Bull. Korean Math. Soc. **60** (2023), No. 3, pp. 611–622
https://doi.org/10.4134/BKMS.b210789
pISSN: 1015-8634 / eISSN: 2234-3016

ON THE HYBRID MEAN VALUE OF GENERALIZED DEDEKIND SUMS, GENERALIZED HARDY SUMS AND KLOOSTERMAN SUMS

Qing Tian and Yan Wang

Abstract. The main purpose of this paper is to study the hybrid mean value problem involving generalized Dedekind sums, generalized Hardy sums and Kloosterman sums. Some exact computational formulas are given by using the properties of Gauss sums and the mean value theorem of the Dirichlet L-function. A result of W. Peng and T. P. Zhang [12] is extended. The new results avoid the restriction that q is a prime.

1. Introduction

Let k be a positive integer. For arbitrary integers h, m, and n, the generalized Dedekind sum is defined by

$$S(h,m,n,k)=\sum_{j=1}^{k}\overline{B}_m\left(\frac{j}{k}\right)\overline{B}_n\left(\frac{hj}{k}\right),$$

where

$$\overline{B}_m(x)=\begin{cases} B_m(x-[x]), & \text{if } x \text{ is not an integer,}\\ 0, & \text{if } x \text{ is an integer}\end{cases}$$

with $B_m(x)$ the m-th Bernoulli polynomial. For $m=n=1$, $S(h,1,1,q)=S(h,q)$ is the classical Dedekind sum, which plays a great role in the study of modular forms theory and has attracted many experts in number theory (see [1,4,5,7]). In [3], Berndt gave certain sums called the classical Hardy sums which are closely connected with Dedekind sums. Some authors [13,14] expressed the Hardy sums in terms of the classical Dedekind sums. Liu and

Received October 29, 2021; Revised January 9, 2023; Accepted February 24, 2023.
2020 *Mathematics Subject Classification.* Primary 11F20, 11L05.
Key words and phrases. Hybrid mean value, Kloosterman sums, generalized Dedekind sums, generalized Hardy sums.
This work is supported by the Natural Science Basic Research Plan in Shaanxi Province of China (NO.2021JQ-495) and the National Natural Science Foundation of China (NO.61902304).

Zhang [10] generalized the Hardy sums as follows:

$$s_1(h,m,k)=\sum_{j=1}^{k}(-1)^{[\frac{hj}{k}]}\overline{B}_m\left(\frac{j}{k}\right),$$

$$s_2(h,m,n,k)=\sum_{j=1}^{k}(-1)^{j}\overline{B}_m\left(\frac{j}{k}\right)\overline{B}_n\left(\frac{hj}{k}\right),$$

$$s_3(h,n,k)=\sum_{j=1}^{k}(-1)^{j}\overline{B}_n\left(\frac{hj}{k}\right),$$

$$s_5(h,m,k)=\sum_{j=1}^{k}(-1)^{j+[\frac{hj}{k}]}\overline{B}_m\left(\frac{j}{k}\right).$$

For $m=n=1$, the sums are reduced to the classical Hardy sums. They also expressed the generalized Hardy sums in terms of generalized Dedekind sums, that is:

Proposition 1.1. *Let h, k be positive integers with $(h,k)=1$. Then*

$$\begin{cases} s_1(h,m,k)=2\cdot S(h,m,1,k)-4\cdot S(\frac{h}{2},m,1,k), & \textit{if } h \textit{ is even number,}\\ s_2(h,m,n,k)=2^m\cdot S(2h,m,n,k)-S(h,m,n,k), & \textit{if } k \textit{ is even number,}\\ s_3(h,n,k)=2\cdot S(h,1,n,k)-4\cdot S(2h,1,n,k), & \textit{if } k,n \textit{ are odd number,}\\ s_5(h,m,k)=2^{m+1}\cdot S(2h,m,1,k)+2^{m+1}\cdot S(h,m,1,2k) & \\ \qquad -(2+2^{m+2})\cdot S(h,m,1,k), & \textit{if } h+k \textit{ is even number,}\end{cases}$$

where $\bar{2}\cdot 2\equiv 1 \bmod q$. Moreover, each one of

$$\begin{cases} s_1(h,m,k)\ (h+m: \textit{even}), & s_2(h,m,n,k)\ (h+m+k: \textit{odd}),\\ s_3(h,n,k)\ (h+k: \textit{odd}), & s_5(h,m,k)\ (h+m+k: \textit{even})\end{cases}$$

is zero.

There are many results on the mean value properties of Dedekind sums ([6,8,9]). Here we concentrate on the hybrid mean value of Dedekind sums and Hardy sums with Kloosterman sums.

Kloosterman sum is defined by

$$K(n,q)=\sideset{}{'}\sum_{c=1}^{q} e\left(\frac{nc+\bar{c}}{q}\right),$$

where $\sum_{c=1}^{\prime q}$ denotes the summation over all c such that $(c,q)=1$, $e(y)=\exp(2\pi i y)$ and $\bar{c}\cdot c\equiv 1 \bmod q$.

Liu and Zhang [11, 15] considered the problems on high power mean value of Dedekind sums with Kloosterman sums, and obtained asymptotic formulas.

Peng [12] and Zhang [16] studied the hybrid mean value involving certain classical Hardy sums and Kloosterman sums

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} K(a,p)K(b,p)s_1(2a\bar{b},1,p)$$

and

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} K(a,p)K(b,p)s_5(a\bar{b},1,p),$$

respectively, where p is a prime. Some computational formulas are given for revealing the cancellation phenomenon between the functions.

However, the results in papers [12] and [16] were proved in the condition q was a prime p. Naturally, one might consider whether the hybrid mean value be extended to generalized Dedekind sums $S(h,m,n,q)$ or certain generalized Hardy sums with Kloosterman sums $K(n,q)$ under the condition of composite number q. If it is true, then what can be expected? These problems may be interesting. In this paper, we shall study the problems and give some exact computational formulas using the properties of Gauss sums and the mean value theorem of the Dirichlet L-function. We shall prove the following:

Theorem 1.1. *Let q be a square-full number, and $m \equiv n \equiv 1 \pmod 2$. Then*

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(\bar{a}b,m,n,q)$$
$$= q^{4-2m-2n}\cdot \sum_{l=0}^{m+n} q^{l}\cdot r_{m,n,l}\cdot \prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{l}}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right),$$

where

$$r_{m,n,l} = B_{m+n-l}\sum_{\substack{a=0\\ a+b\ge m+n-l}}^{m}\sum_{b=0}^{n} B_{m-a}B_{n-b}\frac{\binom{m}{a}\binom{n}{b}\binom{a+b+1}{m+n-l}}{a+b+1},$$

B_m is a Bernoulli number, $\binom{m}{a} = \frac{m!}{a!(m-a)!}$, $\phi_l(q) = \prod_{p|q}(1-\frac{1}{p^l})$, $\prod_{p|q}$ denotes the products of all prime divisors of q and $\phi(q) = q\phi_1(q)$.

Theorem 1.2. *Let q be a square-full number, and $m \equiv 1 \pmod 2$. Then*

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)s_1(2\bar{a}b,m,q)$$
$$= q^{m-2}\cdot \sum_{l=0}^{m+1} q^{l-m}\cdot r_{m,1,l}\cdot \frac{-2^{l}-2}{2^{m-1}+1}\cdot \prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{l}}\right)\left(1-\frac{1}{p^{l-m}}\right).$$

Theorem 1.3. *Let q be a square-full even number, and $m \equiv n \equiv 1 \pmod 2$. Then*

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)s_2(\bar{a}b,m,n,q)$$
$$= q^{2-2m} \cdot \sum_{l=0}^{m+1} r_{m,1,l} \cdot \left(2^m \frac{2^m - 2^{l-1}+1}{2^{m-1}+1} - 1\right) \cdot q^{l-m} \cdot \prod_{p|q}\left(1-\frac{1}{p}\right)$$
$$\times \left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m}}\right).$$

Theorem 1.4. *Let q be a square-full odd number, and $n \equiv 1 \pmod 2$. Then*

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)s_3(\bar{a}b,n,q)$$
$$= q^{2-2n} \cdot \sum_{l=0}^{n+1} r_{1,n,l} \cdot q^l \left(2 - 4 \cdot \frac{2^l - 2^{n+1} - 2}{2^n + 2}\right) \cdot \prod_{p|q}\left(1-\frac{1}{p}\right)$$
$$\times \left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-n}}\right).$$

The present work is a generalization of [12] and [16]. For a general number $q > 2$, we can only get some asymptotic formulas, whether there exist the identities for the hybrid mean value are open problems.

2. Several lemmas

Before starting proof of theorems, several lemmas will be useful.

Lemma 2.1. *Let h, q be positive integers with $q \geq 3$ and $(h,q) = 1$, and $m \equiv n \equiv 1 \pmod 2$. Then we have*

$$S(h,m,n,q) = \frac{-4m!n!}{(2\pi i)^{m+n} q^{m+n-1}} \cdot \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \overline{\chi}(h) L(m,\chi) L(n,\overline{\chi}),$$

where $\sum_{d|q}$ denotes the sums over all divisors of q and $L(m,\chi)$ denotes the Dirichlet L-function corresponding to character $\chi \bmod d$.

Proof. See Theorem 2.3 of [10]. □

Lemma 2.2. *Let $q \geq 3$ be an integer and χ be a non-principal character modulo q. Then we have*

$$\sum_{a=1}^{q}{}' \chi(a)K(a,q) = \tau^2(\chi).$$

Proof. See Theorem 3.1 of [11]. □

Lemma 2.3. *Let q be a square-full number. Then for any non-primitive character χ modulo q, we have the identity*

$$\tau(\chi) = \sum_{a=1}^{q} \chi(a) e\left(\frac{a}{q}\right) = 0.$$

Proof. See Lemma 2.4 of [11]. □

Lemma 2.4. *Let $q \geq 2$ be an integer, $m \equiv n \equiv 1 \pmod 2$. Then we have*

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi})$$
$$= -\frac{(2\pi i)^{m+n}\phi(q)}{4m!n!}\left(\sum_{l=0}^{m+n} r_{m,n,l}\cdot\phi_l(q)\cdot q^{l-m-n} - \frac{B_m B_n \phi_{m+n-1}(q)}{q}\right).$$

Proof. See Theorem 3 of [10]. □

Lemma 2.5. *Let q be a square-full number. Then we have*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi})$$
$$= -\frac{(2\pi i)^{m+n}}{4m!n!}\sum_{l=0}^{m+n} r_{m,n,l}\cdot q^{l-m-n+1}\cdot\prod_{p\mid q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right),$$

where $\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ *denotes the sums over all odd primitive characters modulo q.*

Proof. Noting that q is a square-full number and

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi}) = \sum_{d\mid q}\sideset{}{^*}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} L(m,\chi\chi_q^0)L(n,\overline{\chi}\chi_q^0),$$

by using the Möbius inversion formula, we have

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi}) &= \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m,\chi\chi_q^0)L(n,\overline{\chi}\chi_q^0) \\ &= \sum_{d\mid q}\mu(d)\sum_{\substack{\chi \bmod \frac{q}{d} \\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi}),\end{aligned}$$

where χ_q^0 denotes the principle character modulo q.

According to Lemma 2.4, we obtain that

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi})$$

$$= -\frac{(2\pi i)^{m+n}}{4m!n!}\cdot\left[\sum_{l=0}^{m+n} r_{m,n,l}\cdot\sum_{d|q}\mu(d)\phi(\frac{q}{d})\phi_l\left(\frac{q}{d}\right)\left(\frac{q}{d}\right)^{l-m-n}\right.$$
$$\left.+B_mB_n\cdot\sum_{d|q}\mu(d)\phi(\frac{q}{d})\frac{\phi_{m+n-1}(\frac{q}{d})}{\frac{q}{d}}\right].$$

Since $\mu(n)$, $\phi(n)$ and $\phi_l(n)$ are all multiplicative functions, we get

$$\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right)\phi_l\left(\frac{q}{d}\right)\left(\frac{q}{d}\right)^{l-m-n}$$
$$=\prod_{p|q}\sum_{d|p^\alpha}\mu(d)\phi\left(\frac{p^\alpha}{d}\right)\phi_l\left(\frac{p^\alpha}{d}\right)\left(\frac{p^\alpha}{d}\right)^{l-m-n}$$
$$=q^{l-m-n+1}\cdot\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right).$$

For the same reason,

$$\sum_{d|q}\mu(d)\phi\left(\frac{q}{d}\right)\phi_{m+n-1}\left(\frac{q}{d}\right)\frac{d}{q}=0.$$

Based on the above analysis, we obtain

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi})$$
$$=-\frac{(2\pi i)^{m+n}}{4m!n!}\cdot\sum_{l=0}^{m+n} r_{m,n,l}\cdot q^{l-m-n+1}\cdot\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right).$$

This proves Lemma 2.5. □

Lemma 2.6. *Let q be a square-full number. Then we have the identity*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} L(m,\chi\chi_2^0)L(n,\overline{\chi}\chi_2^0)$$
$$= -\frac{(2\pi i)^{m+n}}{4m!n!}\cdot\sum_{l=0}^{m+n} r_{m,n,l}\cdot\left(1-\frac{1}{2^l}\right)\cdot 2^{l-m-n}\cdot q^{l-m-n+1}\cdot\prod_{p|q}\left(1-\frac{1}{p}\right)$$
$$\times\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right).$$

Proof. By the identity

$$\sum_{\substack{\chi \bmod 2q\\ \chi(-1)=-1}} L(m,\chi)L(n,\overline{\chi}) = \sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}} L(m,\chi\chi_2^0)L(n,\overline{\chi}\chi_2^0)$$

$$= \sum_{d|q} \sideset{}{^*}\sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} L(m, \chi\chi_{2q}^0) L(n, \overline{\chi}\chi_{2q}^0),$$

and the Möbius inversion formula, we have

$$\begin{aligned}
\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m, \chi\chi_2^0) L(n, \overline{\chi}\chi_2^0) &= \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m, \chi\chi_{2q}^0) L(n, \overline{\chi}\chi_{2q}^0) \\
&= \sum_{d|q} \mu(d) \sum_{\substack{\chi \bmod \frac{q}{d} \\ \chi(-1)=-1}} L(m, \chi\chi_{2q}^0) L(n, \overline{\chi}\chi_{2q}^0) \\
&= \sum_{d|q} \mu(d) \sum_{\substack{\chi \bmod \frac{q}{d} \\ \chi(-1)=-1}} L(m, \chi\chi_{\frac{2q}{d}}^0) L(n, \overline{\chi}\chi_{\frac{2q}{d}}^0) \\
&= \sum_{d|q} \mu(d) \sum_{\substack{\chi \bmod \frac{2q}{d} \\ \chi(-1)=-1}} L(m, \chi) L(n, \overline{\chi}).
\end{aligned}$$

According to Lemma 2.5, it follows that

$$\begin{aligned}
&\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} L(m, \chi\chi_2^0) L(n, \overline{\chi}\chi_2^0) \\
&= -\frac{(2\pi i)^{m+n}}{4m!n!} \cdot \left[\sum_{l=0}^{m+n} r_{m,n,l} \cdot \sum_{d|q} \mu(d) \phi\left(\frac{2q}{d}\right) \phi_l\left(\frac{2q}{d}\right) \left(\frac{2q}{d}\right)^{l-m-n+1} \right. \\
&\qquad \left. + \sum_{d|q} \mu(d) \phi\left(\frac{2q}{d}\right) \cdot \frac{B_m B_n \phi_{m+n-1}\left(\frac{2q}{d}\right)}{\frac{2q}{d}} \right] \\
&= -\frac{(2\pi i)^{m+n}}{4m!n!} \cdot \sum_{l=0}^{m+n} r_{m,n,l} \cdot \left(1 - \frac{1}{2^l}\right) \cdot 2^{l-m-n} \cdot q^{l-m-n+1} \cdot \prod_{p|q} \left(1 - \frac{1}{p}\right) \\
&\quad \times \left(1 - \frac{1}{p^l}\right) \left(1 - \frac{1}{p^{l-m-n+1}}\right).
\end{aligned}$$

This proves Lemma 2.6. □

Lemma 2.7. *Let q be a square-full odd number. Then we have*

$$\begin{aligned}
&\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(2) L(m, \chi) L(n, \overline{\chi}) \\
&= -\frac{(2\pi i)^{m+n}}{4m!n!} \cdot \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^{l-m-n+1} \cdot \frac{2^l - 2^{m+n} - 2}{2^m + 2^n} \cdot \prod_{p|q} \left(1 - \frac{1}{p}\right)
\end{aligned}$$

$$\times\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right).$$

Proof. With the Euler infinite product formula (see Theorem 11.6 of [2]), we get

$$L(m,\chi\chi_2^0)=\prod_{p_1}\left(1-\frac{\chi(p_1)\chi_2^0(p_1)}{p_1^m}\right)^{-1}=\prod_{p_1>2}\left(1-\frac{\chi(p_1)}{p_1^m}\right)^{-1}$$
$$=\left(1-\frac{\chi(2)}{2^m}\right)\prod_{p_1}\left(1-\frac{\chi(p_1)}{p_1^m}\right)^{-1}=\left(1-\frac{\chi(2)}{2^m}\right)L(m,\chi),$$

where $\prod\limits_p$ denotes the product over all primes p.

We know that

$$L(m,\chi\chi_2^0)L(n,\overline{\chi}\chi_2^0)=L(m,\chi)L(n,\overline{\chi})\left(1-\frac{\chi(2)}{2^m}\right)\left(1-\frac{\overline{\chi}(2)}{2^n}\right)$$
$$=L(m,\chi)L(n,\overline{\chi})\left[1+\frac{1}{2^{m+n}}-\left(\frac{\chi(2)}{2^m}+\frac{\overline{\chi}(2)}{2^n}\right)\right].$$

Note that $\sideset{}{^*}\sum\limits_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\chi(2)=\sideset{}{^*}\sum\limits_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(2)$, we can get

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\overline{\chi}(2)L(m,\chi)L(n,\overline{\chi})$$
$$=\sideset{}{^*}\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}\left(\frac{1}{2^n}+\frac{1}{2^m}\right)^{-1}\cdot\left(L(m,\chi\chi_2^0)L(n,\overline{\chi}\chi_2^0)-\left(1+\frac{1}{2^{m+n}}\right)\cdot L(m,\chi)L(n,\overline{\chi})\right)$$
$$=-\frac{(2\pi i)^{m+n}}{4m!n!}\cdot\sum_{l=0}^{m+n}r_{m,n,l}\cdot q^{l-m-n+1}\cdot\frac{2^{l-1}-2^{m+n-1}-1}{2^{m-1}+2^{n-1}}\cdot\prod_{p|q}\left(1-\frac{1}{p}\right)$$
$$\times\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m-n+1}}\right).$$

This proves Lemma 2.7. □

3. Proof of theorems

In this section, we shall complete the proof of theorems.

First we give a hybrid mean value formula for generalized Dedekind sums with Kloosterman sums.

The Gauss sum associated with χ modulo q is defined by

$$G(n,\chi)=\sum_{a=1}^{q}\chi(a)e\left(\frac{na}{q}\right)$$

and the classical Gauss sum $\tau(\chi) = G(1, \chi)$.

Note that if χ is a primitive character modulo q, then $|\tau(\chi)| = \sqrt{q}$ and

$$\left| \sum_{a=1}^{q}{}' \chi(a) K(a,q) \right| = |\tau^2(\chi)| = q.$$

Proof of Theorem 1.1. If q is a square-full number and $m \equiv n \equiv 1 \pmod 2$, with the results of Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(\bar{a}b, m, n, q) \\
&= -\frac{4m!n!}{(2\pi i)^{m+n} q^{m+n-1}} \cdot \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)\overline{\chi}(\bar{a}b) L(m,\chi) L(n,\overline{\chi}) \\
&= -\frac{4m!n!}{(2\pi i)^{m+n} q^{m+n-1}} \cdot \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{a=1}^{q}{}' \chi(a) K(a,q) \right|^2 \cdot L(m,\chi) L(n,\overline{\chi}) \\
&= -\frac{4m!n!}{(2\pi i)^{m+n} q^{m+n-1}} \cdot \frac{q^{m+n}}{\phi(q)} \cdot \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} q^2 \cdot L(m,\chi) L(n,\overline{\chi}) \\
&= \frac{q^{4-m-n}}{\phi(q)} \cdot \sum_{l=0}^{m+n} r_{m,n,l} \cdot q^l \cdot \prod_{p|q} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^l}\right)\left(1 - \frac{1}{p^{l-m-n+1}}\right).
\end{aligned}$$

This completes the proof of Theorem 1.1. □

Following a similar idea to the proof of Theorem 1.1, we give some hybrid mean value formulas for generalized Hardy sums and Kloosterman sums, that is we will prove the rest of theorems.

Proof of Theorem 1.2. From Proposition 1.1, Lemma 2.5 and Lemma 2.7, if q is a square-full number and $m \equiv 1 \pmod 2$, we have

$$\begin{aligned}
&\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)s_1(2\bar{a}b, m, q) \\
&= 2\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(2\bar{a}b, m, 1, q) - 4\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(\bar{a}b, m, 1, q) \\
&= -\frac{4m!}{(2\pi i)^{m+1} q^m} \cdot \sum_{d|q} \frac{d^{m+1}}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{a=1}^{q}{}' \chi(a) K(a,q) \right|^2 \cdot (2\overline{\chi}(2) - 4) \cdot L(m,\chi) L(1,\overline{\chi})
\end{aligned}$$

$$= -\frac{4m!}{(2\pi i)^{m+1}q^m} \cdot \frac{q^{m+1}}{\phi(q)} \cdot \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} q^2 \cdot (2\overline{\chi}(2)-4) \cdot L(m,\chi)L(1,\overline{\chi})$$

$$= \frac{q^{3-m}}{\phi(q)} \cdot \sum_{l=0}^{m+1} r_{m,1,l} \cdot q^{l-m} \cdot \frac{2^l - 2^{m+2} - 6}{2^{m-1}+1} \prod_{p|q} \left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m}}\right).$$

This proves Theorem 1.2. □

Similarly, we will deduce the identities involving generalized Hardy sums $s_2(\bar{a}b, m, n, q)$ and $s_3(\bar{a}b, n, q)$ with Kloosterman sums, respectively.

Proof of Theorem 1.3. If q is a square-full even number and $m \equiv n \equiv 1 \pmod 2$, then we have

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)s_2(\bar{a}b, m, n, q)$$

$$= 2^m \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(2\bar{a}b, m, n, q) - \sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(\bar{a}b, m, n, q)$$

$$= -\frac{4m!n!}{(2\pi i)^{m+n}q^{m+n-1}} \cdot \sum_{d|q} \frac{d^{m+n}}{\phi(d)} \cdot \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{a=1}^{q}{}' \chi(a)K(a,q) \right|^2 \cdot (2^m\overline{\chi}(2)-1)L(m,\chi)L(n,\overline{\chi})$$

$$= -\frac{4m!n!}{(2\pi i)^{m+n}q^{m+n-1}} \cdot \frac{q^{m+n}}{\phi(q)} \cdot \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}}^{*} q^2 \cdot (2^m\overline{\chi}(2)-1) \cdot L(m,\chi)L(n,\overline{\chi})$$

$$= \frac{q^{4-m-n}}{\phi(q)} \sum_{l=0}^{m+n} r_{m,n,l} \cdot \left(\frac{2^l - 2^{m+n} - 2^{n-m} - 3}{2^{n-m}+1}\right) \cdot q^l \cdot \prod_{p|q} \left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-m}}\right).$$

This proves Theorem 1.3. □

Proof of Theorem 1.4. If q is a square-full odd number and $n \equiv 1 \pmod 2$, then we have

$$\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)s_3(\bar{a}b, n, q)$$

$$= 2\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(\bar{a}b, 1, n, q) - 4\sum_{a=1}^{q}{}' \sum_{b=1}^{q}{}' K(a,q)K(b,q)S(2\bar{a}b, 1, n, q)$$

$$= -\frac{4n!}{(2\pi i)^{1+n}q^n} \cdot \sum_{d|q} \frac{d^{n+1}}{\phi(d)} \cdot \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \left| \sum_{a=1}^{q}{}' \chi(a)K(a,q) \right|^2 \cdot (2-4\overline{\chi}(2)) \cdot L(1,\chi)L(n,\overline{\chi})$$

$$= -\frac{4n!}{(2\pi i)^{1+n}q^n}\cdot\frac{q^{n+1}}{\phi(q)}\cdot\sum_{\substack{\chi \bmod q\\ \chi(-1)=-1}}^{*} q^2\cdot(2-4\overline{\chi}(2))\cdot L(1,\chi)L(n,\overline{\chi})$$

$$=\frac{q^{3-n}}{\phi(q)}\cdot\sum_{l=0}^{n+1} r_{1,n,l}\cdot q^l\cdot\left(\frac{5\cdot 2^n-2^{l+1}+6}{2^{n-1}+1}\right)\cdot\prod_{p|q}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^l}\right)\left(1-\frac{1}{p^{l-n}}\right).$$

This completes the proof of Theorem 1.4. □

Acknowledgment. The authors express their gratitude to the referee for his (her) very helpful and detailed comments.

References

[1] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Graduate Texts in Mathematics, No. 41, Springer, New York, 1976.

[2] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, New York, 1976.

[3] B. C. Berndt, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan*, J. Reine Angew. Math. **303(304)** (1978), 332–365. https://doi.org/10.1515/crll.1978.303-304.332

[4] L. Carlitz, *The reciprocity theorem for Dedekind sums*, Pacific J. Math. **3** (1953), 523–527. http://projecteuclid.org/euclid.pjm/1103051326

[5] J. B. Conrey, E. Fransen, R. Klein, and C. Scott, *Mean values of Dedekind sums*, J. Number Theory **56** (1996), no. 2, 214–226. https://doi.org/10.1006/jnth.1996.0014

[6] D. Han and W. Zhang, *Some new identities involving Dedekind sums and the Ramanujan sum*, Ramanujan J. **35** (2014), no. 2, 253–262. https://doi.org/10.1007/s11139-014-9591-6

[7] C. H. Jia, *On the mean value of Dedekind sums*, J. Number Theory **87** (2001), no. 2, 173–188. https://doi.org/10.1006/jnth.2000.2580

[8] X. X. Li and W. Zhang, *A hybrid mean value involving a new Gauss sums and Dedekind sums*, Bull. Iranian Math. Soc. **43** (2017), no. 6, 1957–1968.

[9] L. Liu, Z. F. Xu, and N. Wang, *Mean values of generalized Dedekind sums over short intervals*, Acta Arith. **193** (2020), no. 1, 95–108. https://doi.org/10.4064/aa181111-18-2

[10] H. N. Liu and W. Zhang, *Generalized Dedekind sums and Hardy sums*, Acta Math. Sinica (Chinese Ser.) **49** (2006), no. 5, 999–1008.

[11] Y. N. Liu and W. Zhang, *A hybrid mean value related to the Dedekind sums and Kloosterman sums*, Acta Math. Sin. (Engl. Ser.) **27** (2011), no. 3, 435–440. https://doi.org/10.1007/s10114-010-9192-2

[12] W. Peng and T. P. Zhang, *Some identities involving certain Hardy sum and Kloosterman sum*, J. Number Theory **165** (2016), 355–362. https://doi.org/10.1016/j.jnt.2016.01.028

[13] M. R. Pettet and R. Sitaramachandra Rao, *Three-term relations for Hardy sums*, J. Number Theory **25** (1987), no. 3, 328–339. https://doi.org/10.1016/0022-314X(87)90036-9

[14] R. Sitaramachandra Rao, *Dedekind and Hardy sums*, Acta Arith. **48** (1987), no. 4, 325–340. https://doi.org/10.4064/aa-48-4-325-340

[15] W. Zhang and Y. N. Liu, *A hybrid mean value related to the Dedekind sums and Kloosterman sums*, Sci. China Math. **53** (2010), no. 9, 2543–2550. https://doi.org/10.1007/s11425-010-3153-1

[16] H. Zhang and W. Zhang, *On the identity involving certain Hardy sums and Kloosterman sums*, J. Inequal. Appl. **2014** (2014), 52, 9 pp. https://doi.org/10.1186/1029-242X-2014-52

Qing Tian
College of Science
Xi'an University of Architecture and Technology
Xi'an, Shaanxi, 710055, P. R. China
Email address: qingtian@xauat.edu.cn

Yan Wang
College of Science
Xi'an University of Architecture and Technology
Xi'an, Shaanxi, 710055, P. R. China