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FIXED POINT THEOREMS IN CONTROLLED RECTANGULAR METRIC SPACES

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Abstract. In this paper, we introduce an extension of rectangular metric spaces called controlled rectangular metric spaces, by changing the rectangular inequality in the definition of a metric space. We also establish some fixed point theorems for self-mappings defined on such spaces. Our main results extends and improves many results existing in the literature. Moreover, an illustrative example is presented to support the obtained results.

1. Introduction

By a contraction on a metric space (X, d), we understand a mapping $T : X \to X$ satisfying for all $x, y \in X$: $d(Tx, Ty) \leq kd(x, y)$, where k is a real in $[0, 1)$.

In 1922, Banach proved the following theorem.

Theorem 1.1 ([4]). *Let (X, d) be a complete metric space. Let $T : X \to X$ be a contraction. Then:*

(i) *T has a unique fixed point $x \in X$.*
(ii) *For every $x_0 \in X$, the sequence (x_n), where $x_{n+1} = Tx_n$, converges to x.*
(iii) *We have the following estimate: For every $x \in X$, $d(x_n, x) \leq \dfrac{k^n}{1-k} d(x_0, x_1)$, $n \in \mathbb{N}$.*

Several authors generalise the previous theorem in various directions [1, 3, 6, 7, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24].

In 2000, Branciari [5] initiated the notion of rectangular metric space.

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Definition 1.2 ([5]). Let X be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that for all $x, y \in X$ and for all distinct points $u, v \in X$, each of them different from x and y, on has

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all distinct points $x, y \in X$;
(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (the rectangular inequality).

Then (X, d) is called a *rectangular metric space.*

In 2019, Asim et al. [2] introduce the concept of extended b-rectangular metric spaces.

Definition 1.3 ([2]). Let X be a non empty set, and $\theta : X \times X \to [1, +\infty[$. An extended b-rectangular metric is a function $d : X \times X \to [0, +\infty[$ such that for all $x, y \in X$ and all distinct points $u, v \in X$, each of them distinct from x and y one has the following conditions:

(d_1) $d(x, y) = 0$, if and only $x = y$;
(d_2) $d(x, y) = d(y, x)$;
(d_3) $d(x, y) \leq \theta(x, y) [d(x, u) + d(u, v) + d(v, y)]$.

Then (X, d) is called an *extended rectangular b-metric space.*

Example 1.4 ([2]). Consider $X = \{1, 2, 3, 4, 5\}$. Define $\theta : X \times X \to [1, +\infty[$ by

$$\theta(x, y) = x + y + 1 \ \forall x, y \in X.$$

Define

$$d : X \times X \to [0, +\infty[\ \ by$$

- $d(x, x) = 0$ for all $x, y \in X$;
- $d(x, y) = d(y, x)$ for all $x, y \in X$;
- $d(1, 3) = d(2, 5) = 70, d(1, 4) = 1000$ and $d(1, 5) = 1200$;
- $d(1, 2) = d(2, 3) = d(3, 4) = 60, d(3, 5) = d(4, 5) = d(2, 4) = 400.$

Clearly, (X, d) is an extended rectangular b-metric space.

In the next section, we introduce the concept of controlled rectangular metric space and establish some fixed point results for such mappings in the setting of complete controlled rectangular metric spaces which generalize the results of Kannan [10], Reich [20] and Fisher [8] .

2. Main Result

We begin with the following definition.

Definition 2.1. Given a non-empty set X and $\alpha : X \times X \to [1, +\infty[$.

The function $d : X \times X \to [0, +\infty[$ is called a controlled rectangular metric if for all $x, y \in X$ and all distinct points $u, v \in X$; each of them distinct from x and y one has the following conditions:

(d_1) $d(x, y) = 0$, if and only $x = y$;

(d_2) $d(x, y) = d(y, x)$;

(d_3) $d(x, y) \leq \alpha(x, u)\, d(x, u) + \alpha(u, v)\, d(u, v) + \alpha(v, y)\, d(v, y)$.

Then (X, d) is called a *controlled rectangular metric space.*

Remark 2.2. If, for all $x, y \in X$, $\alpha(x, y) = s \geq 1$, then (X, d) is a rectangular b-metric space, which leads us to conclude that every rectangular b-metric space is a controlled rectangular metric space. Also a controlled rectangular metric space is not general an extended rectangular b-metric space.

Example 2.3. Consider $X = \{1, 2, 3, 4\}$. Define $d : X \times X \to [0, +\infty[$ by

- $d(x, x) = 0$ for all $x, y \in X$;
- $d(1, 2) = \frac{1}{2}$, $d(1, 3) = \frac{1}{9}$, $d(1, 4) = \frac{1}{16}$;
- $d(2, 3) = \frac{1}{12}$, $d(2, 4) = \frac{1}{36}$, $d(3, 4) = \frac{1}{49}$.

Define $\alpha : X \times X \to [1, +\infty[$ by

$$\alpha(x, y) = \max\{x, y\}, \forall x, y \in X.$$

Thus, (d_1) and (d_2) are clearly true. We shale prove that (d_3) hold. We have

$$d(1, 2) = \frac{1}{2} \leq \alpha(1, 3)\, d(1, 3) + \alpha(3, 4)\, d(3, 4) + \alpha(4, 2)\, d(4, 2) = 0.58,$$

and

$$d(1, 2) = \frac{1}{2} \leq \alpha(1, 4)\, d(1, 4) + \alpha(4, 3)\, d(4, 3) + \alpha(3, 2)\, d(3, 2) = 0.52.$$

Similarly, other cases can be argued. Thus, for all $x, y \in X$ with distinct $u, v \in X$. We get,

$$d(x, y) \leq \alpha(x, u)\, d(x, u) + \alpha(u, v)\, d(u, v) + \alpha(v, y)\, d(v, y).$$

Hence, (X, d) is controlled rectangular metric space. Note that

$$d(1, 2) = \frac{1}{2} > 0.31 = \alpha(1, 2)\,[d(1, 3) + d(3, 4) + d(4, 2)],$$

that is, d is not an extended rectangular b-metric for the $\alpha = \theta$.

On the other hand, in Example 2.4, we replaced $d(3,4) = 60$ by $d(3,4) = 49$, we prove that d is a extended rectangular b-metric. Note that

$$d(1,4) = 1000 > 992 = \alpha\,(1,2)\,d(1.2) + \alpha\,(2,3)\,d(2.3) + \alpha\,(3,4)\,d(3.4).$$

That is, d is not a controlled rectangular metric for the $\theta = \alpha$.

We define Cauchy and convergent sequences in controlled rectangular metric space as follows:

Definition 2.4. Let (X,d) be a controlled rectangular metric space and $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if and only if $\lim\limits_{n\to+\infty} d\,(x_n, x) = 0$.

Definition 2.5. Let (X,d) be a controlled rectangular metric space and $\{x_n\}$ be a sequence in X. We say that

(i) $\{x_n\}$ is a *Cauchy* if and only if for every $\varepsilon > 0$ there exists a positive integer $N = N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

(ii) (X,d) is *complete* if and only if each Cauchy sequence in X is convergent.

Lemma 2.6. *Let (X,d) be a controlled rectangular metric space and $\{x_n\}$ a Cauchy sequence in X. If $\{x_n\}$ converges to $x \in X$ and converges to $y \in X$, we assume that*

$$\lim_{n\to+\infty} \alpha(x_n, x), \ \lim_{n\to+\infty} \alpha(x, x_n) \textit{ and } \lim_{n,m\to+\infty} \alpha(x_n, x_m)$$

exist and are finite $\forall n, m \in \mathbb{N}$, $n \neq m$, then $x = y$.

Proof. If $\{x_n\}$ a Cauchy sequence in X has two limit point $x, y \in X$, such that

$$\lim_{n\to+\infty} d\,(x_n, x) = 0 \textit{ and } \lim_{n\to+\infty} d\,(x_n, y) = 0.$$

Since, $\{x_n\}$ is Cauchy, then so from (d_3) we have

$$d\,(x,y) \leq \alpha\,(x, x_n)\,d\,(x, x_n) + \alpha\,(x_n, x_{n+1})\,d\,(x_n, x_{n+1}) + \alpha\,(x_{n+1}, y)\,d\,(x_{n+1}, y)\,.$$

By letting $n \to \infty$ in above inequality, we obtain $d\,(x,y) \leq 0$, which implies that $d(x,y) = 0$. Therefore, $x = y$. $\square$

Theorem 2.7. *Let (X,d) be a complete controlled rectangular metric space, and T a self mapping on X. If there exists $k \in\,]0,1[$ such that*

$$d(Ty,Tx) > 0 \Rightarrow d(Ty,Tx) \leq kd(x,y), \forall x, y \in X. \tag{2.1}$$

For $x_0 \in X$, take $x_n = T^n x_0$. Suppose that

$$\sup_{m\geq 1} \lim_{i\to\infty} \alpha\left(x_{i+1}, x_m\right) \frac{\alpha\left(x_{i+1}, x_{i+2}\right) + \alpha\left(x_{i+2}, x_{i+3}\right)}{\alpha\left(x_i, x_{i+1}\right) + \alpha\left(x_{i+1}, x_{i+2}\right)} < \frac{1}{k^2}. \tag{2.2}$$

We assume that, for $x \in X$, we have

$$\lim_{n\to+\infty} \alpha(x_n, x), \ \lim_{n\to+\infty} \alpha(x, x_n) \text{ and } \lim_{n,m\to+\infty} \alpha(x_n, x_m) \tag{2.3}$$

exist and are finite $\forall n, m \in \mathbb{N},\ n \neq m$.

Then T has a unique fixed point in X.

Proof. We choose any x_0 be arbitrary, define the iterative sequence $\{x_n\}$ by

$$x_1 = Tx_0,\ x_2 = Tx_1 ..., x_n = T^n x_0.$$

Step 1. We shall prove that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0.$$

Now, by the hypothesis of theorem, we have

$$\begin{aligned} d\left(x_n, x_{n+1}\right) &= d(Tx_{n-1}, Tx_n) \\ &\leq kd(x_{n-1}, x_n) \\ &\leq k^2 d(x_{n-2}, x_{n-1}) \\ &\leq ... \\ &\leq k^n d(x_0, x_1). \end{aligned}$$

Taking the limit of the above inequality as $n \to \infty$, we deduce that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \tag{2.4}$$

Step 2. We shall prove that

$$\lim_{n\to\infty} d\left(x_n, x_{n+2}\right) = 0.$$

We assume that $x_n \neq x_m$ for every $n, m \in \mathbb{N}$. Indeed, suppose that $x_n = x_m$ for some $n = m + k$ with $k > 0$, so we have $Tx_n = Tx_m$, and

$$d(x_m, x_{m+1}) = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq kd(x_{n-1}, x_n).$$

Since $k \in]0, 1[$, we have

$$d(x_m, x_{m+1}) = d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

Continuing this process, we have

$$d(x_m, x_{m+1}) < d(x_m, x_{m+1}).$$

which is a contradiction. Therefore,

$$d(x_m, x_n) > 0 \ for \ every \ n, m \in \mathbb{N}, n \neq m.$$

Now, substituting $x = x_{n-1}$ and $y = x_{n+1}$ in the (2.1), we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq kd(x_{n-1}, x_{n+1}) \\ &\leq k^2 d(x_{n-2}, x_n) \\ &\leq ... \\ &\leq k^n d(x_0, x_2). \end{aligned}$$

If we take the limit of the above inequality as $n \to +\infty$ we deduce that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.5}$$

Step 3. We shall prove that, x_n is a Cauchy sequence in (X, d) i.e,

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0 \ \ \forall n, m \in \mathbb{N}.$$

Denote by $d_i = d(x_i, x_{i+1})$ for all $i \in \mathbf{N}$. We distinguish two cases.

Case 1: Assume that $m = n+2l+1$ with $l \geq 1$. By property (3) of the controlled rectangular metric spaces, we have

$$\begin{aligned} d(x_n, x_m) &= d(x_n, x_{n+2l+1}) \\ &\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})d(x_{n+2}, x_{n+2l+1}) \\ &\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})\alpha(x_{n+2}, x_{n+3}))d(x_{n+2}, x_{n+3}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})\alpha(x_{n+3}, x_{n+4}))d(x_{n+3}, x_{n+4}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})\alpha(x_{n+4}, x_{n+2l+1}))d(x_{n+4}, x_{n+2l+1}) \\ &\leq \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})\alpha(x_{n+2}, x_{n+3}))d(x_{n+2}, x_{n+3}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})\alpha(x_{n+3}, x_{n+4}))d(x_{n+3}, x_{n+4}) \\ &\quad + \alpha(x_{n+2}, x_{n+2l+1})\alpha(x_{n+4}, x_{n+2l+1}))\alpha(x_{n+4}, x_{n+5})d(x_{n+4}, x_{n+5}) \end{aligned}$$

$$
\begin{aligned}
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\alpha(x_{n+5},x_{n+6})d(x_{n+5},x_{n+6})\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\alpha(x_{n+6},x_{n+2l+1})d(x_{n+6},x_{n+2l+1})\\
\le\;&\alpha(x_n,x_{n+1})d(x_n,x_{n+1})+\alpha(x_{n+1},x_{n+2})d(x_{n+1},x_{n+2})\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+2},x_{n+3}))d(x_{n+2},x_{n+3})\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+3},x_{n+4}))d(x_{n+3},x_{n+4})\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\alpha(x_{n+4},x_{n+5})d(x_{n+4},x_{n+5})\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\alpha(x_{n+5},x_{n+6})d(x_{n+5},x_{n+6})\\
&+...+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\times...\times\alpha(x_{n+2l-2},x_{n+2l+1})\times\\
&\times[\alpha(x_{n+2l-2},x_{n+2l-1})d(x_{n+2l-2},x_{n+2l-1})+\alpha(x_{n+2l-1},x_{n+2l})d(x_{n+2l-1},x_{n+2l})]\\
&+\alpha(x_{n+2},x_{n+2l+1})\times\alpha(x_{n+4},x_{n+2l+1})\times...\times\\
&\times\alpha(x_{n+2l-2},x_{n+2l+1})\alpha(x_{n+2l},x_{n+2l+1})d(x_{n+2l},x_{n+2l+1})
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&d(x_n,x_m)\\
&\le\alpha(x_n,x_{n+1})d_n+\alpha(x_{n+1},x_{n+2})d_{n+1}+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+2},x_{n+3}))d_{n+2}\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+3},x_{n+4}))d_{n+3}\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1})\alpha(x_{n+4},x_{n+5})d_{n+4}\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\alpha(x_{n+5},x_{n+6})d_{n+5}\\
&\quad+...\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\times...\times\alpha(x_{n+2l-2},x_{n+2l+1})\\
&[\alpha(x_{n+2l-2},x_{n+2l+1}))d_{n+2l-2}+\alpha(x_{n+2l-1},x_{n+2l}))d_{n+2l-1}]\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\times...\times\alpha(x_{n+2k-2},x_{n+2l+1})\\
&[\alpha(x_{n+2l},x_{n+2l+1}))d_{n+2l}]\\
&\le\alpha(x_n,x_{n+1})d(x_0,x_1)k^n+\alpha(x_{n+1},x_{n+2})d(x_0,x_1)k^{n+1}\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\left[\alpha(x_{n+2},x_{n+3}))k^{n+2}+\alpha(x_{n+3},x_{n+4}))k^{n+3}\right]d(x_0,x_1)\\
&\quad+...\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\times...\times\alpha(x_{n+2l-2},x_{n+2l+1})\times\\
&\quad\times\left[\alpha(x_{n+2l-2},x_{n+2l-1}))k^{n+2l-2}+\alpha(x_{n+2l-1},x_{n+2l}))k^{n+2l-2}\right]d(x_0,x_1)\\
&\quad+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1})\times...\times
\end{aligned}
$$

$$\begin{aligned}
&\times \alpha(x_{n+2l-2},x_{n+2l+1})\alpha(x_{n+2l},x_{n+2l+1}))k^{n+2l}d(x_0,x_1)\\
\leq\ &\alpha(x_n,x_{n+1})d(x_0,x_1)k^n+\alpha(x_{n+1},x_{n+2})d(x_0,x_1)k^{n+1}\\
&+\alpha(x_{n+2},x_{n+2l+1})\left[\alpha(x_{n+2},x_{n+3}))k^{n+2}+\alpha(x_{n+3},x_{n+4}))k^{n+3}\right]d(x_0,x_1)\\
&+...\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1})\times...\times\alpha(x_{n+2l-2},x_{n+2l+1})\times\\
&\times\left[\alpha(x_{n+2l-2},x_{n+2l-1}))k^{n+2l-2}+\alpha(x_{n+2l-1},x_{n+2l}))k^{n+2l-2}\right]d(x_0,x_1)\\
&+\alpha(x_{n+2},x_{n+2l+1})\alpha(x_{n+4},x_{n+2l+1}))\times...\times\\
&\times\alpha(x_{n+2l-2},x_{n+2l+1})\alpha(x_{n+2l},x_{n+2l+1})\times\\
&\times\left[\alpha(x_{n+2l},x_{n+2l+1})k^{n+2l}d(x_0,x_1)+\alpha(x_{n+2l+1},x_{n+2l+2})k^{n+2l+1}d(x_0,x_1)\right]\\
\leq\ &\alpha(x_n,x_{n+1})d(x_0,x_1)k^n+\alpha(x_{n+1},x_{n+2})d(x_0,x_1)k^{n+1}\\
&+\sum_{i=n+2}^{i=n+2l}\prod_{j=n+2}^{j=i}\alpha(x_j,x_{n+2l+1})\left[\alpha(x_i,x_{i+1})k^i+\alpha(x_{i+1},x_{i+2})k^{i+1}\right]d(x_0,x_1).
\end{aligned}$$

Above, we make use of that $\alpha(x,y)\geq 1$.

$$\textit{Let } S_p=\sum_{i=0}^{i=p}\prod_{j=0}^{j=i}\alpha(x_j,x_{n+2l+1})\left[\alpha(x_i,x_{i+1})k^i+\alpha(x_{i+1},x_{i+1})k^{i+1}\right]d(x_0,x_1).$$

Hence, we have

$$d(x_n,x_m)\leq d(x_0,x_1)\left[\alpha(x_n,x_{n+1})k^n+\alpha(x_{n+1},x_{n+2})k^{n+1}+S_{n+m-1}-S_{n+1}\right].$$

Now, let the term $a_i=\prod_{j=0}^{j=i}\alpha(x_j,x_m)\left[\alpha(x_i,x_{i+1})k^i+\alpha(x_{i+1},x_{i+2})k^{i+1}\right]$.

On the other hand

$$\begin{aligned}
\sup_{m\geq 1}\lim_{i\to\infty}\frac{a_{i+1}}{a_i}&=\sup_{m\geq 1}\lim_{i\to\infty}\alpha\left(x_{i+1},x_m\right)\frac{\alpha\left(x_{i+1},x_{i+2}\right)k+\alpha\left(x_{i+2},x_{i+3}\right)k^2}{\alpha\left(x_i,x_{i+1}\right)+\alpha\left(x_{i+1},x_{i+2}\right)k}\\
&\leq\sup_{m\geq 1}\lim_{i\to\infty}\alpha\left(x_{i+1},x_m\right)\frac{\alpha\left(x_{i+1},x_{i+2}\right)+\alpha\left(x_{i+2},x_{i+3}\right)}{\alpha\left(x_i,x_{i+1}\right)+\alpha\left(x_{i+1},x_{i+2}\right)}<\frac{1}{k^2}.
\end{aligned}$$

Thus the series

$$\sum_{i=n+2}^{i=\infty}\prod_{j=n+2}^{j=i}\alpha(x_j,x_{n+2l+1})\left[\alpha(x_i,x_{i+1})k^i+\alpha(x_{2i+1},x_{i+1})k^{i+1}\right]d(x_0,x_1)$$

is converges. On the other hand

$$\lim_{n\to\infty}\alpha(x_n,x_{n+1})d(x_0,x_1)k^n=\lim_{n\to\infty}\alpha(x_{n+1},x_{n+2})d(x_0,x_1)k^{n+1}=0.$$

We conclude that

$$\lim_{n,m\to\infty}d(x_n,x_m)=0.$$

Case 2: $m = n + 2l$ similar to case 1 we have

$$\begin{aligned}
d(x_n, x_{n+2l}) &\leq \alpha(x_n, x_{n+2})d(x_n, x_{n+2}) + \alpha(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad + \alpha(x_{n+3}, x_{n+2k})d(x_{n+3}, x_{n+2l+1}) \\
&\leq \alpha(x_n, x_{n+2})d(x_n, x_{n+2}) + \alpha(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad + \alpha(x_{n+3}, x_{n+2l}) \times \big[\alpha(x_{n+3}, x_{n+4}))d(x_{n+3}, x_{n+4}) \\
&\quad + \alpha(x_{n+4}, x_{n+5}))d(x_{n+4}, x_{n+5}) + \alpha(x_{n+5}, x_{n+2l}))d(x_{n+5}, x_{n+2l})\big] \\
&\leq \alpha(x_n, x_{n+2})d(x_n, x_{n+2}) + \alpha(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+3}, x_{n+4}))d(x_{n+3}, x_{n+4}) \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+4}, x_{n+5}))d(x_{n+4}, x_{n+5}) \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+5}, x_{n+2l}))d(x_{n+5}, x_{n+2l}) \\
&\leq \alpha(x_n, x_{n+2})d(x_n, x_{n+2}) + \alpha(x_{n+2}, x_{n+3})d(x_{n+2}, x_{n+3}) \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+3}, x_{n+4}))d(x_{n+3}, x_{n+4}) \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+5}, x_{n+2l}))\alpha(x_{n+5}, x_{n+6})d(x_{n+5}, x_{n+6}) \\
&\quad + ... + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+5}, x_{n+2l})) \times ... \times \alpha(x_{n+2l-3}, x_{n+2l}) \times \\
&\quad \times \big[\alpha(x_{n+2l-3}, x_{n+2l-2})d(x_{n+2l-3}, x_{n+2l-2}) \\
&\quad + \alpha(x_{n+2l-2}, x_{n+2l-1})d(x_{n+2l-2}, x_{n+2l-1})\big] \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+5}, x_{n+2l})) \times ... \times \\
&\quad \times \alpha(x_{n+2l-3}, x_{n+2l})\alpha(x_{n+2l-1}, x_{n+2l})d(x_{n+2l-1}, x_{n+2l}) \\
&\leq \alpha(x_n, x_{n+2})k^n d(x_0, x_2) + \alpha(x_{n+2}, x_{n+3})k^{n+2}d(x_0, x_1) \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+3}, x_{n+4})))k^{n+3}d(x_0, x_1) \\
&\quad + ... + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+5}, x_{n+2k})) \times ... \times \alpha(x_{n+2l-3}, x_{n+2l}) \\
&\quad \times \big[\alpha(x_{n+2l-3}, x_{n+2l-2}))k^{n+l-3}d(x_0, x_1) \times \\
&\quad + \alpha(x_{n+2l-2}, x_{n+2k-1}))k^{n+l-2}d(x_0, x_1))\big] \\
&\quad + \alpha(x_{n+3}, x_{n+2l})\alpha(x_{n+5}, x_{n+2l})) \times ... \times \\
&\quad \times \alpha(x_{n+2l-3}, x_{n+2l})\alpha(x_{n+2l-1}, x_{n+2l}) \times \\
&\quad \times \big[\alpha(x_{n+2l-1}, x_{n+2l})k^{n+l-1}d(x_0, x_1) + \alpha(x_{n+2l}, x_{n+2l})k^{n+l}d(x_0, x_1)\big].
\end{aligned}$$

Thus, we conclude

$$d(x_n, x_m) \leq \alpha(x_n, x_{n+2})k^n d(x_0, x_2) + \alpha(x_{n+2}, x_{n+3})k^{n+2}d(x_0, x_1) +$$

$$+\sum_{i=n+3}^{i=n+2l-1}\prod_{j=n+3}^{j=i}\alpha(x_j,x_{n+2l})\left[\alpha(x_i,x_{n+2l})k^i+\alpha(x_{i+1},x_{i+2})k^{i+2}\right]d(x_0,x_1)$$

Above, we make use of that $\alpha(x,y)\geq 1$.

$$Let\ S_q=\sum_{i=0}^{i=q}\prod_{j=0}^{j=i}\alpha(x_j,x_{n+2l})\left[\alpha(x_i,x_{i+1})k^i+\alpha(x_{i+1},x_{i+2})k^{i+1}\right]d(x_0,x_1).$$

Then, we have

$$d(x_n,x_m)\leq d(x_0,x_2)\alpha(x_n,x_{n+2})k^n+d(x_0,x_1)\left[\alpha(x_{n+2},x_{n+3})k^{n+2}+S_{m-1}-S_{n-2}\right]$$

On the other hand

$$\begin{aligned}\sup_{m\geq 1}\lim_{i\to\infty}\frac{a_{i+1}}{a_i}&=\sup_{m\geq 1}\lim_{i\to\infty}\alpha\left(x_{i+1},x_m\right)\frac{\alpha\left(x_{i+1},x_{i+2}\right)k+\alpha\left(x_{i+2},x_{i+3}\right)k^2}{\alpha\left(x_i,x_{i+1}\right)+\alpha\left(x_{i+1},x_{i+2}\right)k}\\&\leq\sup_{m\geq 1}\lim_{i\to\infty}\alpha\left(x_{i+1},x_m\right)\frac{\alpha\left(x_{i+1},x_{i+2}\right)+\alpha\left(x_{i+2},x_{i+3}\right)}{\alpha\left(x_i,x_{i+1}\right)+\alpha\left(x_{i+1},x_{i+2}\right)}<\frac{1}{k^2}\end{aligned}$$

By using the Ratio Test, it is not difficult to that the series

$$\sum_{i=0}^{i=\infty}\prod_{j=0}^{j=i}\alpha(x_j,x_{n+2l+1})\left[\alpha(x_i,x_{i+1})k^i+\alpha(x_{i+1},x_{i+1})k^{i+1}\right]d(x_0,x_1)$$

converges. Hence $d(x_n,x_m)$ converges to zero as $m\to\infty$. Thus, by case 1 and case 2, we have

$$\lim_{n,m\to\infty}d(x_n,x_m)=0. \tag{2.6}$$

We conclude that the sequence x_n is a Cauchy sequence in the complete controlled rectangular metric space (X,d), so x_n converges to some $z\in X$.

We shall show that z is a fixed point of T.

No, we show that $d(Tz,z)=0$. Arguing by contradiction, we assume that $d(Tz,z)>0$. From the controlled rectangular inequality we get,

(2.7)

$$d\left(z,Tz\right)\leq\alpha(z,x_n)d\left(z,x_n\right)+\alpha(x_n,Tx_n)d\left(x_n,Tx_n\right)+\alpha(Tx_n,Tz)d\left(Tx_n,Tz\right).$$

From assumption of the hypothesis, we have

$$d\left(z,Tz\right)\leq\alpha(z,x_n)d\left(z,x_n\right)+\alpha(x_n,Tx_n)d\left(x_n,Tx_n\right)+k\alpha(Tx_n,Tz)d\left(x_n,z\right). \tag{2.8}$$

Therefore,

$$d\left(z,Tz\right)\leq d\left(z,x_n\right)\left[\alpha(z,x_n)+k\alpha(Tx_n,Tz)\right]+\alpha(x_n,Tx_n)d\left(x_n,Tx_n\right). \tag{2.9}$$

By letting $n\to+\infty$ above inequality and using (3.3), we deduce that $d\left(z,Tz\right)=0$ and that is $Tz=0$. Thus T has a fixed point $z\in X$.

Uniqueness: assume there exist two fixed points of T say z and u such that $z \neq u$. By the contractive property of T we have

$$d(z,u) = d(Tz,Tu) \leq kd(z,u).$$

Which is a contradiction. Thus, T has a unique fixed point. □

Theorem 2.8. *Let (X,d) be a complete controlled rectangular metric space, and T a self mapping on X satisfying the following condition: for all $x, y \in X$ there exists $0 < k < \frac{1}{2}$ such that*

$$d(Tx,Ty) \leq k\left[d(x,Tx) + d(y,Ty)\right]. \tag{2.10}$$

Let $x_0 \in X$, take $x_n = T^n x_0$ Also, if

$$\sup_{m\geq 1} \lim_{i\to\infty} \alpha\left(x_{i+1}, x_m\right) \frac{\alpha\left(x_{i+1}, x_{i+2}\right) + \alpha\left(x_{i+2}, x_{i+3}\right)}{\alpha\left(x_i, x_{i+1}\right) + \alpha\left(x_{i+1}, x_{i+2}\right)} < \frac{1}{k^2}. \tag{2.11}$$

We assume that $\lim_{n\to+\infty} \alpha(x_n, x)$, $\lim_{n\to+\infty} \alpha(x, x_n)$ and $\lim_{n,m\to+\infty} \alpha(x_n, x_m)$, exist and are finite for all $n, m \in \mathbb{N}$, $n \neq m$ Such that

$$\lim_{n\to+\infty} \alpha(Tx_n, Tx) < \frac{1}{k}. \tag{2.12}$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define the sequence x_n as follows $x_1 = Tx_0,\ x_2 = Tx_1, ...,\ x_n = Tx_{n-1} = T^n x_0, ...$ Now we prove that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

and

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0.$$

Using the contractive property with $x = x_{n-1}$ and $y = x_n$, we obtain

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq k\left[d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right] \\ \Rightarrow d(x_n, x_{n+1}) &\leq \frac{k}{1-k} d(x_{n-1}, x_n). \end{aligned}$$

Since $0 < k < \frac{1}{2}$, one can easily deduce that $0 < \frac{k}{1-k} < 1$. So, let $\beta = \frac{k}{1-k}$ hence,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \\ &\leq \beta^2 d(x_{n-2}, x_{n-1}) \\ &\leq ... \\ &\leq \beta^n d(x_0, x_1). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.13}$$

Appliyin (2.10) with $x = x_{n-1}$ and $y = x_{n+1}$, we obtain

$$d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \leq k\left[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\right].$$

Thus, by using the fact that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$, we deduce that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.14}$$

Now, similar to proof of Theorem 2.7, we deduce that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete controlled rectangular metric space, we conclude that x_n converges to some z in X.

We shall show that z is a fixed point of T.

No, we show that $d(Tz, z) = 0$. From the controlled rectangular inequality we get,

(2.15)

$$d\left(z, Tz\right) \leq \alpha(z, x_n)d\left(z, x_n\right) + \alpha(x_n, Tx_n)d\left(x_n, Tx_n\right) + \alpha(Tx_n, Tz)d\left(Tx_n, Tz\right).$$

From assumption of the hypothesis, we have

$$\begin{aligned} d\big(z, Tz\big) \leq{}& \alpha(z, x_n)d\big(z, x_n\big) + \alpha(x_n, Tx_n)d\big(x_n, Tx_n\big) \\ &+ k\alpha(Tx_n, Tz)\big[d\big(x_n, Tx_n\big) + d\big(z, Tz\big)\big]. \end{aligned} \tag{2.16}$$

Therefore,

$$\begin{aligned} d\big(z, Tz\big) \leq{}& \frac{1}{1 - k\alpha(Tx_n, Tz)}\big[\alpha(z, x_n)d\big(z, x_n\big) + \alpha(x_n, Tx_n)d\big(x_n, Tx_n\big) \\ &+ \alpha(Tx_n, Tz)d\big(x_n, Tx_n\big)\big]. \end{aligned} \tag{2.17}$$

Letting $n \to \infty$ in (2.17) and using (2.12), we obtain

$$d\left(z, Tz\right) \leq 0.$$

Which is a contradiction. Thus, $z = Tz$.

Uniqueness: assume there exist two fixed points of T say z and u such that $z \neq u$. By the contractive property of T we have

$$d(z, u) = d(Tz, Tu) \leq k\left[d(z, Tz) + d(u, Tu)\right] = 0.$$

Hence $z = u$. □

Theorem 2.9. *Let (X, d) be a complete controlled rectangular metric space, and T a self mapping on X satisfying the following condition: for all $x, y \in X$ there exists $0 < \lambda < \frac{1}{3}$ such that*

$$d(Tx, Ty) \leq \lambda \left[d(x, y) + d(x, Tx) + d(y, Ty)\right]. \tag{2.18}$$

Let $x_0 \in X$, take $x_n = T^n x_0$ Also, if

$$\sup_{m \geq 1} \lim_{i \to \infty} \alpha\left(x_i, x_m\right) \frac{\alpha\left(x_{i+1}, x_{i+2}\right) + \alpha\left(x_{i+2}, x_{i+3}\right)}{\alpha\left(x_i, x_{i+1}\right) + \alpha\left(x_{i+1}, x_{i+2}\right)} < \frac{1}{k^2}. \tag{2.19}$$

We assume that $\lim_{n \to +\infty} \alpha(x_n, x)$, $\lim_{n \to +\infty} \alpha(x, x_n)$ and $\lim_{n, m \to +\infty} \alpha(x_n, x_m)$, exist and are finite for all $n, m \in \mathbb{N}$, $n \neq m$ Such that

(2.20)

$$\lim_{n \to +\infty} \alpha(x_n, T^2 x_n) < \frac{1}{\lambda}, \lim_{n \to +\infty} \alpha(Tx_n, Tx) < \frac{1}{\lambda} \text{ and } \lim_{n \to +\infty} \alpha(Tx, Tx_n) < \frac{1}{\lambda}.$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define the sequence x_n as follows $x_1 = Tx_0$, $x_2 = Tx_1, ..., x_n = Tx_{n-1} = T^n x_0, ...$ Now we prove that

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0$$

and

$$\lim_{n \to \infty} d(x_n, x_{n+2}) = 0.$$

Using the contractive property, we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \lambda \left[d(x_{n-1}, x_n) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})\right].$$

So, we have

$$d(x_n, x_{n+1}) \leq \frac{2\lambda}{1 - \lambda} d(x_{n-1}, x_n).$$

Since $0 < \lambda < \frac{1}{3}$, one can easily deduce that $0 < \frac{2\lambda}{1-\lambda} < 1$. So, let $\beta = \frac{2\lambda}{1-\lambda}$.
Hence,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta d(x_{n-1}, x_n) \\ &\leq \beta^2 d(x_{n-2}, x_{n-1}) \\ &\leq ... \\ &\leq \beta^n d(x_0, x_1). \end{aligned}$$

Therefore,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.21}$$

Applying (2.18) with $x = x_{n-1}$ and $y = x_{n+1}$, we obtain

$$d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \le \lambda\left[d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2} + d(x_{n-1}, x_{n+1})\right]$$

Using the property (3) of the controlled rectangular metric space we get,

$$\begin{aligned} d(x_n, x_{n+2}) &\le \lambda\left[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\right] \\ &\le \lambda\left(d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\right) \\ &\quad + \lambda\big[\alpha(x_{n-1}, x_{n+2})d(x_{n-1}, x_{n+2}) + \alpha(x_{n+2}, x_n)d(x_{n+2}, x_n) \\ &\quad + \alpha(x_n, x_{n+1})d(x_n, x_{n+1})\big] \\ &\le \lambda\left(d(x_{n-1}, x_n) + d(x_{n+1}, x_{n+2})\right) \\ &\quad + \lambda\left[\alpha(x_{n+2}, x_n)d(x_{n+2}, x_n) + \alpha(x_n, x_{n+1})d(x_n, x_{n+1})\right] \\ &\quad + \lambda\alpha(x_{n-1}, x_{n+2})\times \\ &\quad \times \big[\alpha(x_{n-1}, x_n)d(x_{n-1}, x_n) + \alpha(x_n, x_{n+1})d(x_n, x_{n+1}) \\ &\quad + \alpha(x_{n+1}, x_{n+2})d(x_{n+1}, x_{n+2}))\big] \end{aligned}$$

Therefore, we have

$$d(x_n, x_{n+2}) \le d(x_{n-1}, x_n) \left[\frac{2\lambda + \lambda\alpha(x_n, x_{n+1}) + \lambda\alpha(x_{n-1}, x_{n+2})\left[\alpha(x_{n-1}, x_n)) + \alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2})\right]}{1 - \lambda\alpha(x_n, x_{n+2})}\right] \tag{2.22}$$

Letting $n \to \infty$ in (2.22), using (2.20), and (2.21), we obtain

$$\lim_{i\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.23}$$

Now, similarly to prove of Theorem (2.7), we deduce that the sequence $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete controlled rectangular metric space, we conclude that x_n converges to some z in X.

We shall show that z is a fixed point of T. No, we show that $d(Tz, z) = 0$. From the controlled rectangular inequality we get,

$$d\left(z, Tz\right) \le \alpha(z, x_n)d\left(z, x_n\right) + \alpha(x_n, Tx_n)d\left(x_n, Tx_n\right) + \alpha(Tx_n, Tz)d\left(Tx_n, Tz\right). \tag{2.24}$$

From assumption of the hypothesis, we have

$$d(z,Tz) \leq \alpha(z,x_n)d(z,x_n) + \alpha(x_n,Tx_n)d(x_n,Tx_n) + \lambda\alpha(Tx_n,Tz) \left[d(x_n,Tx_n) + d(z,Tz) + d(x_n,z)\right]. \tag{2.25}$$

Letting $n \to \infty$ in (2.25), we obtain

$$d(z,Tz) \leq \lambda \lim_{n\to\infty} \alpha(Tx_n,Tz)\left[d(z,Tz)\right] < d(z,Tz).$$

Which is a contradiction. Thus, $z = Tz$.

Uniqueness: assume there exist two fixed points of T say z and u such that $z \neq u$. By the contractive property of T we have

$$d(z,u) = d(Tz,Tu) \leq \lambda\left[d(z,u) + d(z,Tz) + d(u,Tu)\right] = \lambda d(z,Tz) < d(z,u).$$

Hence $z = u$. □

In the following we prove some new fixed point result for rational contraction of Fisher [8] type in the context of controlled rectangular metric space.

Theorem 2.10. *Let (X,d) be a complete controlled rectangular metric space, and T a self mapping on X satisfying the following condition: for all $x, y \in X$ there exists $\lambda, \beta \in]0,1[$ where $\lambda + \beta < 1$. For $x_0 \in X$, take $x_n = T^n x_0$. Such that*

$$d(Tx,Ty) \leq \lambda d(x,y) + \beta\frac{d(x,Tx) + d(y,Ty)}{1 + d(x,y)}. \tag{2.26}$$

Also, if

$$\sup_{m\geq 1} \lim_{i\to\infty} \alpha(x_{i+1},x_m) \frac{\alpha(x_{i+1},x_{i+2}) + \alpha(x_{i+2},x_{i+3})}{\alpha(x_i,x_{i+1}) + \alpha(x_{i+1},x_{i+2})} < \frac{1}{(\beta+\lambda)^2}. \tag{2.27}$$

We assume that $\lim_{n\to+\infty} \alpha(x_n,x)$, $\lim_{n\to+\infty} \alpha(x,x_n)$ and $\lim_{n,m\to+\infty} \alpha(x_n,x_m)$, exist and are finite for all $n,m \in \mathbb{N}$, $n \neq m$ Such that

(2.28)
$$\lim_{n,m\to+\infty} \alpha(x_n,x_m) < \frac{1}{(\beta+\lambda)}, \lim_{n\to+\infty} \alpha(x_n,x) < \frac{1}{(\beta+\lambda)} \text{ and } \lim_{n\to+\infty} \alpha(x_n,x) < \frac{1}{(\beta+\lambda)}.$$

Then T has a unique fixed point in X.

Proof. Let $x_0 \in X$ and define the sequence x_n as follows $x_1 = Tx_0$, $x_2 = Tx_1, ..., x_n = Tx_{n-1} = T^n x_0, ...$ Now we prove that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

and

$$\lim_{n\to\infty} d(x_n,x_{n+2}) = 0.$$

Using the contractive property we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \lambda d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)} \\ &\leq \lambda d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}), \end{aligned}$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{\lambda}{1-\beta} d(x_{n-1}, x_n) = \gamma d(x_{n-1}, x_n),$$

where $\gamma = \frac{\lambda}{1-\beta}$, then $\gamma \in]0, 1[$. Thus, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \gamma d(x_{n-1}, x_n) \\ &\leq \gamma^2 d(x_{n-2}, x_{n-1}) \\ &\leq \ldots \\ &\leq \gamma^n d(x_0, x_1). \end{aligned}$$

Therefore,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0. \tag{2.29}$$

Applying (2.26) with $x = x_{n-1}$ and $y = x_{n+1}$, we obtain

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Tx_{n-1}, Tx_{n+1}) \\ &\leq \lambda d(x_{n-1}, x_{n+1}) + \beta \frac{d(x_{n-1}, x_n) d(x_{n+1}, x_{n+2})}{1 + d(x_{n-1}, x_n)} \\ &\leq \lambda d(x_{n-1}, x_{n+1}) + \beta d(x_n, x_{n+1}) \\ &\leq \beta d(x_n, x_{n+1}) + \lambda \big[\alpha(x_{n-1}, x_{n+2}) d(x_{n-1}, x_{n+2}) \\ &\quad + \alpha(x_{n+2}, x_n) d(x_{n+2}, x_n) \alpha(x_n, x_{n+1}) d(x_n, x_{n+1})\big] \\ &\leq \beta d(x_n, x_{n+1}) + \lambda \big[\alpha(x_{n+2}, x_n) d(x_{n+2}, x_n) + \alpha(x_n, x_{n+1}) d(x_n, x_{n+1})\big] \\ &\quad + \lambda \alpha(x_{n-1}, x_{n+2}) \big[\alpha(x_{n-1}, x_n) d(x_{n-1}, x_n) \\ &\quad + \alpha(x_n, x_{n+1}) d(x_n, x_{n+1}) \alpha(x_{n+1}, x_{n+2}) d(x_{n+1}, x_{n+2})\big], \end{aligned}$$

which implies

$$d(x_n, x_{n+2}) \leq \frac{d(x_{n-1}, x_n)}{1 - \lambda \alpha(x_{n+2}, x_n)} \tag{2.30}$$
$$\left[\beta + \lambda \alpha(x_n, x_{n+1}) + \alpha(x_{n-1}, x_{n+2}) \left(\alpha(x_{n-1}, x_n) + \alpha(x_n, x_{n+1}) + \alpha(x_{n+1}, x_{n+2})\right)\right].$$

Thus, by using the fact that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$ and using (2.28), we deduce that

$$\lim_{n\to\infty} d(x_n, x_{n+2}) = 0. \tag{2.31}$$

Now, similarly to prove of Theorem (2.7), we deduce that the sequence $\{x_n\}$ is a Cauchy sequence in X. Since (X, d) is a complete controlled rectangular metric space, we conclude that x_n converges to some z in X.

We shall show that z is a fixed point of T. No, we show that $d(Tz, z) = 0$. From the controlled rectangular inequality we get,

$$d(z, Tz) \leq \alpha(z, x_n)d(z, x_n) + \alpha(x_n, Tx_n)d(x_n, Tx_n) + \alpha(Tx_n, Tz)d(Tx_n, Tz). \tag{2.32}$$

From assumption of the hypothesis, we have

$$d(z, Tz) \leq \alpha(z, x_n)d(z, x_n) + \alpha(x_n, Tx_n)d(x_n, Tx_n) + \alpha(Tx_n, Tz) \left[\lambda d(x_n, z) + \beta \frac{d(x_n, Tx_n)\, d(z, Tz)}{1 + d(x_n, z)}\right]. \tag{2.33}$$

By letting $n \to +\infty$ in (2.33), we obtain

$$d(z, Tz) \leq 0.$$

Which is a contradiction. Thus, $z = Tz$.

Uniqueness: assume there exist two fixed points of T say z and u such that $z \neq u$. By the contractive property of T we have

$$d(z, u) = d(Tz, Tu) \leq \lambda d(z, u) + \beta \frac{d(z, Tz)d(u, Tu)}{1 + d(z, u)} = \lambda d(z, u) < d(z, u).$$

Hence $z = u$. □

Example 2.11. Let $X = A \cup B$, where $A = \{\frac{1}{n} : n \in \{2, 3, 4, 5\}\}$ and $B = [1, 2]$. Define $d : X \times X \to [0, +\infty[$ as follows:

$$\begin{cases} d(x, y) = d(y, x) \ for\ all\ x, y \in X; \\ d(x, y) = 0 \Leftrightarrow y = x. \end{cases}$$

and

$$\begin{cases} d\left(\frac{1}{3},\frac{1}{4}\right) = d\left(\frac{1}{4},\frac{1}{5}\right) = 0,04 \\ d\left(\frac{1}{3},\frac{1}{5}\right) = d\left(\frac{1}{4},\frac{1}{6}\right) = 0,09 \\ d\left(\frac{1}{3},\frac{1}{6}\right) = d\left(\frac{1}{5},\frac{1}{6}\right) = 0,36 \\ d\left(x,y\right) = \left(|x-y|\right)^2 \ otherwise. \end{cases}$$

Define mapping $\alpha : X \to X$ by

$$\alpha(x,y) = \begin{cases} \max\{x,y\} + 2 \ if \ x,y \in [1,2] \\ 3 \ otherwise. \end{cases}$$

Then (X,d) is a is controlled rectangular metric space. However we have the following:

1) (X,d) is not a metric space, as

$$d\left(\frac{1}{3},\frac{1}{6}\right) = 0.36 > 0.13 = d\left(\frac{1}{3},\frac{1}{4}\right) + d\left(\frac{1}{4},\frac{1}{6}\right).$$

2) (X,d) is not a controlled metric space, as

$$d\left(\frac{1}{5},\frac{1}{6}\right) = 0.36 > 0.3033 = \alpha\left(\frac{1}{5},\frac{1}{4}\right) d\left(\frac{1}{5},\frac{1}{4}\right) + \alpha\left(\frac{1}{4},\frac{1}{6}\right) d\left(\frac{1}{4},\frac{1}{6}\right).$$

3) (X,d) is not a rectangular metric space, as

$$d\left(\frac{1}{5},\frac{1}{6}\right) = 0.36 > 0.22 = d\left(\frac{1}{5},\frac{1}{3}\right) + d\left(\frac{1}{3},\frac{1}{4}\right) + d\left(\frac{1}{4},\frac{1}{6}\right).$$

Define mapping $T : X \to X$ by

$$T(x) = \begin{cases} x^{\frac{1}{2}} \ if \ x \in [1,2] \\ 1 \ if \ x \in A. \end{cases}$$

Then, $T(x) \in [1,2]$. Let $k = \frac{1}{2}$.

Consider the following possibilities:

case 1: $x,y \in [1,2]$ with $x \neq y$, assume that $x > y$.

$$d(Tx,Ty) = \left[x^{\frac{1}{2}} - y^{\frac{1}{2}}\right]^2.$$

and

$$k.d(x,y)) = \frac{1}{2}\left[x-y\right]^2.$$

On the other hand

$$\begin{aligned} d(Tx,Ty)-k.d(x,y)) &= \left[x^{\frac{1}{2}}-y^{\frac{1}{2}}\right]^2-\frac{1}{2}\left[x-y\right]^2 \\ &= \frac{1}{2}\left(x^{\frac{1}{2}}-y^{\frac{1}{2}}\right)\left(\sqrt{2}x^{\frac{1}{2}}-\sqrt{2}y^{\frac{1}{2}}+x-y\right)\left(\sqrt{2}-x^{\frac{1}{2}}-y^{\frac{1}{2}}\right). \end{aligned}$$

Since $x, y \in [1,2]$, then

$$\left(\sqrt{2}-x^{\frac{1}{2}}-y^{\frac{1}{2}}\right) \leq 0.$$

Which implies that

$$d(Tx,Ty) \leq k.d(x,y)).$$

case 2: $x \in [1,2], y \in A$ or $y \in [1,2], x \in A$.

Therefore, $T(x)=x^{\frac{1}{2}}$, $T(y)=1$, then $d(Tx,Ty)=\left(|x^{\frac{1}{2}}-1|\right)^2=\left(x^{\frac{1}{2}}-1\right)^2$.

Since, $x \geq y$ for all $x \in [1,2], y \in A$. Therefore, $k.d(x,y)=\frac{1}{2}\left(x-y\right)^2$.

On the other hand

$$\begin{aligned} 0 \leq \left(x^{\frac{1}{2}}-1\right)^2 &\leq \left(2^{\frac{1}{2}}-1\right)^2 \\ &\leq \frac{2}{9} \\ &= \frac{1}{2}\left(1-\frac{1}{3}\right)^2 \\ &\leq \frac{1}{2}\left(x-\frac{1}{3}\right)^2 \\ &\leq \frac{1}{2}\left(x-y\right)^2. \end{aligned}$$

Which implies that

$$d(Tx,Ty) \leq k.d(y,Ty)).$$

case 3: $x, y \in A$

$$d(Tx,Ty)=0.$$

Which implies that

$$d(Tx,Ty) \leq k.d(y,Ty)).$$

Note that for each $x \in X$,

$$T^n(x)=\begin{cases} x^{\frac{1}{2^n}} \ if \ x \in [1,2] \\ 1 \ if \ x \in A. \end{cases}$$

Thus we obtain:

$$\lim_{i\to\infty}\sup_{m\geq 1}\alpha\left(x_i,x_m\right)\frac{\alpha\left(x_{i+1},x_{i+2}\right)+\alpha\left(x_{i+2},x_{i+3}\right)}{\alpha\left(x_i,x_{i+1}\right)+\alpha\left(x_{i+1},x_{i+2}\right)}=3<4=\frac{1}{k^2}.$$

On the other hand

$$\lim_{n\to+\infty}\alpha(x_n,x)=\lim_{n\to+\infty}\alpha(x,x_n)\leq 4 \text{ and } \lim_{n,m\to+\infty}\alpha(x_n,x_m)=3\,\forall n,m\in\mathbb{N},\ n\neq m.$$

Therefore, all conditions of Theorem (3.7) are satisfied hence T has a unique fixed point $Z=1$.

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