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COMPACT COMPOSITION OPERATORS ON BESOV SPACES ON THE UNIT BALL

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Abstract. In this paper, we give new necessary and sufficient conditions for the compactness of composition operator on the Besov space and the Bloch space of the unit ball, which, to a certain extent, generalizes the results given by M. Tjani in [10].

1. Introduction

Let ϕ be a holomorphic self-map of the unit ball $\mathbb{B}$ and $H(\mathbb{B})$ the space of all holomorphic functions on $\mathbb{B}$. The composition operator C_ϕ is defined by

$$C_\phi f = f \circ \phi, \quad f \in H(\mathbb{B}).$$

There are intensive studies over the composition operator C_ϕ on the Besov space and the Bloch space, such as [1, 2, 6, 9–11]. In 2003, M. Tjani gave a classical characterization of the compact composition operator on the Besov space of the unit disc $\mathbb{D}$. Let

$$\alpha_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \ \lambda \in \mathbb{D}$$

is the basic conformal automorphism. M. Tjani proved Theorems 1.1 and 1.2 (see [10]).

Theorem 1.1. *Let $1 < p \leq q < \infty$. Then the following are equivalent*:
(i) $C_\phi : \mathcal{B}_p \to \mathcal{B}_q$ *is a compact operator.*
(ii) $N_q(w, \phi) dA(w)$ *is a vanishing q-Carleson measure.*
(iii) $\|C_\phi \alpha_\lambda\|_{\mathcal{B}_q} \to 0$ *as* $|\lambda| \to 1$.

Theorem 1.2. *Let ϕ be a holomorphic self-map of D. Let $X = \mathcal{B}_p$ $(1 < p < \infty)$ or $\mathcal{B}$. Then $C_\phi : X \to \mathcal{B}$ is a compact operator if and only if $\|C_\phi \alpha_\lambda\|_{\mathcal{B}} \to 0$ as $|\lambda| \to 1$.*

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A natural question that aries from this result is what will happen on the unit ball. However, there are not many similar results on the two kinds of spaces of the unit ball, especially on the Besov space.

As for the case of the Bloch space of the unit ball, although the composition operator C_ϕ is always bounded on the Bloch space due to the Schwarz-Pick lemma of the unit ball $\mathbb{B}$ (see [9]), there exist some difficulties in characterizing the compactness of composition operators on the Bloch space. In 2009, by using a smart technique, Chen and Gauthier proved that C_φ is compact on the Bloch space of the unit ball if and only if (see [1])

$$\lim_{|\phi(z)|\to 1}(1-|z|^2)\{H_{\phi(z)}(\phi'(z)z,\phi'(z)z)\}^{\frac{1}{2}}=0.$$

In 2012, J. Dai gave the following theorem (see [2]).

Theorem 1.3. *Let ϕ be a holomorphic mapping of $\mathbb{B}$ into itself. Then the composition operator C_ϕ is compact on the Bloch space $\mathcal{B}$ if and only if*

$$\|C_\phi\varphi_a\|_{\mathcal{B}^*}\to 0 \text{ as } |a|\to 1,$$

where

$$\|F\|_{\mathcal{B}^*}=\Big(\sum_{i=1}^{n}\|f_i\|_{\mathcal{B}}^2\Big),$$

$$B^*=\{F=(f_1,\dots,f_n): f_i\in\mathcal{B},\ i=1,\dots,n\},$$

and $\varphi_a(z)$ are the involutions of the unit ball $\mathbb{B}$.

One may ask whether the norm $\|\cdot\|_{\mathcal{B}^*}$ can be replaced by $\|\cdot\|_{\mathcal{B}}$ in Theorem 1.3. In this paper, we prove that Theorem 1.3 also holds with the other norm.

Mainly motivated by [2] and [10], we give new necessary and sufficient conditions for the compactness of composition operator on the Besov space and the Bloch space of the unit ball.

Let

$$k_\lambda(z)=\frac{1-|\lambda|^2}{1-\langle z,\lambda\rangle},\quad \lambda,z\in\mathbb{B}.$$

Our main results are the following.

Theorem 1.4. *Let $n<p_1\le p_2<\infty$. Define a finite positive Borel measure μ on $\mathbb{B}$ by*

$$\mu(E)=\int_{\phi^{-1}(E)}|\phi'(z)z|^p(1-|z|^2)^{p-n-1}dv(z)$$

for all Borel sets E of $\mathbb{B}$. Then the following are equivalent:

(i) *$C_\phi:\mathcal{B}_{p_1}\to\mathcal{B}_{p_2}$ is a compact operator.*

(ii) *μ is a vanishing p_2-n-1-Carleson measure.*

(iii) *$\|C_\phi k_\lambda\|_{\mathcal{B}_{p_2}}\to 0$ as $|\lambda|\to 1$.*

Theorem 1.5. *Let ϕ be a holomorphic self-map of $\mathbb{B}$. Let $X=\mathcal{B}_p$ $(n<p<\infty)$ or $\mathcal{B}$. Then $C_\phi: X\to\mathcal{B}$ is a compact operator if and only if $\|C_\phi k_\lambda\|_{\mathcal{B}}\to 0$ as $|\lambda|\to 1$.*

Throughout this paper, C denotes a positive constant which may change from one occurrence to the next, the expression $E \approx F$ means that there exists a positive constant C such that $C^{-1}E \leq F \leq CE$.

2. Preliminaries

Let $\mathbb{C}^n$ denote the Euclidean space of complex dimension n ($n \geq 1$). For $z = (z_1, \dots, z_n)$, and $w = (w_1, \dots, w_n)$ in $\mathbb{C}^n$, denote the inner product of z and w by

$$\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n},$$

and write $|z| = \sqrt{\langle z, z\rangle}$. Let $\mathbb{B} = \{z \in \mathbb{C}^n : |z| < 1\}$ be the unit ball of $\mathbb{C}^n$ and dv the normalized Lebesgue measure on $\mathbb{B}$.

Recall that the radial derivative of $f \in H(\mathbb{B})$ is defined by

$$\Re f = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z) = \lim_{r\to 0} \frac{f(z + rz) - f(z)}{r}, \quad r \in \mathbb{R},$$

and the complex gradient of f is

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\Big).$$

It is easy to see that $\Re f(z) = \langle \nabla f(z), \overline{z}\rangle$.

Let $\phi = (\phi_1, \dots, \phi_n)$ be a holomorphic mapping of $\mathbb{B}$ into itself. The symbol $\phi'(z)z$ is denoted by (see [2])

$$\phi'(z)z = (\Re\phi_1(z), \dots, \Re\phi_n(z)).$$

For $f \in H(\mathbb{B})$, by the chain rule, we have

$$\Re(f \circ \phi)(z) = \nabla(f \circ \phi)(z)z = \nabla f(\phi(z))\phi'(z)z.$$

Let

$$Q_f(z) = \sup\Big\{\frac{|\langle \nabla f(z), w\rangle|}{H_z(w, w)^{\frac{1}{2}}} : 0 \neq w \in \mathbb{C}^n\Big\},$$

where $H_z(w, w)$ is the Bergman metric on $\mathbb{B}$ which is defined by

$$H_z(w, w) = \frac{n + 1}{2}\frac{(1 - |z|^2)|w|^2 + |\langle z, w\rangle|^2}{(1 - |z|^2)^2}.$$

The Bloch space $\mathcal{B}$ of the unit ball $\mathbb{B}$ is the space of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{B}} Q_f(z) < \infty.$$

It is easy to see that $\|\cdot\|_{\mathcal{B}}$ is a semi-norm and the Bloch space becomes a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{\mathcal{B}}$. Moreover, for $f \in \mathcal{B}$, by Theorem 3.4 in [12],

$$\begin{aligned}\|f\|_{\mathcal{B}} &\approx \sup\{(1 - |z|^2)|\Re f(z)| : z \in \mathbb{B}\}\\ &\approx \sup\{(1 - |z|^2)|\nabla f(z)| : z \in \mathbb{B}\}.\end{aligned}$$

For each $0 < p < \infty$, the Besov space $\mathcal{B}_p$ is the image of Bergman space A^p_α under a suitable fractional integral operator. When $p \geq 1$, the space $\mathcal{B}_p$ can be equipped with the a (semi-)norm that is invariant under the action of the automorphism group. The space $\mathcal{B}_1$ is the minimal Möbius invariant Banach space. The space $\mathcal{B}_2$ plays the role of the Dirichlet space in high dimensions. And the space $\mathcal{B}_\infty$ is just the Bloch space. See [12] for details.

Suppose $n < p < \infty$ and f is holomorphic on $\mathbb{B}$. Then $f \in \mathcal{B}_p$ if and only if $(1-|z|^2)\mathfrak{R}f(z) \in L^p(\mathbb{B}, d\tau)$ if and only if $(1-|z|^2)\nabla f(z) \in L^p(\mathbb{B}, d\tau)$, where $d\tau(z) = \frac{dv(z)}{(1-|z|^2)^{n+1}}$ (see [12, p. 231]). Then

$$\begin{aligned}\|f\|^p_{\mathcal{B}_p} &= \int_{\mathbb{B}} |\mathfrak{R}f(z)|^p (1-|z|^2)^{p-n-1} dv(z) \\ &\approx \int_{\mathbb{B}} |\nabla f(z)|^p (1-|z|^2)^{p-n-1} dv(z).\end{aligned}$$

It is easy to see that $|f(0)| + \|f\|_{\mathcal{B}_p}$ is a norm on $\mathcal{B}_p$. Define a finite positive Borel measure μ on $\mathbb{B}$ by

$$\mu(E) = \int_{\phi^{-1}(E)} |\phi'(z)z|^p (1-|z|^2)^{p-n-1} dv(z)$$

for all Borel sets E of $\mathbb{B}$. According to [3, p. 163],

$$\begin{aligned}\|C_\phi f\|^p_{\mathcal{B}_p} &= \int_{\mathbb{B}} |\mathfrak{R}(f\circ\phi)(z)|^p (1-|z|^2)^{p-n-1} dv(z) \\ &= \int_{\mathbb{B}} |\nabla f(\phi(z))|^p |\phi'(z)z|^p (1-|z|^2)^{p-n-1} dv(z) \\ &= \int_{\mathbb{B}} |\nabla f(z)|^p d\mu(z). \qquad (1)\end{aligned}$$

3. Main results

The proof of the following lemma follows on similar lines as Lemma 3.8 in [10] or Lemma 2.2 in [9], here we omit the details.

Lemma 3.1. *Let $X, Y = \mathcal{B}_p$ $(n < p < \infty)$ or $\mathcal{B}$. Then $C_\phi : X \to Y$ is a compact operator if and only if for any bounded sequence $\{f_j\}$ in X with $f_j \to 0$ uniformly on compact sets as $n \to \infty$, $\|C_\phi f_j\|_Y \to 0$ as $n \to \infty$.*

For $s, t \in \mathbb{R}$, consider the linear transformations I^t_s defined for $f \in H(\mathbb{B})$ by $I^t_s f(z) = (1-|z|^2)^t D^t_s f(z)$, where D^t_s are the invertible radial differential operators on $H(\mathbb{B})$ of order t for any s defined as coefficient multipliers and discussed in [4, §3]. Every I^0_s is the identity or inclusion. We denote by $D(w, r)$ the ball in the Bergman metric with center $w \in \mathbb{B}$ and radius $r \in (0, \infty)$.

We will need the following lemma obtained in [5].

Lemma 3.2. *Let $q \in \mathbb{R}$ be fixed but unrestricted. Let $p \in (0, \infty)$, $r \in (0, \infty)$, $s \in \mathbb{R}$ be given. The following conditions are equivalent for a positive Borel measure μ on $\mathbb{B}$.*

(i) $\mu(D(w,r)) = o(v_q(D(w,r)))$ *as* $|w| \to 1$ $(w \in \mathbb{B})$.

(ii) *The measure* μ *is a vanishing Carleson measure for* $\mathcal{B}^p_q$, *that is, if* t *satisfies* $q + pt > -1$, *then*

$$\int_{\mathbb{B}} |I^t_s f|^p d\mu = o(\|f\|^p_{\mathcal{B}^p_q}) \ (f \in \mathcal{B}^p_q).$$

(iii) *If* t *satisfies* $q + pt > -1$, *then*

$$(1-|w|^2)^{n+1+q+pt} \int_{\mathbb{B}} \frac{(1-|z|^2)^{pt}}{|1-\langle z,w\rangle|^{(n+1+q+pt)2}} d\mu(z) = o(1) \ \textit{as}\ |w| \to 1 \ (w \in \mathbb{B}).$$

It is clear from Lemma 3.2 that a vanishing Carleson measure μ is independent of p, r, s, t as long as $q + pt > -1$ holds. However, all conditions of Lemma 3.2 depend on q, and we call such a μ also a vanishing q-Carleson measure.

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4. By (i),

$$\begin{aligned}\|C_\phi k_\lambda\|^{p_2}_{\mathcal{B}_{p_2}} &= \int_{\mathbb{B}} |\nabla k_\lambda(z)|^{p_2} d\mu(z) \\ &= |\lambda|^{p_2}(1-|\lambda|^2)^{p_2} \int_{\mathbb{B}} \frac{d\mu(z)}{|1-\langle z,\lambda\rangle|^{2p_2}}.\end{aligned}$$

Since $p_2 > n$, set $t = 0, q = p_2 - n - 1$ in Lemma 3.2, we have (iii)$\Rightarrow$(ii).

Next we show that (i)$\Rightarrow$(iii). Assume that $C_\phi : \mathcal{B}_{p_1} \to \mathcal{B}_{p_2}$ is a compact operator. Since $p_1 > n$, by Proposition 1.4.10 of [8], we have

$$\begin{aligned}\|k_\lambda\|^{p_1}_{\mathcal{B}_{p_1}} &= \int_{\mathbb{B}} |\Re k_\lambda(z)|^{p_1}(1-|z|^2)^{p_1-n-1} dv(z) \\ &= \int_{\mathbb{B}} \frac{(1-|\lambda|^2)^{p_1}(1-|z|^2)^{p_1-n-1}|\langle z,\lambda\rangle|^{p_1}}{|1-\langle z,\lambda\rangle|^{2p_1}} dv(z) \\ &\le C|\lambda|^{p_1} < \infty.\end{aligned}$$

Then the norm of k_λ in $\mathcal{B}_{p_1}$ is

$$k_\lambda(0) + \|k_\lambda\|_{\mathcal{B}_{p_1}} = 1 - |\lambda|^2 + \|k_\lambda\|_{\mathcal{B}_{p_1}} < \infty.$$

Also $k_\lambda \to 0$ as $|\lambda| \to 1$, uniformly on compact sets since $k_\lambda(z) = \frac{1-|\lambda|^2}{1-\langle z,\lambda\rangle}$. Hence, by Lemma 3.1, $\|C_\phi k_\lambda\|_{\mathcal{B}_{p_2}} \to 0$ as $|\lambda| \to 1$.

Finally, let us show that (ii)$\Rightarrow$(i). Let $\{f_j\}$ be a bounded sequence in $\mathcal{B}_{p_1}$ that converges to 0 uniformly on compact sets. Let $0 < r < 1$ be arbitrarily fixed. Then by the proof of Theorem 3 in [7], we have

$$|\nabla f_j(w)|^{p_2} \le \frac{C}{r^{2n}} \int_{\varphi_w(\mathbb{B}_r)} |\nabla f_j(z)|^{p_2} \Big(\frac{1-|w|^2}{1-|\langle z,w\rangle|^2}\Big)^{n+1} dv(z).$$

Together with Fubini's theorem, we have

$$\|C_\phi f_j\|^{p_2}_{\mathcal{B}_{p_2}} = \int_{\mathbb{B}} |\nabla f_j(w)|^{p_2} d\mu(w)$$

$$
\begin{aligned}
&\leq \int_{\mathbb{B}} \frac{C}{r^{2n}} \Big(\int_{\varphi_w(\mathbb{B}_r)} |\nabla f_j(z)|^{p_2} \Big(\frac{1-|w|^2}{1-|\langle z,w\rangle|^2} \Big)^{n+1} dv(z) \Big) d\mu(w) \\
&= \frac{C}{r^{2n}} \int_{\mathbb{B}} |\nabla f_j(z)|^{p_2} \Big(\int_{\mathbb{B}} \Big(\frac{1-|w|^2}{1-|\langle z,w\rangle|^2} \Big)^{n+1} \chi_{\{z: z\in\varphi_w(\mathbb{B}_r)\}}(z) d\mu(w) \Big) dv(z).
\end{aligned}
$$

By Lemma 1.2 in [12], for each $w \in \mathbb{B}$, φ_w satisfies

$$
1 - |\varphi_w(z)|^2 = \frac{(1-|w|^2)(1-|z|^2)}{1-|\langle z,w\rangle|^2}, \; z \in \mathbb{B}.
$$

This implies

$$
1 - |z|^2 > \frac{1}{4}(1-r)(1-|w|^2)
$$

for $z \in \varphi_w(\mathbb{B}_r)$. Note that $1 - |\langle z,w\rangle| > 1 - |z|$, it follows that

$$
\begin{aligned}
\Big(\frac{1-|w|^2}{1-|\langle z,w\rangle|^2} \Big)^{n+1} &< \Big(\frac{4(1-|z|^2)}{(1-r)(1-|z|)^2} \Big)^{n+1} \\
&= \Big(\frac{4(1+|z|)}{(1-r)(1-|z|)} \Big)^{n+1} \\
&< \Big(\frac{16}{(1-r)(1-|z|^2)} \Big)^{n+1}
\end{aligned}
$$

for $z \in \varphi_w(\mathbb{B}_r)$. Recall that $\varphi_w(\mathbb{B}_r) = D(w,R)$, where $\tanh R = r$ (see [12, p. 35] and [8, p. 29]). Since $\chi_{D(w,R)}(z) = \chi_{D(z,R)}(w)$, we have

$$
\begin{aligned}
\|C_\phi f_j\|^{p_2}_{\mathcal{B}_{p_2}} &< C \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{\mathbb{B}} \frac{|\nabla f_j(z)|^{p_2}}{(1-|z|^2)^{n+1}} \Big(\int_{\mathbb{B}} \chi_{D(w,R)}(z) d\mu(w) \Big) dv(z) \\
&= C \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{\mathbb{B}} \frac{|\nabla f_j(z)|^{p_2}}{(1-|z|^2)^{n+1}} \Big(\int_{\mathbb{B}} \chi_{D(z,R)}(w) d\mu(w) \Big) dv(z) \\
&= C \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{|z|>\delta} \frac{|\nabla f_j(z)|^{p_2}}{(1-|z|^2)^{n+1}} \Big(\int_{\mathbb{B}} \chi_{D(z,R)}(w) d\mu(w) \Big) dv(z) \\
&\quad + C \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{|z|\leq\delta} \frac{|\nabla f_j(z)|^{p_2}}{(1-|z|^2)^{n+1}} \Big(\int_{\mathbb{B}} \chi_{D(z,R)}(w) d\mu(w) \Big) dv(z) \\
&= I + II \qquad (2)
\end{aligned}
$$

for any $0 < \delta < 1$.

Fix $\varepsilon > 0$. Since μ is a vanishing $p_2 - n - 1$-Carleson measure, by Lemma 3.2, there exists $\delta > 0$ such that for any $|z| > \delta$, $\mu(D(z,R)) \leq \varepsilon(1-|z|^2)^{p_2}$. It follows that

$$
\begin{aligned}
I &\leq C\varepsilon \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{|z|>\delta} \frac{|\nabla f_j(z)|^{p_2}}{(1-|z|^2)^{n+1}} (1-|z|^2)^{p_2} dv(z) \\
&= C\varepsilon \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{|z|>\delta} |\nabla f_j(z)|^{p_2} (1-|z|^2)^{p_2-n-1} dv(z) \\
&\leq C\varepsilon \frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \|f_j\|^{p_2}_{\mathcal{B}_{p_2}} < C\varepsilon. \qquad (3)
\end{aligned}
$$

If set $g(w) = w_1$, we have

$$\|\phi_1\|_{\mathcal{B}_{p_2}}^{p_2} = \|C_\phi g\|_{\mathcal{B}_{p_2}}^{p_2} = \int_{\mathbb{B}} |\nabla g(w)|^{p_2} d\mu(w) = \int_{\mathbb{B}} d\mu(w).$$

Observe that $(1-|z|^2)^{-n-1} \le (1-\delta^2)^{-n-1}$ when $|z| \le \delta$. Therefore

$$\begin{aligned} II &\le C\frac{16^{n+1}}{r^{2n}(1-r)^{n+1}} \int_{|z|\le\delta} \frac{|\nabla f_j(z)|^{p_2}}{(1-|z|^2)^{n+1}} \Big(\int_{\mathbb{B}} d\mu(w) \Big) dv(z) \\ &\le C\frac{16^{n+1}}{r^{2n}((1-r)(1-\delta^2))^{n+1}} \int_{|z|\le\delta} |\nabla f_j(z)|^{p_2} \|\phi_1\|_{\mathcal{B}_{p_2}}^{p_2} dv(z) \\ &< C\varepsilon \qquad (4) \end{aligned}$$

for j large enough, since $|\nabla f_j(z)| \to 0$ uniformly on compact sets. Combining (2), (3) and (4) we obtain that $\|C_\phi f_j\|_{\mathcal{B}_{p_2}} < C\varepsilon$ for j large enough. Therefore $\|C_\phi f_j\|_{\mathcal{B}_{p_2}} \to 0$ as $j \to \infty$, and Lemma 3.1 yields that $C_\phi : \mathcal{B}_{p_1} \to \mathcal{B}_{p_2}$ is a compact operator. This finishes the proof of Theorem 1.4. □

Then we can prove Theorem 1.5.

Proof of Theorem 1.5. First, suppose that $C_\phi : X \to \mathcal{B}$ is a compact operator. Since we have proved $\{k_\lambda : \lambda \in \mathbb{B}\}$ is a bounded set in $\mathcal{B}_p$ in Theorem 1.4, and it is easy to see that $\{k_\lambda : \lambda \in \mathbb{B}\}$ is a bounded set in $\mathcal{B}$, so $\{k_\lambda : \lambda \in \mathbb{B}\}$ is a bounded set in X. Obviously, $k_\lambda \to 0$ uniformly on compact sets as $|\lambda| \to 1$. Thus by Lemma 3.1,

$$\lim_{|\lambda|\to 1} \|C_\phi k_\lambda\|_{\mathcal{B}} = 0.$$

Conversely, suppose

$$\lim_{|\lambda|\to 1} \|C_\phi k_\lambda\|_{\mathcal{B}} = 0.$$

Let $\{f_j\}$ be a bounded sequence in X such that $f_j \to 0$ uniformly on compact sets as $n \to \infty$. We will show that

$$\lim_{|\lambda|\to 1} \|C_\phi f_j\|_{\mathcal{B}} = 0.$$

Let $\varepsilon > 0$ be given and fix $0 < \delta < 1$ such that if $|\lambda| > \delta$, then $\|C_\phi k_\lambda\|_{\mathcal{B}} < \varepsilon$. Hence for any $z_0 \in \mathbb{B}$ such that $|\phi(z_0)| > \delta$, $\|C_\phi k_{\phi(z_0)}\|_{\mathcal{B}} < \varepsilon$. In particular,

$$|\nabla k_{\phi(z_0)}(\phi(z_0))||\phi'(z_0)z_0|(1-|z_0|^2) < \varepsilon,$$

that is,

$$\frac{1-|\phi(z_0)|^2}{|1-\langle\phi(z_0),\phi(z_0)\rangle|^2}|\phi(z_0)||\phi'(z_0)z_0|(1-|z_0|^2) < \varepsilon,$$

i.e.,

$$\frac{|\phi'(z_0)z_0|}{1-|\phi(z_0)|^2}|\phi(z_0)|(1-|z_0|^2) < \varepsilon.$$

Therefore, for any $j \in N$ and $z_0 \in \mathbb{B}$ such that $|\phi(z_0)| > \delta$,

$$\Re(f_j\circ\phi)(z_0)(1-|z_0|^2) = |\nabla f_j(\phi(z_0))||\phi'(z_0)z_0|(1-|z_0|^2)$$

$$< \varepsilon|\nabla f_j(\phi(z_0))|\frac{(1-|\phi(z_0)|^2)}{|\phi(z_0)|}$$

$$< \frac{\varepsilon\|f_j\|_{\mathcal{B}}}{\delta}. \tag{5}$$

Since the set $A = \{w : |w| \leq \delta\}$ is a compact subset of $\mathbb{B}$ and $|\nabla f_j(w)| \to 0$ uniformly on compact sets, $\sup_{w\in A}|\nabla f_j(w)| \to 0$ as $j \to \infty$. Therefore we may choose j_0 large enough so that $|\nabla f_j(\phi(z))| < \varepsilon$ for any $j \geq j_0$ and any $z \in \mathbb{B}$ such that $|\phi(z)| \leq \delta$. Then, for all such z,

$$\begin{aligned}
\Re(f_j\circ\phi)(z)(1-|z|^2) &= |\nabla f_j(\phi(z))||\phi'(z)z|(1-|z|^2)\\
&< \varepsilon|\phi'(z)z|(1-|z|^2)\\
&= \varepsilon(1-|z|^2)\Big(\sum_{i=1}^n|\Re\phi_i(z)|^2\Big)^{\frac{1}{2}}\\
&\leq C\varepsilon\Big(\sum_{i=1}^n\|\phi_i\|_{\mathcal{B}}^2\Big)^{\frac{1}{2}}, \qquad (6)
\end{aligned}$$

where $j \geq j_0$. Thus, (5) and (6) yield $\|f_j\circ\phi\|_{\mathcal{B}} < C\varepsilon$ for $j \geq j_0$. Then $\|C_\phi f_j\|_{\mathcal{B}} \to 0$ as $j \to \infty$. Hence by Lemma 3.1, $C_\phi : X \to \mathcal{B}$ is a compact operator. □

An immediate consequence of Theorem 1.5 along with Lemma 3.1 and Theorem 1.4 is the following proposition.

Proposition 3.3. *Let $n < p_1 \leq p_2 < \infty$. Then if $C_\phi : \mathcal{B}_{p_1} \to \mathcal{B}_{p_2}$ is a compact operator, then so is $C_\phi : \mathcal{B} \to \mathcal{B}$.*

Proposition 3.4. *Let $n < r < p_2$, $n < p_1 \leq p_2 < \infty$. Suppose that $C_\phi : \mathcal{B}_r \to \mathcal{B}_r$ is a bounded operator. Then $C_\phi : \mathcal{B}_{p_1} \to \mathcal{B}_{p_2}$ is a compact operator if and only if $C_\phi : \mathcal{B} \to \mathcal{B}$ is a compact operator.*

Proof. First, suppose that C_ϕ is a compact operator on the Bloch space. Since $C_\phi : \mathcal{B}_r \to \mathcal{B}_r$ is a bounded operator, for any $\lambda \in \mathbb{B}$,

$$\begin{aligned}
\|C_\phi k_\lambda\|_{\mathcal{B}_{p_2}}^{p_2} &= \int_{\mathbb{B}}|\nabla k_\lambda(\phi(z))|^{p_2}|\phi'(z)z|^{p_2}(1-|z|^2)^{p_2-n-1}dv(z)\\
&= \int_{\mathbb{B}}|\nabla k_\lambda(\phi(z))|^{r}|\phi'(z)z|^{r}(1-|z|^2)^{r-n-1}\\
&\quad(|\nabla k_\lambda(\phi(z))||\phi'(z)z|(1-|z|^2))^{(p_2-r)}dv(z)\\
&\leq \|C_\phi k_\lambda\|_{\mathcal{B}}^{p_2-r}\|C_\phi k_\lambda\|_{\mathcal{B}_r}^{r}\\
&\leq C\|C_\phi k_\lambda\|_{\mathcal{B}}^{p_2-r}. \qquad (7)
\end{aligned}$$

Therefore (7) and Theorem 1.5 yield that $\|C_\phi k_\lambda\|_{\mathcal{B}_{p_2}} \to 0$ as $|\lambda| \to 1$. Thus by Theorem 1.4, $C_\phi : \mathcal{B}_{p_1} \to \mathcal{B}_{p_2}$ is a compact operator. The converse follows from Proposition 3.3. This finishes the proof of the proposition. □

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