - 1 & b_{22} & b_{23} \\ a_{32} & b_{32} & b_{33} \end{pmatrix}, \ C = \begin{pmatrix} a_{13} & b_{13} - 1 & c_{13} \\ a_{23} + 1 & b_{23} & c_{23} \\ a_{33} - 1 & b_{33} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$B - AB + BA = 0, \ A - BC + CB = 2I, \ B - C + AC - CA = 0.$$

It means that

$$B = [[B, C], B], \ -B + C = [[B, C], C].$$

That is, $\{B, C, [B, C]\}$ generates a Lie subalgebra of $\mathbb{R}^{3\times 3}$. Then

$$[[B,C],[B,C]] + [[C,[B,C]],B] + [[[B,C],B],C] = 0.$$

It gives $2[B, C] = 0$, furthermore $B = C = [B, C] = 0$, which is impossible.

That is, there is no $\omega$-left-symmetric algebra $V_\omega$ such that $(V_\omega, [\cdot,\cdot])$ is type $VII_T$.

### 3.5. $(V_\omega, [\cdot,\cdot])$ is type $VIII_a$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1,e_2] = -e_3,\ [e_2,e_3] = e_1 - ae_2,\ [e_3,e_1] = ae_1 + e_2,$$
$$\omega(e_1,e_2) = -2a,\ \omega(e_2,e_3) = 0,\ \omega(e_3,e_1) = 0,\ a > 0.$$

Clearly, $(V_\omega, [\cdot,\cdot])$ is simple. Moreover we have

$$A = \begin{pmatrix} a_{11} & a_{12} & c_{11}-a \\ a_{21} & a_{22} & c_{21}-1 \\ a_{31} & a_{32} & c_{31} \end{pmatrix},\ B = \begin{pmatrix} a_{12} & b_{12} & c_{12}+1 \\ a_{22} & b_{22} & c_{22}-a \\ a_{32}+1 & b_{32} & c_{32} \end{pmatrix},\ C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$-C - AB + BA = -2aI,\quad A - aB - BC + CB = 0,$$
$$aA + B + AC - CA = 0.$$

Then we have

$$-A + aB = [[B,A],B],\ aA + B = [[B,A],A].$$

That is, $\{B, A, [B, A]\}$ generates a Lie subalgebra of $\mathbb{R}^{3\times 3}$. Then we know

$$[[B,A],[B,A]] + [[A,[B,A]],B] + [[[B,A],B],A] = 0.$$

It follows that $2a[B, A] = 0$, so $B = A = [B, A] = 0$, which is impossible.

That is, there is no $\omega$-left-symmetric algebra $V_\omega$ such that $(V_\omega, [\cdot,\cdot])$ is type $VIII_a$.

### 3.6. $(V_\omega, [\cdot,\cdot])$ is type $VIII_{Ta}$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1,e_2] = ae_2 - e_3,\ [e_2,e_3] = e_1,\ [e_3,e_1] = e_2 - ae_3,$$
$$\omega(e_1,e_2) = 0,\ \omega(e_2,e_3) = 2a,\ \omega(e_3,e_1) = 0,\ a > 0.$$

Clearly, $(V_\omega, [\cdot,\cdot])$ is simple for $a \neq 1$. Moreover we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},\ B = \begin{pmatrix} a_{12} & b_{12} & b_{13} \\ a_{22}-a & b_{22} & b_{23} \\ a_{32}+1 & b_{32} & b_{33} \end{pmatrix},\ C = \begin{pmatrix} a_{13} & b_{13}-1 & c_{13} \\ a_{23}+1 & b_{23} & c_{23} \\ a_{33}-a & b_{33} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$aB - C - AB + BA = 0,\ A - BC + CB = 2aI,\ B - aC + AC - CA = 0.$$

It means that

$$aB - C = [[B, C], B], \ -B + aC = [[B, C], C].$$

That is, $\{B, C, [B, C]\}$ generates a Lie subalgebra of $\mathbb{R}^{3\times 3}$. Then we know

$$[[B, C], [B, C]] + [[C, [B, C]], B] + [[[B, C], B], C] = 0.$$

It gives $2[B, C] = 0$, then $B = C$. Since it is impossible for $B = C = 0$, we have $A = 2I$ and $a = 1$. Then we have

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \ B = C = \begin{pmatrix} 0 & a_1 & a_1 + 1 \\ 1 & a_2 & a_2 \\ 1 & a_3 & a_3 \end{pmatrix}.$$

That is, if $V_\omega$ is an $\omega$-left-symmetric algebra such that $(V_\omega, [\cdot, \cdot])$ is type $VIII_{Ta}$, then $a = 1$ and there exists a basis $\{e_1, e_2, e_3\}$ of $V_\omega$ such that

$$\begin{cases} e_1e_1 = 2e_1, \ e_1e_2 = 2e_2, \ e_1e_3 = 2e_3, \\ e_2e_1 = e_3e_1 = e_2 + e_3, \\ e_2e_2 = e_3e_2 = a_1e_1 + a_2e_2 + a_3e_3, \\ e_2e_3 = e_3e_3 = (a_1 + 1)e_1 + a_2e_2 + a_3e_3, \\ \omega(e_1, e_2) = 0, \ \omega(e_2, e_3) = 2, \ \omega(e_3, e_1) = 0. \end{cases}$$

### 3.7. $(V_\omega, [\cdot, \cdot])$ is type $VIII_{Na}$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1, e_2] = ae_2 - e_3, \ [e_2, e_3] = e_1 - ae_2, \ [e_3, e_1] = ae_1 + e_2 - ae_3,$$
$$\omega(e_1, e_2) = -2a, \ \omega(e_2, e_3) = 2a, \ \omega(e_3, e_1) = 0, \ a > 0.$$

Clearly $(V_\omega, [\cdot, \cdot])$ is simple. Moreover we have

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ B = \begin{pmatrix} a_{12} & b_{12} & b_{13} \\ a_{22} - a & b_{22} & b_{23} \\ a_{32} + 1 & b_{32} & b_{33} \end{pmatrix}, \ C = \begin{pmatrix} a_{13} + a & b_{13} - 1 & c_{13} \\ a_{23} + 1 & b_{23} + a & c_{23} \\ a_{33} - a & b_{33} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$aB - C - AB + BA = -2aI, \quad A - aB - BC + CB = 2aI,$$
$$aA + B - aC + AC - CA = 0.$$

That is,

$$aB - C + 2aI = [A, B], \ A - aB - 2aI = [B, C], \ -aA - B + aC = [A, C].$$

Let $L$ be the Lie subalgebra of $\mathbb{R}^{3\times 3}$ generated by $\{A, B, C, I\}$ with the dimension $k$.

Case 1: $k = 4$. Then we have

$$\mathrm{ad}_A = \begin{pmatrix} 0 & 0 & -a & 0 \\ 0 & a & -1 & 0 \\ 0 & -1 & a & 0 \\ 0 & 2a & 0 & 0 \end{pmatrix}, \ \mathrm{ad}_B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -a & 0 & -a & 0 \\ 1 & 0 & 0 & 0 \\ -2a & 0 & -2a & 0 \end{pmatrix}, \ \mathrm{ad}_C = \begin{pmatrix} a & -1 & 0 & 0 \\ 1 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 2a & 0 & 0 \end{pmatrix}.$$

By $\mathrm{ad}_{[A,B]} = \mathrm{ad}_{aB-C+2aI}$, we have $a = 0$, which is impossible.

Case 2: $k = 3$. We will discuss the following cases:

(1) If $\{A, B, C\}$ is a basis of $L$, let $2aI = xA + yB + zC$, then we have

$$xA + (y + a)B + (z - 1)C = [A, B],$$
$$(1 - x)A - (y + a)B - zC = [B, C], \quad -aA - B + aC = [A, C].$$

Under this basis, we have

$$\mathrm{ad}_A = \begin{pmatrix} 0 & x & -a \\ 0 & y+a & -1 \\ 0 & z-1 & a \end{pmatrix}, \ \mathrm{ad}_B = \begin{pmatrix} -x & 0 & 1-x \\ -y-a & 0 & -y-a \\ 1-z & 0 & -z \end{pmatrix}, \ \mathrm{ad}_C = \begin{pmatrix} a & x-1 & 0 \\ 1 & y+a & 0 \\ -a & z & 0 \end{pmatrix}.$$

Then by $\mathrm{ad}_{-aA-B+aC} = \mathrm{ad}_{[A,C]}$, we have

$$\begin{pmatrix} a^2+x & -a & a^2+x-1 \\ 2a+y & 0 & 2a+y \\ -a^2+z-1 & a & -a^2+z \end{pmatrix} = \begin{pmatrix} a^2+x & a+y-ax-az & a^2+x-1 \\ 2a+y & -x-z & 2a+y \\ -a^2+z-1 & ax-y-a+az & -a^2+z \end{pmatrix}.$$

It follows that $x = -z$ and $y = -2a$. Then $2aI = x(A - C) - 2aB$, so $\mathrm{ad}_{x(A-C)-2aB} = 0$. That is,

$$\begin{pmatrix} ax & x & ax-2a \\ -x-2a^2 & 0 & -x-2a^2 \\ -2a-ax & -x & -ax \end{pmatrix} = 0,$$

it follows that $x = 0$ and $a = 0$, which is impossible.

(2) If $\{A, B, 2aI\}$ is a basis of $L$, assume that $\{A, B, C\}$ is linear dependent by (1), then $C = xA + yB$. It follows that

$$-xA + (-y + a)B + 2aI = [A, B], \quad A - aB - 2aI = -x[A, B],$$
$$(ax - a)A - (ay - 1)B = y[A, B].$$

It gives $y = 0$, which is impossible. Similarly, we can show that $\{A, C, 2aI\}$ and $\{C, B, 2aI\}$ are not the basis of $L$.

Case 3: $k = 2$. We discuss the following cases.

(1) $\{A, 2aI\}$, or $\{B, 2aI\}$, or $\{C, 2aI\}$ is basis of $L$. For any case, then we have

$$aB - C + 2aI = 0, \ A - aB - 2aI = 0, \ -aA - B + aC = 0.$$

It gives $A = C$ and $B = 0$, which is impossible.

(2) $\{A, B\}$, or $\{A, C\}$, $\{C, B\}$ is a basis of $L$. For the first case, let $C = xA + yB$ and $2aI = pA + qB$. Then we have

$$\begin{cases} 1 - p = x(x - p), & \text{(i)} \\ a + q = x(q + a - y), & \text{(ii)} \end{cases} \quad \text{and} \quad \begin{cases} a(x - 1) = y(p - x), & \text{(iii)} \\ ay - 1 = y(q + a - y). & \text{(iv)} \end{cases}$$

By (i), we have $p = x+1$ or $x = 1$. If $x = 1$, then $y = 0$ by (ii), but (iv) doesn't hold. So $p = x + 1$, then $y = a(x - 1)$ by (iii). Then

$$ay - 1 = y(q + a - y) = a(x - 1)(q + a - y) = a(a + q) - a(q + a - y) = ay,$$

which is impossible.

Clearly $k \neq 1$. That is, there is no $\omega$-left-symmetric algebra $V_\omega$ such that $(V_\omega, [\cdot,\cdot])$ is type $VIII_{Na}$.

### 3.8. $(V_\omega, [\cdot,\cdot])$ is type $IX_a$ as an $\omega$-Lie algebra

Then there is a basis $\{e_1, e_2, e_3\}$ such that

$$[e_1,e_2] = e_3,\ [e_2,e_3] = e_1 - ae_2,\ [e_3,e_1] = ae_1 + e_2,$$
$$\omega(e_1,e_2) = 2a,\ \omega(e_2,e_3) = 0,\ \omega(e_3,e_1) = 0,\ a > 0.$$

Clearly $(V_\omega, [\cdot,\cdot])$ is simple. Moreover we have

$$A = \begin{pmatrix} a_{11} & b_{11} & c_{11} - a \\ a_{21} & b_{21} & c_{21} - 1 \\ a_{31} & b_{31} + 1 & c_{31} \end{pmatrix},\ B = \begin{pmatrix} b_{11} & b_{12} & c_{12} + 1 \\ b_{21} & b_{22} & c_{22} - a \\ b_{31} & b_{32} & c_{32} \end{pmatrix},\ C = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

By (3.2), we have

$$C - 2aI - AB + BA = 0, \quad A - aB - BC + CB = 0,$$
$$aA + B + AC - CA = 0.$$

It means that

$$A - aB = [B,[A,B]],\ -aA - B = [A,[A,B]].$$

That is, $\{B, A, [A,B]\}$ generates a Lie subalgebra of $\mathbb{R}^{3\times 3}$. Then we know

$$[[A,B],[A,B]] + [[B,[A,B]],A] + [[[A,B],A],B] = 0.$$

It gives $2a[A,B] = 0$, then $[A,B] = 0$, $C = 2aI$ and $a^2 + 1 = 0$, which is impossible.

That is, there is no $\omega$-left-symmetric algebra $V_\omega$ such that $(V_\omega, [\cdot,\cdot])$ is type $IX_a$.

In summary, we have the structure theorem of $\omega$-left-symmetric algebras in dimension 3.

**Theorem 3.3.** *Let $V_\omega$ be an $\omega$-left-symmetric algebra over $\mathbb{R}$ in dimension 3. Then one of the following cases holds*:

(1) *$V_\omega$ is a left-symmetric algebra.*
(2) *$(V_\omega, [\cdot,\cdot])$ is type $VI_N$, and there exists a basis $\{e_1, e_2, e_3\}$ of $V_\omega$ such that*

$$\begin{cases} e_1e_1 = e_2e_1 = a_1e_1 + a_2e_2 + a_3e_3 = 2e_1 - e_3e_1, \\ e_1e_2 = e_2e_2 = (a_1 - 1)e_1 + (a_2 + 1)e_2 + a_3e_3 = 2e_2 - e_3e_2, \\ e_1e_3 = e_2e_3 = (2 - a_1)e_1 + (1 - a_2)e_2 + (1 - a_3)e_3 = 2e_3 - e_3e_3, \\ \omega(e_1,e_2) = 0,\ \omega(e_2,e_3) = 2,\ \omega(e_3,e_1) = -2. \end{cases}$$

(3) $(V_\omega, [\cdot,\cdot])$ *is type* $VIII_{Ta}$ *with* $a = 1$*, and there exists a basis* $\{e_1, e_2, e_3\}$ *of* $V_\omega$ *such that*

$$\begin{cases} e_1e_1 = 2e_1,\ e_1e_2 = 2e_2,\ e_1e_3 = 2e_3, \\ e_2e_1 = e_3e_1 = e_2 + e_3, \\ e_2e_2 = e_3e_2 = a_1e_1 + a_2e_2 + a_3e_3, \\ e_2e_3 = e_3e_3 = (a_1+1)e_1 + a_2e_2 + a_3e_3, \\ \omega(e_1,e_2) = 0,\ \omega(e_2,e_3) = 2,\ \omega(e_3,e_1) = 0. \end{cases}$$

## 4. The isomorphism of $\omega$-left-symmetric algebras

**Definition 4.1.** Let $V_\omega$ and $V_\Omega$ be $\omega$-left-symmetric algebras over $\mathbb{F}$. If there is a linear isomorphism $\rho : V_\omega \to V_\Omega$ such that

$$\rho(xy) = \rho(x)\rho(y),\ \forall x, y \in V_\omega,$$

then $\rho$ is called an isomorphism from $V_\omega$ to $V_\Omega$. Furthermore, if $\omega(x,y) = \Omega(\rho(x), \rho(y))$, then $\rho$ is called an $\omega$-isomorphism.

Denote by $\mathrm{Isom}(V_\omega, V_\Omega)$ and $\mathrm{Isom}_{\omega,\Omega}(V_\omega, V_\Omega)$ the sets of isomorphisms and $\omega$-isomorphisms from $V_\omega$ to $V_\Omega$, respectively. Clearly

$$\mathrm{Isom}_{\omega,\Omega}(V_\omega, V_\Omega) \subseteq \mathrm{Isom}(V_\omega, V_\Omega).$$

**Proposition 4.2.** *Let* $V_\omega$ *and* $V_\Omega$ *be* $\omega$*-left-symmetric algebras with* $\dim V_\omega = \dim V_\Omega \geq 1$*. Then*

$$\mathrm{Isom}(V_\omega, V_\Omega) = \mathrm{Isom}_{\omega,\Omega}(V_\omega, V_\Omega).$$

*Proof.* It is enough to prove $\mathrm{Isom}(V_\omega, V_\Omega) \subseteq \mathrm{Isom}_{\omega,\Omega}(V_\omega, V_\Omega)$. For any $\rho \in \mathrm{Isom}(V_\omega, V_\Omega)$,

$$\begin{aligned} \omega(x,y)\rho(z) &= \rho((xy)z - x(yz) - (yx)z + y(xz)) \\ &= (\rho(x)\rho(y))\rho(z) - \rho(x)(\rho(y)\rho(z)) - (\rho(y)\rho(x))\rho(z) + \rho(y)(\rho(x)\rho(z)) \\ &= \Omega(\rho(x), \rho(y))\rho(z) \end{aligned}$$

by the definitions of $\omega$-left-symmetric algebras and isomorphisms. For any $x, y \in V_\omega$, there exists $0 \neq z \in V_\omega$, then $\rho(z) \neq 0$. Hence we have $\Omega(\rho(x), \rho(y)) = \omega(x,y)$, i.e., $\rho \in \mathrm{Isom}_{\omega,\Omega}(V_\omega, V_\Omega)$. □

In the following, we will compute $\mathrm{Isom}(V_\omega, V_\omega)$ for $V_\omega$ in cases (2) and (3) of Theorem 3.3, and then discuss the classification up to an isomorphism (i.e., $\omega$-isomorphism by Proposition 4.2). We first give a simple fact.

**Lemma 4.3.** *Let* $V_\omega$ *be an* $\omega$*-left-symmetric algebra and* $\rho \in \mathrm{Isom}(V_\omega, V_\omega)$*. If* $x \in V_\omega$ *such that* $l_x = kI$*, then* $l_{\rho(x)} = kI$*.*

*Proof.* Since $\rho \in \mathrm{Isom}(V_\omega, V_\omega)$, we have $\rho(x)\rho(y) = \rho(xy) = k\rho(y)$ for any $y \in V_\omega$. Thus $l_{\rho(x)} = kI$. □

**Case (2) in Theorem 3.3.** Then $\{f_1 = e_1, f_2 = e_2 - e_1, f_3 = e_3 + e_2\}$ is a basis of $V_\omega$ such that

$$\begin{cases} f_1f_1 = b_1f_1 + b_2f_2 + b_3f_3, \\ f_1f_2 = -f_2,\ f_1f_3 = 2f_1 + f_2 + f_3, \\ f_2f_1 = f_2f_2 = f_2f_3 = 0, \\ f_3f_1 = 2f_1,\ f_3f_2 = 2f_2,\ f_3f_3 = 2f_3. \end{cases}$$

Here $b_1, b_2, b_3$ are arbitrary real numbers. Assume that $\rho \in \mathrm{Isom}(V_\omega, V_\omega)$. By Lemma 4.3 and the algebraic structure, we must have

$$\rho(f_2) = bf_2,\ \rho(f_3) = -af_2 + f_3,\ b \neq 0.$$

Furthermore, by $\rho(f_1f_2) = \rho(f_1)\rho(f_2)$ and $\rho(f_1f_3) = \rho(f_1)\rho(f_3)$, we have

$$\rho(f_1) = f_1 + (a + \frac{1-b}{2})f_2.$$

Moreover, $\rho(f_1f_1) = \rho(f_1)\rho(f_1)$ if and only if for the coefficient of $f_2$,

$$(b_1 - b_3 + 1)a = \frac{(2b_2 - b_1 - 1)(1 - b)}{2}.$$

Set $f_i' = \rho(f_i)$. Then we have the following cases:

(1) If $2b_2 - b_3 \neq 0$, set $a = \frac{b_2(2b_2 - b_1 - 1)}{2b_2 - b_3}$ and $\frac{1-b}{2} = \frac{b_2(b_1 - b_3 + 1)}{2b_2 - b_3}$, we have

$$\begin{cases} f_1'f_1' = b_1f_1' + b_3f_3',\ f_1'f_2' = -f_2',\ f_1'f_3' = 2f_1' + f_2' + f_3', \\ f_2'f_1' = f_2'f_2' = f_2'f_3' = 0,\ f_3'f_1' = 2f_1',\ f_3'f_2' = 2f_2',\ f_3'f_3' = 2f_3'. \end{cases}$$

For this case, $\omega$-left-symmetric algebras with different $(b_1, b_3)$ are not isomorphic.

(2) If $2b_2 - b_3 = 0$ and $2b_2 = b_1 + 1$, then $a$ and $b \neq 0$ are arbitrary. Furthermore taking $a$ and $b$ such $b_2 = a + \frac{1-b}{2}$, we have

$$\begin{cases} f_1'f_1' = (2b_2 - 1)f_1' + 2b_2f_3',\ f_1'f_2' = -f_2',\ f_1'f_3' = 2f_1' + f_2' + f_3', \\ f_2'f_1' = f_2'f_2' = f_2'f_3' = 0,\ f_3'f_1' = 2f_1',\ f_3'f_2' = 2f_2',\ f_3'f_3' = 2f_3'. \end{cases}$$

It is a special case of (1).

(3) If $2b_2 - b_3 = 0$ and $2b_2 \neq b_1 + 1$, then $a + \frac{1-b}{2} = 0$. Furthermore we have

$$\begin{cases} f_1'f_1' = b_1f_1' + b_2f_2' + 2b_2f_3',\ f_1'f_2' = -f_2',\ f_1'f_3' = 2f_1' + f_2' + f_3', \\ f_2'f_1' = f_2'f_2' = f_2'f_3' = 0,\ f_3'f_1' = 2f_1',\ f_3'f_2' = 2f_2',\ f_3'f_3' = 2f_3'. \end{cases}$$

For this case, $\omega$-left-symmetric algebras with different $(b_1, b_2)$ are not isomorphic.

**Case (3) in Theorem 3.3.** Then $\{f_1 = e_1, f_2 = e_2 - e_3, f_3 = e_3\}$ is a basis of $V_\omega$ such that

$$\begin{cases} f_1f_1 = 2f_1,\ f_1f_2 = 2f_2,\ f_1f_3 = 2f_3, \\ f_2f_1 = f_2f_2 = f_2f_3 = 0, \\ f_3f_1 = f_2 + 2f_3,\ f_3f_2 = -f_1, \\ f_3f_3 = b_1f_1 + b_2f_2 + b_3f_3. \end{cases}$$

Here $b_1 = a_1 + 1$, $b_2 = a_2$ and $b_3 = a_3$ are arbitrary real numbers. Assume that $\rho \in \mathrm{Isom}(V_\omega, V_\omega)$. By Lemma 4.3 and the algebraic structure, we must have

$$\rho(f_1) = f_1 + af_2,\ \rho(f_2) = bf_2,\ b \neq 0.$$

Furthermore, by $\rho(f_3f_1) = \rho(f_3)\rho(f_1)$ and $\rho(f_3f_2) = \rho(f_3)\rho(f_2)$, we have

$$a = 0,\ \rho(f_3) = (\frac{b}{2} - \frac{1}{2b})f_2 + \frac{1}{b}f_3.$$

Moreover, $\rho(f_3f_3) = \rho(f_3)\rho(f_3)$ means

$$b_1f_1 + b_2bf_2 + b_3((\frac{b}{2} - \frac{1}{2b})f_2 + \frac{1}{b}f_3) = -(\frac{1}{2} - \frac{1}{2b^2})f_1 + \frac{1}{b^2}(b_1f_1 + b_2f_2 + b_3f_3).$$

Then we have the following cases:

(1) If $b_3 \neq 0$, then $b = 1$. Thus $\rho = I$.
(2) If $b_3 = 0$ and $b_2 \neq 0$, then $b = 1$. Thus $\rho = I$.
(3) If $b_2 = b_3 = 0$, $b_1 \neq -\frac{1}{2}$, then $b = \pm 1$.
(4) If $b_2 = b_3 = 0$, $b_1 = -\frac{1}{2}$, then $b \neq 0$.

It follows that $\omega$-left-symmetric algebras with different $(b_1, b_2, b_3)$ are not isomorphic.

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