Bull. Korean Math. Soc. **60** (2023), No. 3, pp. 575–591
https://doi.org/10.4134/BKMS.b210652
pISSN: 1015-8634 / eISSN: 2234-3016

# EXTENSIONS OF MULTIPLE LAURICELLA AND HUMBERT'S CONFLUENT HYPERGEOMETRIC FUNCTIONS THROUGH A HIGHLY GENERALIZED POCHHAMMER SYMBOL AND THEIR RELATED PROPERTIES

Ritu Agarwal, Junesang Choi, Naveen Kumar, and Rakesh K. Parmar

Abstract. Motivated by several generalizations of the Pochhammer symbol and their associated families of hypergeometric functions and hypergeometric polynomials, by choosing to use a very generalized Pochhammer symbol, we aim to introduce certain extensions of the generalized Lauricella function $F_A^{(n)}$ and the Humbert's confluent hypergeometric function $\Psi^{(n)}$of $n$ variables with, as their respective particular cases, the second Appell hypergeometric function $F_2$ and the generalized Humbert's confluent hypergeometric functions $\Psi_2$ and investigate their several properties including, for example, various integral representations, finite summation formulas with an $s$-fold sum and integral representations involving the Laguerre polynomials, the incomplete gamma functions, and the Bessel and modified Bessel functions. Also, pertinent links between the major identities discussed in this article and different (existing or novel) findings are revealed.

## 1. Introduction and preliminaries

The literature on special functions contains diverse extensions (or generalizations) and applications of the gamma function $\Gamma(z)$, the incomplete gamma function the Beta function $B(\alpha, \beta)$ (see, e.g., [1, 3, 4, 7–15, 21, 26, 27, 29, 47]), the confluent hypergeometric and hypergeometric functions ${}_1F_1(z)$ and ${}_2F_1(z)$, and the generalized hypergeometric functions ${}_rF_s(z)$ with $r$ numerator and $s$ denominator parameters (see, e.g., [16, 17, 19, 23, 25, 31–34, 37–40, 43, 44] and

Received September 1, 2021; Accepted March 10, 2023.
2020 *Mathematics Subject Classification.* Primary 33B15, 33B20, 33C05, 33C15, 33C20; Secondary 33B99, 33C99, 60B99.
*Key words and phrases.* Generalized gamma functions, generalized Pochhammer symbol, generalized Gauss hypergeometric function, Lauricella functions, Appell functions, Laguerre polynomials, incomplete gamma functions, Bessel and modified Bessel functions, generalized second Appell function in two variables, generalized multivariable Lauricella functions, generalized multivariable Humbert's confluent hypergeometric function.

the references cited therein). In particular, for an appropriately bounded sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ of essentially arbitrary (real or complex) numbers, Srivastava et al. [43, Eq. (2.1)] recently considered the function $\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z)$ given by

$$\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z) := \begin{cases} \sum_{\ell=0}^{\infty} \kappa_\ell \dfrac{z^\ell}{\ell!} \quad (|z| < R;\ 0 < R < \infty;\ \kappa_0 := 1), \\ \mathfrak{M}_0\ z^\omega\ \exp(z) \left[1 + O\left(\dfrac{1}{z}\right)\right] \\ \big(\Re(z) \to \infty;\ \mathfrak{M}_0 > 0;\ \omega \in \mathbb{C}\big) \end{cases} \tag{1}$$

for some suitable constants $\mathfrak{M}_0$ and $\omega$ depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. In terms of the function $\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z)$ defined by (1), Srivastava et al. [43] proposed and explored the following astonishingly profound generalizations of the extended Gamma function:

$$\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(z) = \int_0^\infty t^{z-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t - \frac{p}{t}\right)\, \mathrm{d}t$$

$$\big(\Re(z) > 0;\ \Re(p) \geqq 0\big).$$

In this section and throughout, let $\mathbb{C}$, $\mathbb{Z}$, and $\mathbb{N}$ represent the sets of complex numbers, integers, and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_{\leq\eta}$ be the set of integers less than or equal to a fixed $\eta \in \mathbb{Z}$. R. Srivastava has proposed and conducted a thorough analysis of the following new family of generalized hypergeometric functions [37, p. 2203, Eq. (64)]:

$$\begin{aligned} &{}_rF_s\left[\begin{matrix}(\alpha_1, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \alpha_2, \ldots, \alpha_r; \\ \beta_1, \ldots, \beta_s;\end{matrix}\ z\right] \\ &= \sum_{n=0}^{\infty} \frac{(\alpha_1, p; \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_n\, (\alpha_2)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n}\, \frac{z^n}{n!}, \end{aligned} \tag{2}$$

in terms of the generalized Pochhammer symbol $(\lambda, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_\nu$ defined by [37, p. 2203, Eq. (63)]:

$$(\lambda, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_\nu := \frac{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\lambda+\nu)}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\lambda)} \qquad \big(\Re(p) > 0;\ \lambda, \nu \in \mathbb{C}\big), \tag{3}$$

or, equivalently, by means of an integral representation as follows:

$$(\lambda, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_\nu = \frac{1}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\lambda)} \int_0^\infty t^{\lambda+\nu-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t - \frac{p}{t}\right) \mathrm{d}t \tag{4}$$

$$\big(\Re(p) > 0;\ \Re(\lambda+\nu) > 0\big).$$

In this paper, by using the generalized Pochhammer symbol in (3), we aim to further extend the multiple Lauricella function $F_A^{(n)}$ (see, e.g., [41, p. 33, Eq. (1)]) and the multiple Humbert's confluent hypergeometric function $\Psi_2^{(n)}$ (see, e.g., [41, p. 34, Eq. (9)]) together with, as their respective particular cases, the second Appell hypergeometric function $F_2$ (see, e.g., [41, p. 23, Eq. (3)]) and the generalized Humbert's confluent hypergeometric function $\Psi_2$ (see, e.g., [41, p. 26, Eq. (22)]) and make a systematic investigation of their several properties such as various integral representations, finite summation formulas with an $s$-fold sum and integral representations involving the Laguerre polynomials, the incomplete gamma functions, and the Bessel and modified Bessel functions. Also, we make pertinent links between this paper's major findings and a variety of (existing or novel) identities.

## 2. Further extensions

In this part, using the generalized Pochhammer symbol in (3), we propose further extensions of the four functions described in the previous section, as defined in the following definition.

**Definition.** Let $\alpha$, $\beta_1, \ldots, \beta_n \in \mathbb{C}$ and $\gamma_1, \ldots, \gamma_n \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. Also let $p \in \mathbb{C}$ be such that $\Re(p) \geq 0$. Then

(i) A generalization $\mathscr{F}_A^{(n)}$ of the Lauricella's hypergeometric function $F_A^{(n)}$ of $n$ variables is defined by

$$\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n] := \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha; p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+\cdots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma_1)_{m_1} \cdots (\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}. \tag{5}$$

Here if $\Re(\alpha) > 0$ and $\Re(p) > 0$, then (5) converges absolutely for all $x_1, \ldots, x_n \in \mathbb{C}$; if $p = 0$, then (5) converges absolutely for $|x_1| + \cdots + |x_n| < 1$.

(ii) A generalization $\mathscr{F}_2$ of the Appell hypergeometric function $F_2$ is defined by

$$\begin{aligned} &\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] \\ &:= \sum_{m_1,m_2=0}^{\infty} \frac{(\alpha; p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+m_2} (\beta_1)_{m_1} (\beta_2)_{m_2}}{(\gamma_1)_{m_1} (\gamma_2)_{m_2}} \frac{x_1^{m_1}}{m_1!} \frac{x_2^{m_2}}{m_2!} \\ &= \sum_{m_1=0}^{\infty} \frac{(\alpha)_{m_1} (\beta_1)_{m_1}}{(\gamma_1)_{m_1}} \, {}_2F_1 \left[ \begin{matrix} (\alpha + m_1, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_2; \\ \gamma_2; \end{matrix} \; x_2 \right] \frac{x_1^{m_1}}{m_1!}. \end{aligned} \tag{6}$$

Here (6) is a particular case of (5) when $n = 2$; the function ${}_2F_1$ is a particular case of (2) when $n = 1$, that is (see [37]):

$${}_2F_1 \left[ \begin{matrix} (\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1; \\ \gamma_1; \end{matrix} \; x_1 \right] := \sum_{m_1=0}^{\infty} \frac{(\alpha; p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1} (\beta_1)_{m_1}}{(\gamma_1)_{m_1}} \frac{x_1^{m_1}}{m_1!}.$$

(iii) A generalization $\mathscr{P}_2^{(n)}$ of the Humbert's confluent hypergeometric function $\Psi_2^{(n)}$ of $n$ variables is defined by

$$\mathscr{P}_2^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}); \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n] \tag{7}$$

$$:= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(\alpha;p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+\cdots+m_n}}{(\gamma_1)_{m_1}\cdots(\gamma_n)_{m_n}} \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!}$$

$$= \lim_{\min\{|\beta_1|,\ldots,|\beta_n|\}\to\infty} \left\{ \mathscr{F}_A^{(n)} \left[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;\frac{x_1}{\beta_1},\ldots,\frac{x_n}{\beta_n}\right]\right\}.$$

Here (7) converges absolutely for all $x_1, \ldots, x_n \in \mathbb{C}$ and is a confluent form of (5).

(iv) A generalization $\mathscr{P}_2$ of the Humbert's confluent hypergeometric function $\Psi_2$ of 2 variables is defined by

$$\mathscr{P}_2[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0});\gamma_1,\gamma_2;x_1,x_2] \tag{8}$$

$$:= \sum_{m_1,\,m_2=0}^{\infty} \frac{(\alpha;p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+m_2}}{(\gamma_1)_{m_1}(\gamma_2)_{m_2}} \frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}$$

$$= \lim_{\min\{|\beta_1|,|\beta_2|\}\to\infty} \left\{ \mathscr{F}_2 \left[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),\beta_1,\beta_2;\gamma_1,\gamma_2;\frac{x_1}{\beta_1},\frac{x_2}{\beta_2}\right]\right\}.$$

Here (8) is a particular case of (7) when $n = 2$ and is a confluent form of (6).

*Remark* 2.1. Choi et al. [18] used the incomplete Pochhammer symbol (see [18, Eq. (1.6)]) to introduce and investigate the families of the incomplete Lauricella hypergeometric functions $\gamma_A^{(n)}$ and $\Gamma_A^{(n)}$ of $n$ variables (see Equations (2.1) and (2.2) in [18]). Here, it may be obvious that the generalized functions in (5), (6) and (7) using the generalized Pochhammer symbol (3) (or (4)) differ from those in [18, Equations (2.1) and (2.2)], mainly because the generalized Pochhammer symbol (3) (or (4)) and the incomplete Pochhammer symbol (see [18, Eq. (1.6)]) are distinct.

## 3. Integral representations

In this section, by using the integral representation (4) of the generalized Pochhammer symbol $(\lambda;p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_\nu$, we construct certain integral representations of the generalized Lauricella function $\mathscr{F}_A^{(n)}$ in (5) and the generalized Humbert's confluent hypergeometric function $\mathscr{P}_2^{(n)}$. We also present corresponding integral representations for the generalized Appell function $\mathscr{F}_2$ and the generalized Humbert's confluent hypergeometric function $\mathscr{P}_2$. We further offer certain integral representations for $\mathscr{F}_A^{(n)}$ and $\mathscr{F}_2$ whose integrands include a finite product of a function (with different arguments) chosen among the Laguerre polynomials $L_n^{(\alpha)}(x)$, the incomplete gamma function $\gamma(\kappa,x)$, and the Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$.

**Theorem 3.1.** *Let* $\min\{\Re(p), \Re(\alpha)\} > 0$. *Also let* $\beta_1,\ldots,\beta_n \in \mathbb{C}$ *and* $\gamma_1,\ldots,\gamma_n \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Further let the function* $\Theta(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z)$ *be the same as in* (1). *Then*

$$\mathscr{F}_A^{(n)}[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n] \tag{9}$$

$$= \frac{1}{\Gamma_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha)} \int_0^\infty t^{\alpha-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right) \prod_{j=1}^n {}_1F_1\begin{bmatrix}\beta_j; \\ \gamma_j; \end{bmatrix} x_j t\,\mathrm{d}t$$

$$\left(x_1,\, \ldots,\, x_n \in \mathbb{C}\right).$$

*Proof.* Replace the term

$$(\alpha; p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+\cdots+m_n} \tag{10}$$

in the multiple summation (5) by the integral representation of (10):

$$\begin{aligned}(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+\cdots+m_n} \\ = \frac{1}{\Gamma_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha)} \int_0^\infty t^{\alpha+m_1+\cdots+m_n-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right) \mathrm{d}t \\ \left(\Re(p) > 0;\ \Re(\alpha) > 0\right),\end{aligned} \tag{11}$$

which can be derived from (4). Then we want to interchange the order of summation and integration in the resulting identity. The most straightforward approach to justify this procedure is to consider the integral to be a Lebesgue integral. Due to the series's absolute convergence under the constraints, each term in the series may be considered to be nonnegative. Now we can apply Lévi's convergence theorem (see [2, p. 268, Theorem 10.25]) to obtain the desired identity (9). □

**Theorem 3.2.** *Let* $\min\{\Re(p), \Re(\alpha)\} > 0$. *Also let* $\gamma_1, \ldots, \gamma_n \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. *Further let the function* $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z\right)$ *be the same as in* (1). *Then*

$$\mathscr{P}_2^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}); \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= \frac{1}{\Gamma_p^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha)} \int_0^\infty t^{\alpha-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right) \prod_{j=1}^n {}_0F_1(-\,; \gamma_j; x_j t)\,\mathrm{d}t$$

$$\left(x_1,\, \ldots,\, x_n \in \mathbb{C}\right).$$

*Proof.* By substituting the first equality in (7) for (5), the proof would proceed concurrently with that of Theorem 3.1. The specifics have been removed. □

**Theorem 3.3.** *Let* $\min\{\Re(p), \Re(\alpha), \Re(\beta_1), \ldots, \Re(\beta_n)\} > 0$. *Also let* $\gamma_1, \ldots, \gamma_n \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. *Further let the function* $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; z\right)$ *be the same as in* (1). *Then the following* $(n+1)$*-tuple integral representation for* $\mathscr{F}_A^{(n)}$ *in* (5) *holds true.*

$$\begin{aligned}\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n] \\ = \frac{1}{\Gamma^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha) \cdot \prod\limits_{j=1}^n \Gamma(\beta_j)}\end{aligned} \tag{12}$$

$$\times \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-\sum_{j=1}^{n} t_j} \cdot s^{\alpha-1} \cdot \prod_{j=1}^{n} t_j^{\beta_j - 1} \cdot \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -s - \frac{p}{s}\right)$$
$$\times \prod_{j=1}^{n} {}_0F_1(-;\gamma_j; x_j s t_j)\, \mathrm{d}s\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

*Proof.* Using the integral representation (11) and the following integral representation for the usual Pochhammer symbol:

$$(\lambda)_n = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-u}\, u^{\lambda+n-1}\, \mathrm{d}u \quad (n \in \mathbb{N}_0,\ \Re(\lambda+n) > 0) \tag{13}$$

in each of Pochhammer symbols $(\beta_1)_{m_1}, \ldots, (\beta_n)_{m_n}$ in (5), we can obtain the desired identity (12). □

**Theorem 3.4.** *Let* $\min\{\Re(p),\ \Re(\alpha),\ \Re(\beta_1),\ \ldots,\Re(\beta_n)\} > 0$. *Also let* $\gamma_1, \ldots, \gamma_n \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Then the following n-tuple integral representation for* $\mathscr{F}_A^{(n)}$ *in* (5) *holds true.*

$$\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n] \tag{14}$$
$$= \frac{1}{\prod_{j=1}^{n} \Gamma(\beta_j)} \int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-\sum_{j=1}^{n} t_j} \cdot \prod_{j=1}^{n} t_j^{\beta_j-1}$$
$$\times\ \mathscr{P}_2^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}); \gamma_1, \ldots, \gamma_n; x_1 t_1, \ldots, x_n t_n]\, \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

*Proof.* Using the integral representation (13) in each of the Pochhammer symbols $(\beta_1)_{m_1}, \ldots, (\beta_n)_{m_n}$ in (5) and applying (7), we can get the desired multiple integral representation (14). □

**Theorem 3.5.** *Let* $\min\{\Re(p),\ \Re(\alpha)\} > 0$. *Also let* $\Re(\gamma_j) > \Re(\beta_j) > 0$ $(j = 1, \ldots, n)$. *Then the following n-tuple integral representation for* $\mathscr{F}_A^{(n)}$ *in* (5) *holds true.*

$$\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \gamma_1, \ldots, \gamma_n; x_1, \ldots, x_n] \tag{15}$$
$$= \frac{1}{\prod_{j=1}^{n} B(\beta_j, \gamma_j - \beta_j)} \int_0^1 \cdots \int_0^1 \prod_{j=1}^{n} t_j^{\beta_j-1}(1-t_j)^{\gamma_j-\beta_j-1}$$
$$\times\ {}_1F_0\big[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}); \text{---}\,; x_1 t_1 + \cdots + x_n t_n\big]\ \mathrm{d}t_1 \cdots \mathrm{d}t_n.$$

*Proof.* Using the following integral representation for a quotient of Pochhammer symbols which can also be expressed in terms of the Beta functions $B(\alpha, \beta)$:

$$\frac{(\beta)_\nu}{(\gamma)_\nu} = \frac{B(\beta+\nu, \gamma-\beta)}{B(\beta, \gamma-\beta)} = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta+\nu-1}(1-t)^{\gamma-\beta-1}\, \mathrm{d}t$$
$$\big(\Re(\gamma) > \Re(\beta) > \max\{0, -\Re(\nu)\}\big)$$

in (5) and the series rearrangement identity (see [42, p. 52, Eq. 1.6(3)]):

$$\sum_{m_1,\ldots,m_n=0}^{\infty} \Omega\left(m_1+\cdots+m_n\right) \frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} = \sum_{m=0}^{\infty} \Omega(m) \frac{(x_1+\cdots+x_m)^m}{m!},$$

and applying (2) in the resulting identity, we can get the desired multiple integral representation (15). □

When $n = 2$, the special case of Theorems 3.1–3.4 instantly yields integral representations for the generalized Appell function $\mathscr{F}_2$ in (6) and Humbert's confluent function $\mathscr{P}_2$ in (8), as stated in Corollaries 3.6–3.9.

**Corollary 3.6.** *Let* $\min\{\Re(p), \Re(\alpha)\} > 0$. *Also let* $\beta_1, \beta_2 \in \mathbb{C}$ *and* $\gamma_1, \gamma_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Then the following integral representation for* $\mathscr{F}_2$ *in* (6) *holds true.*

$$\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] = \frac{1}{\Gamma^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)}$$
$$\times \int_0^\infty t^{\alpha-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right)\ {}_1F_1\left[\begin{array}{c}\beta_1;\\ \gamma_1;\end{array} x_1 t\right]\ {}_1F_1\left[\begin{array}{c}\beta_2;\\ \gamma_2;\end{array} x_2 t\right] \mathrm{d}t.$$

**Corollary 3.7.** *Let* $\min\{\Re(p), \Re(\alpha)\} > 0$. *Also let* $\gamma_1, \gamma_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Then the following integral representation for* $\mathscr{P}_2$ *in* (8) *holds true.*

$$\mathscr{P}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}); \gamma_1, \gamma_2; x_1, x_2] = \frac{1}{\Gamma^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)}$$
$$\times \int_0^\infty t^{\alpha-1}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right)\ {}_0F_1(-\,;\gamma_1; x_1 t)\, {}_0F_1(-\,;\gamma_2; x_2 t)\,\mathrm{d}t.$$

**Corollary 3.8.** *Let* $\min\{\Re(p), \Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0$. *Also let* $\gamma_1, \gamma_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Then the following triple integral representation for* $\mathscr{F}_2$ *in* (6) *holds true.*

$$\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2] := \frac{1}{\Gamma^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)}$$
$$\times \int_0^\infty\int_0^\infty\int_0^\infty e^{-t_1-t_2}\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -s-\frac{p}{s}\right) s^{\alpha-1}\, t_1^{\beta_1-1} t_2^{\beta_2-1}$$
$$\times\, {}_0F_1(\text{——}\,;\gamma_1; x_1 s t_1)\, {}_0F_1(\text{——}\,;\gamma_2; x_2 s t_2)\,\mathrm{d}s\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

**Corollary 3.9.** *Let* $\min\{\Re(p), \Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0$. *Also let* $\gamma_1, \gamma_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Then the following double integral representation for* $\mathscr{F}_2$ *in* (6) *holds true.*

$$\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \gamma_1, \gamma_2; x_1, x_2]$$
$$= \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty\int_0^\infty e^{-t_1-t_2}\, t_1^{\beta_1-1} t_2^{\beta_2-1}$$
$$\times\, \mathscr{P}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}); \gamma_1, \gamma_2; x_1 t_1, x_2 t_2]\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

*Remark* 3.10. A number of elementary and special functions are expressed in terms of (generalized) hypergeometric functions (see, e.g., [35], [20, p. 265, Eq. (3.2)], and [6, 22, 24, 28, 45, 46]). For example, recall those of the Laguerre polynomial $L_n^{(\alpha)}(x)$ of order (index) $\alpha$ and degree $n$ in $x$, the incomplete gamma function $\gamma(\kappa, x)$, the Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$:

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1(-n; \alpha+1; x) \quad (\alpha \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}), \tag{16}$$

$$\gamma(\kappa, x) = \frac{1}{\kappa}\, x^\kappa\, {}_1F_1(\kappa; \kappa+1; -x) \quad (\kappa \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}), \tag{17}$$

$$J_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{——}\,; \nu+1; -\frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}) \tag{18}$$

and

$$I_\nu(z) = \frac{(\frac{z}{2})^\nu}{\Gamma(\nu+1)}\, {}_0F_1\left(\text{——}\,; \nu+1; \frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}). \tag{19}$$

Now, by applying the relationships (16) and (17) to the integral formulas (9), and by using the relationships (18) and (19) in the integral formula (12), we can present several interesting integral representations for the generalized Lauricella hypergeometric function $\mathscr{F}_A^{(n)}$ in (5) as asserted in Corollaries 3.11 and 3.12 (without proof).

**Corollary 3.11.** *Let* $\min\{\Re(p), \Re(\alpha)\} > 0$. *Also let* $m_1, \ldots, m_n \in \mathbb{N}_0$ *and* $\beta_1, \ldots, \beta_n \in \mathbb{C} \setminus \mathbb{Z}_{\leq -1}$. *Further let* $\Re(\alpha) > \sum_{j=1}^n \Re(\beta_j)$ *for* (20). *Then each of the following integral representations for* $\mathscr{F}_A^{(n)}$ *in* (5) *holds true.*

$$\begin{aligned}
&\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -m_1, \ldots, -m_n; \beta_1+1, \ldots, \beta_n+1; x_1, \ldots, x_n] \\
&= \frac{\prod_{j=1}^n m_j!}{\Gamma_p(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}) \prod_{j=1}^n (\beta_j+1)_{m_j}(\alpha)} \\
&\quad \times \int_0^\infty \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t - \frac{p}{t}\right)\, t^{\alpha-1} \prod_{j=1}^n L_{m_j}^{(\beta_j)}(x_j t)\, \mathrm{d}t
\end{aligned}$$

*and*

$$\begin{aligned}
&\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \beta_1+1, \ldots, \beta_n+1; -x_1, \ldots, -x_n] \\
&= \frac{\prod_{j=1}^n \beta_j\, x_j^{-\beta_j}}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)}
\end{aligned} \tag{20}$$

$$\times \int_0^\infty \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right) t^{\alpha-\sum\limits_{j=1}^{n}\beta_j-1} \prod_{j=1}^{n} \gamma(\beta_j, x_j t)\,\mathrm{d}t.$$

**Corollary 3.12.** *Let* $\min\{\Re(p), \Re(\alpha), \Re(\beta_1), \ldots, \Re(\beta_n)\} > 0$*. Also let* $\gamma_1, \ldots, \gamma_n \in \mathbb{C}\setminus\mathbb{Z}_{\leq -1}$*. Further let* $\Re(\alpha) > \frac{1}{2}\sum_{j=1}^{n}\Re(\gamma_j)$ *and* $\Re(\beta_j) > \frac{1}{2}\Re(\gamma_j)$ $(j = 1, \ldots, n)$*. Then each of the following* $(n+1)$*-tuple integral representations for* $\mathscr{F}_A^{(n)}$ *in* (5) *holds true.*

$$\begin{aligned}
&\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \gamma_1+1, \ldots, \gamma_n+1; -x_1, \ldots, -x_n] \\
&= \frac{\prod\limits_{j=1}^{n}\Gamma(\gamma_j+1)\cdot\prod\limits_{j=1}^{n} x_j^{-\frac{\gamma_j}{2}}}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)\cdot\prod\limits_{j=1}^{n}\Gamma(\beta_j)} \\
&\quad\times \int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-\sum\limits_{j=1}^{n}t_j} s^{\alpha-\frac{1}{2}\sum\limits_{j=1}^{n}\gamma_j-1} \prod_{j=1}^{n} t_j^{\beta_j-\frac{\gamma_j}{2}-1} \\
&\qquad\times \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -s-\frac{p}{s}\right)\cdot\prod_{j=1}^{n} J_{\gamma_j}\left(2\sqrt{x_j s t_j}\right)\,\mathrm{d}s\,\mathrm{d}t_1\cdots\mathrm{d}t_n
\end{aligned}$$

*and*

$$\begin{aligned}
&\mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \ldots, \beta_n; \gamma_1+1, \ldots, \gamma_n+1; x_1, \ldots, x_n] \\
&= \frac{\prod\limits_{j=1}^{n}\Gamma(\gamma_j+1)\cdot\prod\limits_{j=1}^{n} x_j^{-\frac{\gamma_j}{2}}}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)\cdot\prod\limits_{j=1}^{n}\Gamma(\beta_j)} \\
&\quad\times \int_0^\infty\int_0^\infty\cdots\int_0^\infty e^{-\sum\limits_{j=1}^{n}t_j} s^{\alpha-\frac{1}{2}\sum\limits_{j=1}^{n}\gamma_j-1} \prod_{j=1}^{n} t_j^{\beta_j-\frac{\gamma_j}{2}-1} \\
&\qquad\times \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -s-\frac{p}{s}\right)\cdot\prod_{j=1}^{n} I_{\gamma_j}\left(2\sqrt{x_j s t_j}\right)\,\mathrm{d}s\,\mathrm{d}t_1\cdots\mathrm{d}t_n.
\end{aligned}$$

The particular cases of the results in Corollaries 3.11 and 3.12 when $n = 2$ are given in Corollaries 3.13 and 3.14, respectively.

**Corollary 3.13.** *Let* $\min\{\Re(p), \Re(\alpha)\} > 0$*. Also let* $m_1, m_2 \in \mathbb{N}_0$ *and* $\beta_1, \beta_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq -1}$*. Further let* $\Re(\alpha) > \Re(\beta_j)+\Re(\beta_2)$ *for* (21)*. Then each of the following integral representations for* $\mathscr{F}_2$ *in* (6) *holds true.*

$$\begin{aligned}
&\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -m_1, -m_2; \beta_1+1, \beta_2+1; x_1, , x_2] \\
&= \frac{m_1!m_2!}{(\beta_1+1)_{m_1}(\beta_2+1)_{m_2}\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)}
\end{aligned}$$

$$\times \int_0^\infty t^{\alpha-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right)\, L_{m_1}^{(\beta_1)}(x_1 t) L_{m_2}^{(\beta_2)}(x_2 t)\,\mathrm{d}t$$

*and*

$$\begin{aligned} (21)\qquad & \mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \beta_1+1, \beta_2+1; -x_1, -x_2] \\ &= \frac{\beta_1\beta_2 x_1^{-\beta_1} x_2^{-\beta_2}}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)} \\ &\times \int_0^\infty t^{\alpha-\beta_1-\beta_2-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -t-\frac{p}{t}\right)\,\gamma(\beta_1, x_1 t)\gamma(\beta_2, x_2 t)\,\mathrm{d}t. \end{aligned}$$

**Corollary 3.14.** *Let* $\min\{\Re(p), \Re(\alpha), \Re(\beta_1), \Re(\beta_2)\} > 0$. *Also let* $\gamma_1, \gamma_2 \in \mathbb{C}\setminus\mathbb{Z}_{\leq -1}$. *Further let* $\Re(\alpha) > \frac{1}{2}\sum_{j=1}^{2}\Re(\gamma_j)$ *and* $\Re(\beta_j) > \frac{1}{2}\Re(\gamma_j)$ $(j = 1, 2)$. *Then each of the following triple integral representations for* $\mathscr{F}_2$ *in* (6) *holds true.*

$$\begin{aligned} &\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \gamma_1+1, \gamma_2+1; -x_1, -x_2] \\ &= \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\, x_1^{-\frac{\gamma_1}{2}} x_2^{-\frac{\gamma_2}{2}}}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\times \int_0^\infty\int_0^\infty\int_0^\infty e^{-t_1-t_2} s^{\alpha-\frac{\gamma_1}{2}-\frac{\gamma_2}{2}-1} t_1^{\beta_1-\frac{\gamma_1}{2}-1} t_1^{\beta_2-\frac{\gamma_2}{2}-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -s-\frac{p}{s}\right) \\ &\qquad \times J_{\gamma_1}\left(2\sqrt{x_1 s t_1}\,\right) J_{\gamma_2}\left(2\sqrt{x_2 s t_2}\,\right)\,\mathrm{d}s\,\mathrm{d}t_1\mathrm{d}t_2 \end{aligned}$$

*and*

$$\begin{aligned} &\mathscr{F}_2[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), \beta_1, \beta_2; \gamma_1+1, \gamma_2+1; x_1, x_2] \\ &= \frac{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)\, x_1^{-\frac{\gamma_1}{2}} x_2^{-\frac{\gamma_2}{2}}}{\Gamma_p^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \\ &\times \int_0^\infty\int_0^\infty\int_0^\infty e^{-t_1-t_2} s^{\alpha-\frac{\gamma_1}{2}-\frac{\gamma_2}{2}-1} t_1^{\beta_1-\frac{\gamma_1}{2}-1} t_1^{\beta_2-\frac{\gamma_2}{2}-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -s-\frac{p}{s}\right) \\ &\qquad \times I_{\gamma_1}\left(2\sqrt{x_1 s t_1}\,\right) I_{\gamma_2}\left(2\sqrt{x_2 s t_2}\,\right)\,\mathrm{d}s\,\mathrm{d}t_1\mathrm{d}t_2. \end{aligned}$$

## 4. Finite summation formulas involving $\mathscr{F}_A^{(n)}$

The following theorems investigate several finite summation formulas for the generalized Lauricella function $\mathscr{F}_A^{(n)}$.

**Theorem 4.1.** *Let* $m \in \mathbb{N}_0$ *and* $\Re(p) \geq 0$. *Also let* $\alpha, \beta_3, \ldots, \beta_n \in \mathbb{C}$ *and* $\gamma_3, \ldots, \gamma_n \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$. *Then the following finite summation formula for* $\mathscr{F}_A^{(n)}$ *holds true for all* $x_1, \ldots, x_n \in \mathbb{C}$.

$$(22)\qquad \sum_{k=0}^{m} \mathscr{F}_A^{(n)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -k, -m+k, \beta_3, \ldots, \beta_n;$$

$$1, 1, \gamma_3, \ldots, \gamma_n; x_1, \ldots, x_n]$$

$$= (m+1)\mathscr{F}_A^{(n-1)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -m, \beta_3, \ldots, \beta_n; 2, \gamma_3, \ldots, \gamma_n; x_1 + x_2, x_3, \ldots, x_n].$$

*Proof.* We make use of the integral representation (9) and the following well-known identity for the Laguerre polynomials (see, e.g., [35, p. 209, Eq. (3)]):

$$\sum_{k=0}^{m} L_k^{(\lambda)}(x) L_{m-k}^{(\mu)}(y) = L_m^{(\lambda+\mu+1)}(x+y) \tag{23}$$

for $\lambda = \mu = 0$. Thus, in view of the ${}_1F_1$ representation (16) for the Laguerre polynomials, we get the desired finite summation formula (22). □

**Theorem 4.2.** *Let $\alpha \in \mathbb{C}$ and $\Re(p) \geq 0$. Also let $m_1, \ldots, m_s \in \mathbb{N}_0$. Then the following multiple finite summation formula for $F_A^{(n)}$ holds true for all $x_1, \ldots, x_{2s} \in \mathbb{C}$.*

$$\sum_{k_1=0}^{m_1} \cdots \sum_{k_s=0}^{m_s} \mathscr{F}_A^{(2s)}\big[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -k_1, -m_1 + k_1, \ldots, -k_s, -m_s + k_s; 1, \ldots, 1; x_1, \ldots, x_{2s}\big] \tag{24}$$

$$= (m_1+1)\cdots(m_s+1)\mathscr{F}_A^{(s)}[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -m_1, \ldots, -m_s; 2, \ldots, 2; x_1 + x_2, \ldots, x_{2s-1} + x_{2s}].$$

*Proof.* By iterating the technique used to prove the finite summation formula (22), which is based upon the identity (23) and the integral representation (9), the required multiple summation formula (24) is obtained using the ${}_1F_1$ representation (16) for the Laguerre polynomials. □

**Theorem 4.3.** *Let $m \in \mathbb{N}_0$, $\alpha \in \mathbb{C} \setminus \{1\}$, and $\Re(p) \geq 0$. Also let $\beta_3, \ldots, \beta_n \in \mathbb{C}$, $\gamma_3, \ldots, \gamma_n \in \mathbb{C}\setminus\mathbb{Z}_{\leq 0}$, and $\lambda \in \mathbb{C}\setminus\mathbb{Z}_{\leq -1}$. Then the following finite summation formula for $\mathscr{F}_A^{(n)}$ holds true for all $x_1, \ldots, x_n \in \mathbb{C}$ with $x_1 \neq x_2$.*

$$\sum_{k=0}^{m} \binom{\lambda+k}{k} \mathscr{F}_A^{(n)}\big[(\alpha, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -k, -k, \beta_3, \ldots, \beta_n; \lambda+1, \lambda+1, \gamma_3, \ldots, \gamma_n; x_1, \ldots, x_n\big] \tag{25}$$

$$= \frac{(\lambda+1)_{m+1}}{m!(\alpha-1)} (x_1 - x_2)^{-1} \times \mathscr{F}_A^{(n)}\big[(\alpha-1, p, \{\kappa_\ell\}_{\ell\in\mathbb{N}_0}), -m, -m-1, \beta_3, \ldots, \beta_n; \lambda+1, \lambda+1, \gamma_3, \ldots, \gamma_n; x_1, \ldots, x_n\big] + x_1 \rightleftharpoons x_2,$$

*where $x_1 \rightleftharpoons x_2$ indicates the presence of a second term that originates from the first term by interchanging $x_1$ and $x_2$.*

*Proof.* Applying the relationship (16) and the following known result (see, e.g., [35, p. 206, Eq. (10)]):

$$\begin{aligned}&\sum_{k=0}^{m} \frac{k!}{(\lambda+1)_k}\, L_k^{(\lambda)}(x) L_k^{(\lambda)}(y)\\ &= \frac{(m+1)!}{(\lambda+1)_m}\,(x-y)^{-1}\left[L_m^{(\lambda)}(x)L_{m+1}^{(\lambda)}(y) - L_{m+1}^{(\lambda)}(x)L_m^{(\lambda)}(y)\right]\end{aligned}$$

to the integral representation (9), we get the desired finite summation formula (25). □

*Remark* 4.4. By suitably iterating the process in Theorem 4.3, we obtain

$$\begin{aligned}&\sum_{k_1=0}^{m_1}\sum_{k_2=0}^{m_2}\sum_{k_3=0}^{m_3}\binom{a_1+k_1}{k_1}\binom{a_2+k_2}{k_2}\binom{a_3+k_3}{k_3}\\ &\times \mathscr{F}_A^{(n)}\big[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),-k_1,-k_1,-k_2,-k_2,-k_3,-k_3,\beta_7,\ldots,\beta_n;\\ &\qquad a_1+1,a_1+1,a_2+1,a_2+1,a_3+1,a_3+1,\gamma_7,\ldots,\gamma_n;x_1,\ldots,x_n\big]\\ &= \frac{(a_1+1)_{m_1+1}(a_2+1)_{m_2+1}(a_3+1)_{m_3+1}}{m_1!(\alpha-1)(\alpha-2)(\alpha-3)}(x_1-x_2)^{-1}\Big[(x_3-x_4)^{-1}\Big\{(x_5-x_6)^{-1}\\ &\times \mathscr{F}_A^{(n)}\big[(\alpha-3,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),-m_1,-m_1-1,-m_2,-m_2-1,-m_3,-m_3-1,\\ &\qquad \beta_7,\ldots,\beta_n;a_1+1,a_1+1,a_2+1,a_2+1,a_3+1,a_3+1,\gamma_7,\ldots,\gamma_n;\\ &\qquad x_1,\ldots,x_n\big]+x_5\rightleftharpoons x_6\Big\}+x_3\rightleftharpoons x_4\Big]+x_1\rightleftharpoons x_2,\end{aligned}$$

where the right-hand side obviously has $2^3$ terms. Similarly, we can derive a more general multiple finite summation formula for $\mathscr{F}_A^{(n)}$ in the following form:

$$\begin{aligned}&\sum_{k_1=0}^{m_1}\cdots\sum_{k_s=0}^{m_s}\binom{a_1+k_1}{k_1}\cdots\binom{a_s+k_s}{k_s}\\ &\times \mathscr{F}_A^{(2s)}[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),-k_1,-k_1,\ldots,-k_s,-k_s;\\ &\qquad a_1+1,a_1+1,\ldots,a_s+1,a_s+1;x_1,\ldots,x_{2s}],\end{aligned}$$

whose detailed expression is being left as an exercise for the interested reader.

## 5. The generalized Lauricella function $\mathscr{F}_A^{(2s)}$ as an $s$-fold sum

By interpreting the first two ${}_1F_1$ functions occurring on the right-hand side of (9) as a Cauchy product, it is found that the generalized Lauricella function $\mathscr{F}_A^{(n)}$ can be expressed as a series whose terms are composed of ${}_3F_2$ and $\mathscr{F}_A^{(n-2)}$ as follows:

(26)
$$\begin{aligned}&\mathscr{F}_A^{(n)}[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n]\\ &= \sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta_1)_m}{(\gamma_1)_m}\frac{x_1^m}{m!}\,{}_3F_2\left[\begin{matrix}-m,\,1-\gamma_1-m,\,\beta_2;\\ 1-\beta_1-m,\,\gamma_2;\end{matrix}\; -\frac{x_2}{x_1}\right]\end{aligned}$$

$$\times \mathscr{F}_A^{(n-2)}[(\alpha+m,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),\beta_3,\ldots,\beta_n;\gamma_3,\ldots,\gamma_n;x_3,\ldots,x_n]$$
$$(p \geqq 0;\ |x_1|+\cdots+|x_n|<1 \quad \text{when} \quad p=0).$$

More generally, by iterating the same process $s$ times in (9) with $n$ replaced by $2s$ as in the last summation formula (26), $\mathscr{F}_A^{(2s)}$ would be finally expressed as a multiple series whose terms are $s$-tuple products of the hypergeometric ${}_3F_2$ functions:

$$\mathscr{F}_A^{(2s)}[(\alpha,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}),\beta_1,\ldots,\beta_{2s};\gamma_1,\ldots,\gamma_{2s};x_1,\ldots,x_{2s}]$$
$$= \sum_{m_1,\ldots,m_n=0}^{\infty} (\alpha;p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_{m_1+\cdots+m_n}$$
$$\times \prod_{j=1}^{s}\left\{\frac{(\beta_{2j-1})_{m_j}}{(\gamma_{2j-1})_{m_j}}\frac{x_{2j-1}^{m_j}}{m_j!}\,{}_3F_2\left[\begin{matrix}-m_j,\,1-\gamma_{2j-1}-m_j,\,\beta_{2j};\\ 1-\beta_{2j-1}-m_j,\,\gamma_{2j};\end{matrix}\ -\frac{x_{2j}}{x_{2j-1}}\right]\right\}$$
$$(p \geqq 0;\ |x_1|+\cdots+|x_{2s}|<1 \quad \text{when} \quad p=0).$$

## 6. Concluding remarks and further observations

In our present investigation, with the help of the generalized Pochhammer symbol $(\lambda,p,\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})_\nu$ in (3), we have introduced the generalized Lauricella function $\mathscr{F}_A^{(n)}$ and the generalized Humbert's confluent hypergeometric function $\mathscr{P}^{(n)}$ of $n$ variables, whose particular cases when $n=2$ have also been considered to be named and denoted as the generalized Appell function $\mathscr{F}_2$ and the generalized Humbert's confluent hypergeometric function $\mathscr{P}_2$ of two variables. Then we have investigated several properties of these newly introduced functions such as diverse integral representations and finite summation formulas for $\mathscr{F}_A^{(n)}$. Also, we have obtained some integral representations involving the Laguerre polynomials $L_n^{(\alpha)}(x)$, the incomplete gamma function $\gamma(\kappa,x)$, and the Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$. Certain specific instances of the findings given in this article, which may be derived by specializing the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$'s choices, reduce to the corresponding known or novel results for the classical Lauricella and Appell functions (see, e.g., [5], [41], [42]). In particular, if we set

$$\kappa_\ell = 1,$$

the functions in (5) and (6) immediately yield the known generalized Lauricella function $F_A^{(n)}$ of $n$ variables and generalized Appell function $F_2$ of two variables [44].

Again, if we choose

$$\kappa_\ell = \frac{(\rho)_\ell}{(\sigma)_\ell} \qquad (\ell\in\mathbb{N}_0)\,,$$

the functions in (5) and (6) may give new extensions for the generalized Lauricella function $F_A^{(n)}$ of $n$ variables and the extended Appell function $F_2$.

Thus, the numerous findings given in this article are certain to be novel and extend previously published simpler results (see, e.g., [5], [41] and [42]).

We conclude this article by making the following further observations:

- The special case of (26) when $n = 2$ can easily be rewritten for the generalized Appell function $\mathscr{F}_2$.
- The special case of (26) when $\kappa_\ell = 1$ and $p = 0$ yields a known result (see, e.g., [42, p. 181, Problem 38(ii)]).
- By setting $\kappa_\ell = 1$ and $p = 0$ in the results presented in Sections 4 and 5, we can obtain the corresponding known identities due to Srivastava [36] and Padmanabham and Srivastava [30].
- By taking $\kappa_\ell = \frac{(\rho)_\ell}{(\sigma)_\ell}$ $(\ell \in \mathbb{N}_0)$ in the results given in this paper, we may develop corresponding new results for generalized Lauricella function $F_A^{(n)}$, generalized Humbert's confluent hypergeometric function $\Psi^{(n)}$ of $n$ variables, generalized Appell function $F_2$ and Humbert's confluent function $\Psi_2$ of two variables.

**Acknowledgements.** The fourth-named author has carried out this research under the project MATRICS, SERB, Department of Science & Technology (DST), India (File No. MTR/2019/001328).

## References

[1] R. Agarwal, R. K. Parmar, and S. D. Purohit, *Operators of fractional calculus and associated integral transforms of the* $(\mathfrak{r}, \mathfrak{s})$*-extended Bessel-Struve kernel function*, Int. J. Appl. Comput. Math. **6** (2020), no. 6, Paper No. 175, 18 pp. https://doi.org/10.1007/s40819-020-00927-x

[2] T. M. Apostol, *Mathematical Analysis*, second edition, Addison-Wesley Publishing Co., Reading, MA, 1974.

[3] R. A. Askey, *The q-gamma and q-beta functions*, Applicable Anal. **8** (1978/79), no. 2, 125–141. https://doi.org/10.1080/00036817808839221

[4] R. A. Askey, *Ramanujan's extensions of the gamma and beta functions*, Amer. Math. Monthly **87** (1980), no. 5, 346–359. https://doi.org/10.2307/2321202

[5] W. N. Bailey, *Generalized hypergeometric series*, Cambridge Tracts in Mathematics and Mathematical Physics, No. 32, Stechert-Hafner, Inc., New York, 1964.

[6] B. C. Carlson, *Special Functions of Applied Mathematics*, Academic Press, New York, 1977.

[7] M. A. Chaudhry, *Transformation of the extended gamma function* $\Gamma_{0,2}^{2,0}[(b, x)]$ *with applications to astrophysical thermonuclear functions*, Astrophys. Space Sci. **262** (1998/99), no. 3, 263–270. https://doi.org/10.1023/A:1001843927721

[8] M. A. Chaudhry, A. Qadir, M. Rafique, and S. M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math. **78** (1997), no. 1, 19–32. https://doi.org/10.1016/S0377-0427(96)00102-1

[9] M. A. Chaudhry, N. M. Temme, and E. J. M. Veling, *Asymptotics and closed form of a generalized incomplete gamma function*, J. Comput. Appl. Math. **67** (1996), no. 2, 371–379. https://doi.org/10.1016/0377-0427(95)00018-6

[10] M. A. Chaudhry and S. M. Zubair, *Generalized incomplete gamma functions with applications*, J. Comput. Appl. Math. **55** (1994), no. 1, 99–124. https://doi.org/10.1016/0377-0427(94)90187-2

[11] M. A. Chaudhry and S. M. Zubair, *On the decomposition of generalized incomplete gamma functions with applications of Fourier transforms*, J. Comput. Appl. Math. **59** (1995), no. 3, 253–284. https://doi.org/10.1016/0377-0427(94)00026-W

[12] M. A. Chaudhry and S. M. Zubair, *On an extension of generalized incomplete gamma functions with applications*, J. Austral. Math. Soc. Ser. B **37** (1996), no. 3, 392–405. https://doi.org/10.1017/S0334270000010730

[13] M. A. Chaudhry and S. M. Zubair, *On a connection between the generalized incomplete gamma functions and their extensions*, J. Austral. Math. Soc. Ser. B **38** (1997), no. 4, 581–589. https://doi.org/10.1017/S0334270000000874

[14] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[15] M. A. Chaudhry and S. M. Zubair, *Extended incomplete gamma functions with applications*, J. Math. Anal. Appl. **274** (2002), no. 2, 725–745. https://doi.org/10.1016/S0022-247X(02)00354-2

[16] J. Choi and R. K. Parmar, *Fractional integration and differentiation of the* $(p, q)$*-extended Bessel function*, Bull. Korean Math. Soc. **55** (2018), no. 2, 599–610. https://doi.org/10.4134/BKMS.b170193

[17] J. Choi and R. K. Parmar, *Fractional calculus of the* $(p, q)$*–extended Struve function*, Far East J. Math. Sci. **103** (2018), no. 2, 541–559 . https://doi.org/10.17654/MS103020541

[18] J. Choi, R. K. Parmar, and H. M. Srivastava, *The incomplete Lauricella functions of several variables and associated properties and formulas*, Kyungpook Math. J. **58** (2018), no. 1, 19–35. https://doi.org/10.5666/KMJ.2018.58.1.19

[19] J. Choi, A. K. Rathie, and R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Math. J. **36** (2014), no. 2, 357–385. https://doi.org/10.5831/HMJ.2014.36.2.357

[20] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[21] W. Gautschi, *A computational procedure for the incomplete Gamma function*, ACTM Trans. Math. Software **5** (1979), 466–481. https://doi.org/10.1145/355853.355863

[22] Y. L. Luke, *Mathematical Functions and Their Approximations*, Academic Press, Inc., New York, 1975.

[23] M.-J. Luo, R. K. Parmar, and R. K. Raina, *On extended Hurwitz-Lerch zeta function*, J. Math. Anal. Appl. **448** (2017), no. 2, 1281–1304. https://doi.org/10.1016/j.jmaa.2016.11.046

[24] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Third Enlarged edition, Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtingung der Anwendungsgebiete, Bd. **5**, Springer-Verlag, Berlin, Heidelberg and New York, 1966.

[25] D. J. Maširević, R. K. Parmar, and T. K. Pogány, $(p, q)$*-extended Bessel and modified Bessel functions of the first kind*, Results Math. **72** (2017), no. 1-2, 617–632. https://doi.org/10.1007/s00025-016-0649-1

[26] A. R. Miller, *Reductions of a generalized incomplete gamma function, related Kampé de Fériet functions, and incomplete Weber integrals*, Rocky Mountain J. Math. **30** (2000), no. 2, 703–714. https://doi.org/10.1216/rmjm/1022009290

[27] A. R. Miller and I. S. Moskowitz, *On certain generalized incomplete gamma functions*, J. Comput. Appl. Math. **91** (1998), no. 2, 179–190. https://doi.org/10.1016/S0377-0427(98)00031-4

[28] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (Editors), *NIST Handbook of Mathematical Functions*, [With 1 CD-ROM (Windows, Macintosh and UNIX)], US Department of Commerce, National Institute of Standards and Technology, Washington, D.C., 2010.

[29] E. Özergin, M. A. Özarslan, and A. Altın, *Extension of gamma, beta and hypergeometric functions*, J. Comput. Appl. Math. **235** (2011), no. 16, 4601–4610. https://doi.org/10.1016/j.cam.2010.04.019

[30] P. A. Padmanabham and H. M. Srivastava, *Summation formulas associated with the Lauricella function* $F_A^{(r)}$, Appl. Math. Lett. **13** (2000), no. 1, 65–70. https://doi.org/10.1016/S0893-9659(99)00146-9

[31] R. K. Parmar, P. Chopra, and R. B. Paris, *On an extension of extended beta and hypergeometric functions*, J. Class. Anal. **11** (2017), no. 2, 91–106. https://doi.org/10.7153/jca-2017-11-07

[32] R. K. Parmar and T. K. Pogány, *On* $(p, q)$*-extension of further members of Bessel-Struve functions class*, Miskolc Math. Notes **20** (2019), no. 1, 451–463. https://doi.org/10.18514/MMN.2019.2608

[33] R. K. Parmar and T. K. Pogány, *On Mathieu-type series for the unified Gaussian hypergeometric functions*, Appl. Anal. Discrete Math. **14** (2020), no. 1, 138–149.

[34] R. K. Parmar, T. K. Pogány, and R. K. Saxena, *On properties and applications of* $(p, q)$*-extended* $\tau$*-hypergeometric functions*, C. R. Math. Acad. Sci. Paris **356** (2018), no. 3, 278–282. https://doi.org/10.1016/j.crma.2017.12.014

[35] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.

[36] H. M. Srivastava, *On a summation formula for the Appell function* $F_2$, Proc. Cambridge Philos. Soc. **63** (1967), 1087–1089. https://doi.org/10.1017/s0305004100042146

[37] R. Srivastava, *Some generalizations of Pochhammer's symbol and their associated families of hypergeometric functions and hypergeometric polynomials*, Appl. Math. Inf. Sci. **7** (2013), no. 6, 2195–2206. https://doi.org/10.12785/amis/070609

[38] R. Srivastava, *Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions*, Appl. Math. Comput. **243** (2014), 132–137. https://doi.org/10.1016/j.amc.2014.05.074

[39] H. M. Srivastava, A. Çetinkaya, and . Kıymaz, *A certain generalized Pochhammer symbol and its applications to hypergeometric functions*, Appl. Math. Comput. **226** (2014), 484–491. https://doi.org/10.1016/j.amc.2013.10.032

[40] R. Srivastava and N. E. Cho, *Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials*, Appl. Math. Comput. **234** (2014), 277–285. https://doi.org/10.1016/j.amc.2014.02.036

[41] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications, Horwood, Chichester, 1985.

[42] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Series: Mathematics and its Applications, Horwood, Chichester, 1984.

[43] H. M. Srivastava, R. K. Parmar, and P. Chopra, *A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions*, Axioms **1** (2012), no. 3, 238–258. https://doi.org/10.3390/axioms1030238

[44] H. M. Srivastava, R. K. Parmar, and M. M. Joshi, *Extended Lauricella and Appell functions and their associated properties*, Adv. Stud. Contemporary Math. **25** (2015), no. 2, 151–165.

[45] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge, England, 1944.

[46] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions*: *With an Account of the Principal Transcendental Functions*, fourth edition. Reprinted., Cambridge Univ. Press, New York, 1962.

[47] Z.-H. Yang and Y.-M. Chu, *Asymptotic formulas for gamma function with applications*, Appl. Math. Comput. **270** (2015), 665–680. https://doi.org/10.1016/j.amc.2015.08.087

Ritu Agarwal
Department of Mathematics
Malaviya National Institute of Technology
Jaipur 302017, Rajasthan, India
*Email address*: ragarwal.maths@mnit.ac.in

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 38066, Korea
*Email address*: junesangchoi@gmail.com

Naveen Kumar
Department of Mathematics
Malaviya National Institute of Technology
Jaipur 302017, Rajasthan, India
*Email address*: naveenahrodia@gmail.com

Rakesh K. Parmar
Department of HEAS (Mathematics)
University College of Engineering and Technology
Bikaner 334004, Rajasthan, India
and
Department of Mathematics
Ramanujan School of Mathematical Sciences
Pondicherry University-A Central University
Pondicherry 605014, India
*Email address*: rakeshparmar27@gmail.com, rakeshparmar@pondiuni.ac.in