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HYPERSTABILITY OF A GENERAL QUINTIC FUNCTIONAL EQUATION AND A GENERAL SEPTIC FUNCTIONAL EQUATION

Sun-Sook Jin and Yang-Hi Lee

Abstract. In this paper, we present some hyperstability results for a general quintic functional equation and a general septic functional equation.

1. Introduction

Throughout this paper, let V, X and Y be a real vector space, a real normed space, and a real Banach space, respectively. For a mapping f from V to Y, we consider the functional equation

$$\sum_{i=0}^{n} {}_nC_i(-1)^{n-i}f(x+iy)=0, \tag{1.1}$$

where $n \in \mathbb{N}$. Observe that a solution mapping $f : V \to Y$ of (1.1) is a generalized polynomial mapping of degree at most $n-1$ in the sence of J. Baker in [1]. So, for $n = 2, 3, 4, 5, 6, 7$, and 8, we call the functional equation (1.1) a Jensen, a general quadratic, a general cubic, a general quartic, a general quintic, a general sextic, and a general septic functional equation, respectively.

Recall the stability problem of a functional equation was started by S. M. Ulam[14] and D. H. Hyers[4], and many mathematicians have generalized by showing the stability results of various kind of functional equations (see [1, 2, 5, 6, 11, 13], etc.). Prior to this paper, for the functional equation (1.1), with $n = 3, 4, 5, 6$, and 7, stability problems were studied by Y.-H. Lee[7, 8, 9, 10], by following;

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Theorem 1.1. [10] *Let $p \neq 1,2,3,4,5$ be a fixed nonnegative real number and $\theta > 0$ be a real number. Suppose that $f : X \to Y$ is a mapping such that*

$$\left\| \sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x+iy) \right\| \leq \theta(\|x\|^p + \|y\|^p) \tag{1.2}$$

for all $x, y \in X$. Then there exists a solution mapping F of the functional equation

$$\sum_{i=0}^{6} {}_6C_i(-1)^{6-i} F(x+iy) = 0$$

such that $\|f(x) - f(0) - F(x)\|$ is less than or equal to

$$\begin{cases} \left(\frac{K}{45\cdot 2^{2p}(2^p-2)} + \frac{(128+44\cdot 2^p)K}{45(2^p-32)(2^p-8)2^{2p}} + \frac{8}{2^p(2^p-16)(2^p-4)}\right)\theta\|x\|^p & \text{if } 5<p, \\ \left(\frac{(2\cdot 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{8}{2^p(2^p-16)(2^p-4)}\right)\theta\|x\|^p & \text{if } 4<p<5, \\ \left(\frac{(2\cdot 2^p-1)K}{45(2^p-8)(2^p-2)2^p} + \frac{2K}{45(32-2^p)2^p} + \frac{8}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p & \text{if } 3<p<4, \\ \left(\frac{K}{90\cdot 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{8}{(2^p-4)(16-2^p)}\right)\theta\|x\|^p & \text{if } 2<p<3, \\ \left(\frac{K}{90\cdot 2^p(2^p-2)} + \frac{(128-2^p)K}{90(32-2^p)(8-2^p)2^p} + \frac{8}{(16-2^p)(4-2^p)}\right)\theta\|x\|^p & \text{if } 1<p<2, \\ \left(\frac{(22+2^p)K}{720(8-2^p)(2-2^p)} + \frac{K}{720(32-2^p)} + \frac{8}{(16-2^p)(4-2^p)}\right)\theta\|x\|^p & \text{if } 0 \leq p<1 \end{cases}$$

for all $x \in X\backslash\{0\}$ and $F(0) = 0$, where $K = 182 + 38 \cdot 2^p + 6 \cdot 3^p$.

In this article, we present some hyperstability results for the general quintic functional equation

$$Cf(x,y) := \sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x+iy) = 0 \tag{1.3}$$

and the general septic functional equation

$$Df(x,y) := \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x+iy) = 0, \tag{1.4}$$

which are special parts of the study of stability problems of functional equations as proving a solution of a functional equation differing slightly from the given equation is exactly the solution. Precisely, let $p < 0$ and

$\theta > 0$ be real numbers, and let the mapping $f : X \to Y$ satisfy the inequality

$$\|Cf(x,y)\|\,,\ \|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X\backslash\{0\}$, then we will show that f must be a solution of

$$Cf(x,y) = 0,\ Df(x,y) = 0$$

for all $x, y \in X$, respectively. A solution mapping of the general quintic functional equation is called a general quintic mapping and the solution mapping. On the other hand, we call a solution of the general septic functional equation a general septic mapping.

Moreover, the hyperstability problems of the other type functional equations were studied by E. Gselmann[3], and A. Najati et al.[12].

2. Hyperstability of a general quintic functional equation

Throughout this paper, for $f : V \to Y$, we use the following abbreviations,

$$\tilde{f}(x) := f(x) - f(0), \quad f_o(x) := \frac{f(x) - f(-x)}{2}, \quad f_e(x) := \frac{f(x) + f(-x)}{2}$$

for all $x \in V$.

Lemma 2.1. *For a given mapping $f : V \to Y$, let f_1, f_2, f_3, f_4, f_5 : $V \to Y$ be the mappings defined by following;*

$$f_1(x) := \frac{1}{L}\begin{vmatrix} f_o(x) & 1 & 1 \\ f_o(2x) & 8 & 32 \\ f_o(4x) & 8^2 & 32^2 \end{vmatrix} = \frac{4(256f_o(x) - 40f_o(2x) + f_o(4x))}{720},$$

$$f_2(x) := \frac{1}{L'}\begin{vmatrix} f_e(x) & 1 \\ f_e(2x) & 16 \end{vmatrix} = \frac{16f_e(x) - f_e(2x)}{12},$$

$$f_3(x) := \frac{1}{L}\begin{vmatrix} 1 & f_o(x) & 1 \\ 2 & f_o(2x) & 32 \\ 2^2 & f_o(4x) & 32^2 \end{vmatrix} = -\frac{5(64f_o(x) - 34f_o(2x) + f_o(4x))}{720},$$

$$f_4(x) := \frac{1}{L'}\begin{vmatrix} 1 & f_e(x) \\ 4 & f_e(2x) \end{vmatrix} = -\frac{4f_e(x) - f_e(2x)}{12},$$

$$f_5(x) := \frac{1}{L}\begin{vmatrix} 1 & 1 & f_o(x) \\ 2 & 8 & f_o(2x) \\ 2^2 & 8^2 & f_o(4x) \end{vmatrix} = \frac{16f_o(x) - 10f_o(2x) + f_o(4x)}{720}$$

for all $x \in V$*, where* $L := \begin{vmatrix} 1 & 1 & 1 \\ 2 & 8 & 32 \\ 2^2 & 8^2 & 32^2 \end{vmatrix}$ *and* $L' := \begin{vmatrix} 1 & 1 \\ 4 & 16 \end{vmatrix}$*. Then we can show that* $f_o(x) = f_1(x) + f_3(x) + f_5(x)$ *and* $f_e(x) = f_2(x) + f_4(x)$*, and so*

$$f(x) = f_o(x) + f_e(x) = \sum_{i=1}^{5} f_i(x) \tag{2.1}$$

for all $x \in V$*.*

Proof. We note that $L,\, L' \neq 0$. By the uniqueness of solution (stated in Cramer's rule), the family $\{f_1(x),\, f_3(x),\, f_5(x)\}$ is the only solution to the system of nonhomogeneous linear equations

$$\begin{cases} f_1(x) + f_3(x) + f_5(x) = f_o(x), \\ 2f_1(x) + 8f_3(x) + 32f_5(x) = f_o(2x), \\ 2^2 f_1(x) + 8^2 f_3(x) + 32^2 f_5(x) = f_o(4x) \end{cases} \tag{2.2}$$

for all $x \in V$. It follows from the first equation of (2.2) that $f_o(x) = f_1(x) + f_3(x) + f_5(x)$ for all $x \in V$. Similarly, we can show that $f_e(x) = f_2(x) + f_4(x)$ for all $x \in V$.

□

Moreover, in this section, we use the following definitions.

Definition 2.2. For a given mapping $f : V \to Y$, we define the mappings $Cf : V^2 \to Y$ and $\Gamma f,\, \Delta f : V \to Y$ as following;

$$Cf(x,y) := \sum_{i=0}^{6} {}_6C_i(-1)^{6-i} f(x+iy),$$
$$\Gamma f(x) := Cf_o(-4x, 2x) + 6Cf_o(6x, -x) + 36Cf_o(-x, x) + 70Cf_o(-2x, x),$$
$$\Delta f(x) := Cf_e(-2x, x) + 3Cf_e(-3x, x)$$

for all $x, y \in V$.

By laborious computation, we can get the equalities in the following lemma.

Lemma 2.3. *Let* $f : V \to Y$ *be an arbitrarily given mapping. Then the equalities*

$$\Gamma\tilde{f}(x) = \Gamma f(x) = f_o(8x) - 42f_o(4x) + 336f_o(2x) - 512f_o(x), \tag{2.3}$$

$$\Delta\tilde{f}(x) = \tilde{f}_e(4x) - 20\tilde{f}_e(2x) + 64\tilde{f}_e(x) \tag{2.4}$$

hold for all $x \in V$*.*

Now we will prove the hyperstability of the quintic functional equation $Cf(x,y)=0$ by following;

THEOREM 2.4. *Let $p<0$ and $\theta>0$ be real numbers. Suppose that $f:X\to Y$ is a mapping such that*

$$\|Cf(x,y)\| \le \theta(\|x\|^p+\|y\|^p) \tag{2.5}$$

for all $x,y\in X\backslash\{0\}$. Then $Cf(x,y)=0$ for all $x,y\in X$, and so f is a generalized polynomial mapping of degree at most 5.

Proof. Since $\tilde{f}(0)=0$ and $C\tilde{f}(x,y)=Cf(x,y)$ for all $x,y\in X$, by (2.5) we get

$$\|C\tilde{f}(x,y)\|,\ \|C\tilde{f}_o(x,y)\|,\ \|C\tilde{f}_e(x,y)\| \le \theta(\|x\|^p+\|y\|^p) \tag{2.6}$$

for all $x,y\in X\backslash\{0\}$. Together with the definitions of $\Gamma\tilde{f}$ and $\Delta\tilde{f}$ in Definition 2.1, it follows that

$$\|\Gamma\tilde{f}(x)\| \le 720K'\theta\|x\|^p, \tag{2.7}$$

$$\|\Delta\tilde{f}(x)\| \le 12K\theta\|x\|^p \tag{2.8}$$

for all $x\in X\backslash\{0\}$, where

$$K' := \frac{6\cdot 6^p+4^p+71\cdot 2^p+148}{720} \quad \text{and} \quad K := \frac{3\cdot 3^p+2^p+4}{12}.$$

Now, using the definitions of $f_1,\ f_2,\ \cdots$, and f_5 in Lemma 2.1, together with the properties (2.3), (2.4), (2.7) and (2.8), we can show the following inequalities

$$\left\|\frac{\tilde{f}_1(2^ix)}{2^i}-\frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}}\right\| = \left\|-\frac{4\Gamma\tilde{f}(2^ix)}{720\cdot 2^{i+1}}\right\| \le \frac{4K'\theta\|2^ix\|^p}{2^{i+1}},$$

$$\left\|\frac{\tilde{f}_2(2^ix)}{4^i}-\frac{\tilde{f}_2(2^{i+1}x)}{4^{i+1}}\right\| = \left\|\frac{\Delta\tilde{f}(2^ix)}{12\cdot 4^{i+1}}\right\| \le \frac{K\theta\|2^ix\|^p}{4^{i+1}},$$

$$\left\|\frac{\tilde{f}_3(2^ix)}{8^i}-\frac{\tilde{f}_3(2^{i+1}x)}{8^{i+1}}\right\| = \left\|\frac{5\Gamma\tilde{f}(2^ix)}{720\cdot 8^{i+1}}\right\| \le \frac{5K'\theta\|2^ix\|^p}{8^{i+1}},$$

$$\left\|\frac{\tilde{f}_4(2^ix)}{16^i}-\frac{\tilde{f}_4(2^{i+1}x)}{16^{i+1}}\right\| = \left\|-\frac{\Delta\tilde{f}(2^ix)}{12\cdot 16^{i+1}}\right\| \le \frac{K\theta\|2^ix\|^p}{16^{i+1}},$$

$$\left\|\frac{\tilde{f}_5(2^ix)}{32^i}-\frac{\tilde{f}_5(2^{i+1}x)}{32^{i+1}}\right\| = \left\|-\frac{\Gamma\tilde{f}(2^ix)}{720\cdot 32^{i+1}}\right\| \le \frac{K'\theta\|2^ix\|^p}{32^{i+1}}$$

for all $x \in X\backslash\{0\}$. Notice, for each $k = 1, 2, \cdots, 5$, it is clear that

$$\frac{\tilde{f}_k(2^n x)}{2^{kn}} - \frac{\tilde{f}_k(2^{n+m}x)}{2^{k(n+m)}} = \sum_{i=n}^{n+m-1} \left(\frac{\tilde{f}_k(2^i x)}{2^{ki}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{k(i+1)}} \right)$$

for all $x \in X$, and so we easily obtain

$$\begin{aligned} \left\| \frac{\tilde{f}_1(2^n x)}{2^n} - \frac{\tilde{f}_1(2^{n+m}x)}{2^{n+m}} \right\| &\le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_1(2^i x)}{2^i} - \frac{\tilde{f}_1(2^{i+1}x)}{2^{i+1}} \right\| \\ &\le \sum_{i=n}^{n+m-1} \frac{4K'\theta\|2^i x\|^p}{2^{i+1}}, \end{aligned} \tag{2.9}$$

$$\begin{aligned} \left\| \frac{\tilde{f}_2(2^n x)}{4^n} - \frac{\tilde{f}_2(2^{n+m}x)}{4^{n+m}} \right\| &\le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_2(2^i x)}{2^i} - \frac{\tilde{f}_2(2^{i+1}x)}{2^{i+1}} \right\| \\ &\le \sum_{i=n}^{n+m-1} \frac{K\theta\|2^i x\|^p}{4^{i+1}}, \end{aligned} \tag{2.10}$$

$$\begin{aligned} \left\| \frac{\tilde{f}_3(2^n x)}{8^n} - \frac{\tilde{f}_3(2^{n+m}x)}{8^{n+m}} \right\| &\le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_3(2^i x)}{2^i} - \frac{\tilde{f}_3(2^{i+1}x)}{2^{i+1}} \right\| \\ &\le \sum_{i=n}^{n+m-1} \frac{5K'\theta\|2^i x\|^p}{8^{i+1}}, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \left\| \frac{\tilde{f}_4(2^n x)}{16^n} - \frac{\tilde{f}_4(2^{n+m}x)}{16^{n+m}} \right\| &\le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_4(2^i x)}{2^i} - \frac{\tilde{f}_4(2^{i+1}x)}{2^{i+1}} \right\| \\ &\le \sum_{i=n}^{n+m-1} \frac{K\theta\|2^i x\|^p}{16^{i+1}}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \left\| \frac{\tilde{f}_5(2^n x)}{32^n} - \frac{\tilde{f}_5(2^{n+m}x)}{32^{n+m}} \right\| &\le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_5(2^i x)}{2^i} - \frac{\tilde{f}_5(2^{i+1}x)}{2^{i+1}} \right\| \\ &\le \sum_{i=n}^{n+m-1} \frac{K'\theta\|2^i x\|^p}{32^{i+1}} \end{aligned} \tag{2.13}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. They imply that

$$\left\| \sum_{k=1}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}} - \sum_{k=1}^{5} \frac{\tilde{f}_k(2^{n+m} x)}{2^{k(n+m)}} \right\| \leq \sum_{i=n}^{n+m-1} \left(\frac{4K'}{2^{i+1}} + \frac{K}{4^{i+1}} + \frac{5K'}{8^{i+1}} + \frac{K}{16^{i+1}} + \frac{K'}{32^{i+1}} \right) \theta \|2^i x\|^p \tag{2.14}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N}\cup\{0\}$. Since $p < 0$, the inequalities (2.9)-(2.14) lead us to show that all the sequences $\left\{\frac{\tilde{f}_1(2^n x)}{2^n}\right\}, \cdots, \left\{\frac{\tilde{f}_5(2^n x)}{2^{5n}}\right\}$, and $\left\{\sum_{k=1}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}}\right\}$ are Cauchy sequences for all $x \in X\backslash\{0\}$. Moreover, since Y is complete and $\tilde{f}(0) = 0$, the sequences $\left\{\frac{\tilde{f}_1(2^n x)}{2^n}\right\}, \cdots, \left\{\frac{\tilde{f}_5(2^n x)}{2^{5n}}\right\}$, and $\left\{\sum_{k=1}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}}\right\}$ converge for all $x \in X$, too. Hence, for each $k = 1, 2, \cdots, 5$, we can define the mappings $F_k, F : X \to Y$ by

$$F_k(x) := \lim_{n\to\infty} \frac{\tilde{f}_k(2^n x)}{2^{kn}}, \qquad F(x) := \lim_{n\to\infty} \sum_{k=1}^{5} \frac{\tilde{f}_k(2^n x)}{2^{kn}}$$

for all $x \in X$. Now, letting $n = 0$ and passing the limit $m \to \infty$ in (2.14). Since $\tilde{f}(x) = \sum_{k=1}^{5} \tilde{f}_k(x)$ for all $x \in X$, see (2.1), we get the inequality

$$\|\tilde{f}(x) - F(x)\| \leq \left(\frac{4K'}{2 - 2^p} + \frac{K}{4 - 2^p} + \frac{5K'}{8 - 2^p} + \frac{K}{16 - 2^p} + \frac{K'}{32 - 2^p} \right) \theta \|x\|^p \tag{2.15}$$

for all $x \in X$. By (2.6) and the definition of $\tilde{f}_1$ in Lemma 2.1, we easily get

$$\begin{aligned}
\|CF_1(x, y)\| &= \lim_{n\to\infty} \left\| \frac{C\tilde{f}_1(2^n x, 2^n y)}{2^n} \right\| \\
&= \lim_{n\to\infty} \left\| \frac{256C\tilde{f}_o(2^n x, 2^n y)}{180 \cdot 2^n} - \frac{40C\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{180 \cdot 2^n} + \frac{C\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{180 \cdot 2^n} \right\| \\
&\leq \lim_{n\to\infty} \frac{\left(256 \cdot 2^{np} + 40 \cdot 2^{(n+1)p} + 2^{(n+2)p}\right) \theta(\|x\|^p + \|y\|^p)}{180 \cdot 2^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$, since $p < 0$. Observe that

$$CF_1(x,0) = \sum_{i=0}^{6} {}_6C_i(-1)^{6-i}F_1(x) = 0$$
$$CF_1(0,y) = \sum_{i=0}^{6} {}_6C_i(-1)^{6-i}F_1(iy) = \sum_{i=0}^{6} {}_6C_{6-i}(-1)^{i}F_1((6-i)y)$$
$$= \sum_{i=0}^{6} {}_6C_i(-1)^{6-i}F_1((6-i)y) = CF_1(6y,-y) = 0$$

for all $x \in X$ and $y \in X\backslash\{0\}$. Therefore, we have that $CF_1(x,y) = 0$ for all $x, y \in X$. Similarly we can show that $CF_2(x,y) = \cdots = CF_5(x,y) = 0$ for all $x, y \in X$. Hence we have $CF(x,y) = \sum_{k=1}^{5} CF_k(x,y) = 0$ for all $x, y \in X$. Now we use the equation

$$C\tilde{f}((1-n)x, nx) = C\tilde{f}((1-n)x, nx) - CF((1-n)x, nx)$$
$$= \sum_{i=0}^{6} {}_6C_i(-1)^{6-i}\left(\tilde{f}((1-n)x + inx) - F((1-n)x + inx)\right)$$
$$= \tilde{f}((1-n)x) - F((1-n)x) - 6\left(\tilde{f}(x) - F(x)\right)$$
$$+ \sum_{i=2}^{6} {}_6C_i(-1)^{6-i}\left(\tilde{f}((1-n)x + inx) - F((1-n)x + inx)\right),$$

together with (2.6), (2.15) and the property $p < 0$, to show that

$$6\|\tilde{f}(x) - F(x)\|$$
$$\leq \lim_{n\to\infty} \|C\tilde{f}((1-n)x, nx)\| + \lim_{n\to\infty} \left\|\tilde{f}((1-n)x) - F((1-n)x)\right\|$$
$$+ \sum_{i=2}^{6} \lim_{n\to\infty} \left\|{}_6C_i\left(\tilde{f}(((i-1)n+1)x) - F(((i-1)n+1)x\right)\right\|$$
$$\leq \lim_{n\to\infty}\left(|1-n|^p + n^p + M|1-n|^p + M\sum_{i=2}^{6} {}_6C_i\,|(i-1)n+1|^p\right)\theta\|x\|^p$$
$$= 0$$

for all $x \in X\backslash\{0\}$, where $M := \frac{4K'}{2-2^p} + \frac{K}{4-2^p} + \frac{5K'}{8-2^p} + \frac{K}{16-2^p} + \frac{K'}{32-2^p}$. Moreover, since $\tilde{f}(0) = 0 = F(0)$, we get $\tilde{f}(x) = F(x)$ for all $x \in X$. Therefore we conclude that, $Cf(x,y) = C\tilde{f}(x,y) = CF(x,y) = 0$ for all $x, y \in X$, and so f becomes a quintic mapping as desired. □

3. Hyperstability of a general septic functional equation

LEMMA 3.1. *For a given mapping* $f : V \to Y$, *let* $f_6,\ f_7,\ f_8,\ f_9,\ f_{10},\ f_{11},\ f_{12} : V \to Y$ *be the mappings defined by followings;*

$$f_6(x) := \frac{1}{N}\begin{vmatrix} f_o(x) & 1 & 1 & 1 \\ f_o(2x) & 8 & 32 & 128 \\ f_o(4x) & 8^2 & 32^2 & 128^2 \\ f_o(8x) & 8^3 & 32^3 & 128^3 \end{vmatrix}$$

$$= \frac{64\,(32768 f_o(x) - 5376 f_o(2x) + 168 f_o(4x) - f_o(8x))}{1451520},$$

$$f_7(x) := \frac{1}{N'}\begin{vmatrix} f_e(x) & 1 & 1 \\ f_e(2x) & 16 & 64 \\ f_e(4x) & 16^2 & 64^2 \end{vmatrix} = \frac{4(1024 f_e(x) - 80 f_e(2x) + f_e(4x))}{2880},$$

$$f_8(x) := \frac{1}{N}\begin{vmatrix} 1 & f_o(x) & 1 & 1 \\ 2 & f_o(2x) & 32 & 128 \\ 2^2 & f_o(4x) & 32^2 & 128^2 \\ 2^3 & f_o(8x) & 32^3 & 128^3 \end{vmatrix}$$

$$= -\frac{84(8192 f_o(x) - 4416 f_o(2x) + 162 f_o(4x) - f_o(8x))}{1451520},$$

$$f_9(x) := \frac{1}{N'}\begin{vmatrix} 1 & f_e(x) & 1 \\ 4 & f_e(2x) & 64 \\ 4^2 & f_e(4x) & 64^2 \end{vmatrix} = -\frac{5(256 f_e(x) - 68 f_e(2x) + f_e(4x))}{2880},$$

$$f_{10}(x) := \frac{1}{N}\begin{vmatrix} 1 & 1 & f_o(x) & 1 \\ 2 & 8 & f_o(2x) & 128 \\ 2^2 & 8^2 & f_o(4x) & 128^2 \\ 2^3 & 8^3 & f_o(8x) & 128^3 \end{vmatrix}$$

$$= \frac{21(2048 f_o(x) - 1296 f_o(2x) + 138 f_o(4x) - f_o(8x))}{1451520},$$

$$f_{11}(x) := \frac{1}{N'}\begin{vmatrix} 1 & 1 & f_e(x) \\ 4 & 16 & f_e(2x) \\ 4^2 & 16^2 & f_e(4x) \end{vmatrix} = \frac{64 f_e(x) - 20 f_e(2x) + f_e(4x)}{2880}$$

$$f_{12}(x) := \frac{1}{N}\begin{vmatrix} 1 & 1 & 1 & f_o(x) \\ 2 & 8 & 32 & f_o(2x) \\ 2^2 & 8^2 & 32^2 & f_o(4x) \\ 2^3 & 8^3 & 32^3 & f_o(8x) \end{vmatrix}$$

$$= -\frac{512 f_o(x) - 336 f_o(2x) + 42 f_o(4x) - f_o(8x)}{1451520}$$

for all $x \in V$, *where*

$$N := \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 8 & 32 & 128 \\ 2^2 & 8^2 & 32^2 & 128^2 \\ 2^3 & 8^3 & 32^3 & 128^3 \end{vmatrix} \quad \textit{and} \quad N' := \begin{vmatrix} 1 & 1 & 1 \\ 4 & 16 & 64 \\ 4^2 & 16^2 & 64^2 \end{vmatrix}.$$

Then we have

$$f(x) = f_o(x) + f_e(x) = \sum_{i=6}^{12} f_i(x) \tag{3.1}$$

for all $x \in V$.

Proof. We note that $N,\ N' \neq 0$. The uniqueness of solution (stated in Cramer's rule) implies that the family $\{f_6(x),\ f_8(x),\ f_{10}(x),\ f_{12}(x)\}$ is the only solution to the system of nonhomogeneous linear equations

$$\begin{cases} f_6(x) + f_8(x) + f_{10}(x) + f_{12}(x) = f_o(x), \\ 2f_6(x) + 8f_8(x) + 32f_{10}(x) + 128f_{12}(x) = f_o(2x), \\ 2^2 f_6(x) + 8^2 f_8(x) + 32^2 f_{10}(x) + 128^2 f_{12}(x) = f_o(4x), \\ 2^3 f_6(x) + 8^3 f_8(x) + 32^3 f_{10}(x) + 128^3 f_{12}(x) = f_o(8x) \end{cases} \tag{3.2}$$

for all $x \in V$. Therefore, it follows from the first equation of (3.2) that $f_o(x) = f_6(x)+f_8(x)+f_{10}(x)+f_{12}(x)$ for all $x \in V$. In the similar way, we get $f_e(x) = f_7(x)+f_9(x)+f_{11}(x)$ for all $x \in V$. So we have obtained the equation (3.1)

□

In this section, we use the following definitions.

Definition 3.2. For a given mapping $f : V \to Y$, we define the mappings $Df : V^2 \to Y$ and $\tilde{\Gamma} f,\ \tilde{\Delta} f : V \to Y$ as

$$\begin{aligned} Df(x,y) &:= \sum_{i=0}^{8} {}_8C_i(-1)^{8-i} f(x+iy), \\ \tilde{\Gamma} f(x) &:= Df_o(16x,-2x) + 8Df_o(-2x,2x) + 36Df_o(-4x,2x) \\ &\quad + 120Df_o(-6x,2x) + 160Df_o(8x,-x) + 1280Df_o(-x,x) \\ &\quad + 4032Df_o(-2x,x) + 5376Df_o(-3x,x), \\ \tilde{\Delta} f(x) &:= Df_e(8x,-x) + 8Df_e(-x,x) + 36Df_e(-2x,x) + 120Df_e(-3x,x) \\ &\quad + 123Df_e(-4x,x) \end{aligned}$$

for all $x, y \in V$.

By laborious computation we can get the equalities in the following lemma.

Lemma 3.3. *Let $f : V \to Y$ be an arbitrarily given mapping. Then the equalities*

$$\tilde{\Gamma}\tilde{f}(x) = \tilde{\Gamma}f(x) = f_o(16x) - 170f_o(8x) + 5712f_o(4x) - 43520f_o(2x) + 65536f_o(x), \tag{3.3}$$

$$\tilde{\Delta}\tilde{f}(x) = \tilde{f}_e(8x) - 84\tilde{f}_e(4x) + 1344\tilde{f}_e(2x) - 4096\tilde{f}_e(x) \tag{3.4}$$

hold for all $x \in V$.

We show the hyperstability of the septic functional equation $Df(x, y) = 0$ by following;

Theorem 3.4. *Let $p < 0$ and $\theta > 0$ be real numbers. Suppose that $f : X \to Y$ is a mapping such that*

$$\|Df(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{3.5}$$

for all $x, y \in X\backslash\{0\}$. Then

$$Df(x, y) = 0$$

for all $x, y \in X$, and thus the mapping f is a generalized polynomial of degree at most 7.

Proof. Observe that $D\tilde{f}(x, y) = Df(x, y)$ and we have

$$\|D\tilde{f}(x, y)\|,\ \|D\tilde{f}_o(x, y)\|,\ \|D\tilde{f}_e(x, y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{3.6}$$

for all $x \in X\backslash\{0\}$ by (3.5). By the definitions of $\tilde{\Gamma}\tilde{f}$ and $\tilde{\Delta}\tilde{f}$ in Definition 3.1, it follows that

$$\|\tilde{\Gamma}\tilde{f}(x)\| \le 1451520\tilde{K}'\theta\|x\|^p, \tag{3.7}$$

$$\|\tilde{\Delta}\tilde{f}(x)\| \le 2880\tilde{K}\theta\|x\|^p \tag{3.8}$$

for all $x \in X\backslash\{0\}$, where

$$\tilde{K}' := \frac{16^p + 160 \cdot 8^p + 120 \cdot 6^p + 36 \cdot 4^p + 5376 \cdot 3^p + 4205 \cdot 2^p + 12128}{1451520},$$

$$\tilde{K} := \frac{8^p + 123 \cdot 4^p + 120 \cdot 3^p + 36 \cdot 2^p + 296}{2880}.$$

From the definitions of $f_6, f_7, \ldots, f_{12}$ in Lemma 3.1, together with the properties (3.3), (3.4), (3.7) and (3.8), we obtain the inequalities

$$\left\|\frac{\tilde{f}_6(2^i x)}{2^i} - \frac{\tilde{f}_6(2^{i+1}x)}{2^{i+1}}\right\| = \left\|\frac{64\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 2^{i+1}}\right\| \leq \frac{64\tilde{K}'\theta\|2^i x\|^p}{2^{i+1}},$$

$$\left\|\frac{\tilde{f}_7(2^i x)}{4^i} - \frac{\tilde{f}_7(2^{i+1}x)}{4^{i+1}}\right\| = \left\|-\frac{4\tilde{\Delta}\tilde{f}(2^i x)}{2880 \cdot 4^{i+1}}\right\| \leq \frac{4\tilde{K}\theta\|2^i x\|^p}{4^{i+1}},$$

$$\left\|\frac{\tilde{f}_8(2^i x)}{8^i} - \frac{\tilde{f}_8(2^{i+1}x)}{8^{i+1}}\right\| = \left\|\frac{-84\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 8^{i+1}}\right\| \leq \frac{84\tilde{K}'\theta\|2^i x\|^p}{8^{i+1}},$$

$$\left\|\frac{\tilde{f}_9(2^i x)}{16^i} - \frac{\tilde{f}_9(2^{i+1}x)}{16^{i+1}}\right\| = \left\|\frac{5\tilde{\Delta}\tilde{f}(2^i x)}{2880 \cdot 16^{i+1}}\right\| \leq \frac{5\tilde{K}\theta\|2^i x\|^p}{16^{i+1}},$$

$$\left\|\frac{\tilde{f}_{10}(2^i x)}{32^i} - \frac{\tilde{f}_{10}(2^{i+1}x)}{32^{i+1}}\right\| = \left\|\frac{21\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 32^{i+1}}\right\| \leq \frac{21\tilde{K}'\theta\|2^i x\|^p}{32^{i+1}},$$

$$\left\|\frac{\tilde{f}_{11}(2^i x)}{64^i} - \frac{\tilde{f}_{11}(2^{i+1}x)}{64^{i+1}}\right\| = \left\|-\frac{\tilde{\Delta}\tilde{f}(2^i x)}{2880 \cdot 64^{i+1}}\right\| \leq \frac{\tilde{K}\theta\|2^i x\|^p}{64^{i+1}},$$

$$\left\|\frac{\tilde{f}_{12}(2^i x)}{128^i} - \frac{\tilde{f}_{12}(2^{i+1}x)}{128^{i+1}}\right\| = \left\|-\frac{\tilde{\Gamma}\tilde{f}(2^i x)}{1451520 \cdot 128^{i+1}}\right\| \leq \frac{\tilde{K}'\theta\|2^i x\|^p}{128^{i+1}}$$

for all $x \in X \backslash \{0\}$. For each $k = 6, 7, \cdots, 12$, it is clear that

$$\frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}} - \frac{\tilde{f}_k(2^{n+m}x)}{2^{(k-5)(n+m)}} = \sum_{i=n}^{n+m-1}\left(\frac{\tilde{f}_k(2^i x)}{2^{(k-5)i}} - \frac{\tilde{f}_k(2^{i+1}x)}{2^{(k-5)(i+1)}}\right)$$

for all $x \in X$, so we obtain that

$$\left\| \frac{\tilde{f}_6(2^n x)}{2^n} - \frac{\tilde{f}_6(2^{n+m} x)}{2^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_6(2^i x)}{2^i} - \frac{\tilde{f}_6(2^{i+1} x)}{2^{i+1}} \right\| \le \sum_{i=n}^{n+m-1} \frac{64\tilde{K}'\theta\|2^i x\|^p}{2^{i+1}}, \tag{3.9}$$

$$\left\| \frac{\tilde{f}_7(2^n x)}{4^n} - \frac{\tilde{f}_7(2^{n+m} x)}{4^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_7(2^i x)}{2^i} - \frac{\tilde{f}_7(2^{i+1} x)}{2^{i+1}} \right\| \le \sum_{i=n}^{n+m-1} \frac{4\tilde{K}\theta\|2^i x\|^p}{4^{i+1}}, \tag{3.10}$$

$$\left\| \frac{\tilde{f}_8(2^n x)}{8^n} - \frac{\tilde{f}_8(2^{n+m} x)}{8^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_8(2^i x)}{2^i} - \frac{\tilde{f}_8(2^{i+1} x)}{2^{i+1}} \right\| \le \sum_{i=n}^{n+m-1} \frac{84\tilde{K}'\theta\|2^i x\|^p}{8^{i+1}}, \tag{3.11}$$

$$\left\| \frac{\tilde{f}_9(2^n x)}{16^n} - \frac{\tilde{f}_9(2^{n+m} x)}{16^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_9(2^i x)}{2^i} - \frac{\tilde{f}_9(2^{i+1} x)}{2^{i+1}} \right\|, \le \sum_{i=n}^{n+m-1} \frac{5\tilde{K}\theta\|2^i x\|^p}{16^{i+1}} \tag{3.12}$$

$$\left\| \frac{\tilde{f}_{10}(2^n x)}{32^n} - \frac{\tilde{f}_{10}(2^{n+m} x)}{32^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_{10}(2^i x)}{2^i} - \frac{\tilde{f}_{10}(2^{i+1} x)}{2^{i+1}} \right\|, \le \sum_{i=n}^{n+m-1} \frac{21\tilde{K}'\theta\|2^i x\|^p}{32^{i+1}} \tag{3.13}$$

$$\left\| \frac{\tilde{f}_{11}(2^n x)}{64^n} - \frac{\tilde{f}_{11}(2^{n+m} x)}{64^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_{11}(2^i x)}{2^i} - \frac{\tilde{f}_{11}(2^{i+1} x)}{2^{i+1}} \right\|, \le \sum_{i=n}^{n+m-1} \frac{\tilde{K}\theta\|2^i x\|^p}{64^{i+1}} \tag{3.14}$$

$$\left\| \frac{\tilde{f}_{12}(2^n x)}{128^n} - \frac{\tilde{f}_{12}(2^{n+m} x)}{128^{n+m}} \right\| \le \sum_{i=n}^{n+m-1} \left\| \frac{\tilde{f}_{12}(2^i x)}{2^i} - \frac{\tilde{f}_{12}(2^{i+1} x)}{2^{i+1}} \right\| \le \sum_{i=n}^{n+m-1} \frac{\tilde{K}'\theta\|2^i x\|^p}{128^{i+1}} \tag{3.15}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N} \cup \{0\}$. They imply that

$$(3.16)\qquad \left\|\sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}} - \sum_{k=6}^{12} \frac{\tilde{f}_k(2^{n+m} x)}{2^{(k-5)(n+m)}}\right\| \leq \sum_{i=n}^{n+m-1} \left(\frac{64\tilde{K}'}{2^{i+1}} + \frac{4\tilde{K}}{4^{i+1}} + \frac{84\tilde{K}'}{8^{i+1}} + \frac{5\tilde{K}}{16^{i+1}} + \frac{21\tilde{K}'}{32^{i+1}} + \frac{\tilde{K}}{64^{i+1}} + \frac{\tilde{K}'}{128^{i+1}}\right)\theta\|2^i x\|^p$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N}\cup\{0\}$. Since $p < 0$, the inequalities (3.9)-(3.16) lead us to prove that all the sequences $\left\{\frac{\tilde{f}_6(2^n x)}{2^n}\right\}, \cdots, \left\{\frac{\tilde{f}_{12}(2^n x)}{2^{7n}}\right\}$, and $\left\{\sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}}\right\}$ are Cauchy sequences for all $x \in X\backslash\{0\}$. Moreover, since Y is complete and $\tilde{f}(0) = 0$, the sequences

$$\left\{\frac{\tilde{f}_6(2^n x)}{2^n}\right\}, \cdots, \left\{\frac{\tilde{f}_{12}(2^n x)}{2^{7n}}\right\}, \quad \text{and} \quad \left\{\sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}}\right\}$$

converge for all $x \in X$, too. Hence, for each $k = 6, 7, \cdots, 12$, we can define mappings $F_k, \hat{F} : X \to Y$ by

$$F_k(x) := \lim_{n\to\infty} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}},$$

$$\hat{F}(x) := \lim_{n\to\infty} \sum_{k=6}^{12} \frac{\tilde{f}_k(2^n x)}{2^{(k-5)n}}$$

for all $x \in X$. Now, letting $n = 0$ and passing the limit $m \to \infty$ in (3.16), by

$$\tilde{f}(x) = \sum_{k=6}^{12} \tilde{f}(x)$$

in (3.1), we get the inequality

$$(3.17)\qquad \|\tilde{f}(x) - \hat{F}(x)\| \leq \left(\frac{64\tilde{K}'}{2 - 2^p} + \frac{4\tilde{K}}{4 - 2^p} + \frac{84\tilde{K}'}{8 - 2^p} + \frac{5\tilde{K}}{16 - 2^p} + \frac{21\tilde{K}'}{32 - 2^p} + \frac{\tilde{K}}{64 - 2^p} + \frac{\tilde{K}'}{128 - 2^p}\right)\theta\|x\|^p$$

for all $x \in X$. By (3.6), we easily have that

$$\begin{aligned}
\|DF_6(x,y)\| &= \lim_{n\to\infty}\left\|\frac{D\tilde{f}_6(2^n x, 2^n y)}{2^n}\right\| \\
&= \lim_{n\to\infty}\left\|\frac{32768 D\tilde{f}_o(2^n x, 2^n y)}{2835\cdot 8\cdot 2^n} - \frac{5376 D\tilde{f}_o(2^{n+1} x, 2^{n+1} y)}{2835\cdot 8\cdot 2^n}\right. \\
&\quad \left. + \frac{168 D\tilde{f}_o(2^{n+2} x, 2^{n+2} y)}{2835\cdot 8\cdot 2^n} - \frac{D\tilde{f}_o(2^{n+3} x, 2^{n+3} y)}{2835\cdot 8\cdot 2^n}\right\| \\
&\le \lim_{n\to\infty}\frac{32768\cdot 2^{np}\theta(\|x\|^p+\|y\|^p)}{2835\cdot 8\cdot 2^n} + \lim_{n\to\infty}\frac{5376\cdot 2^{(n+1)p}\theta(\|x\|^p+\|y\|^p)}{2835\cdot 8\cdot 2^n} \\
&\quad + \lim_{n\to\infty}\frac{168\cdot 2^{(n+2)p}\theta(\|x\|^p+\|y\|^p)}{2835\cdot 8\cdot 2^n} + \lim_{n\to\infty}\frac{2^{(n+3)p}\theta(\|x\|^p+\|y\|^p)}{2835\cdot 8\cdot 2^n} \\
&= 0
\end{aligned}$$

for all $x, y \in X\backslash\{0\}$. Observe that

$$DF_6(x,0) = \sum_{i=0}^{8} {}_8C_i(-1)^{8-i}F_6(x) = 0$$

for all $x \in X$, as well as

$$\begin{aligned}
DF_6(0,y) &= \sum_{i=0}^{8} {}_8C_i(-1)^{8-i}F_6(iy) = \sum_{i=0}^{8} {}_8C_{8-i}(-1)^{i}F_6((8-i)y) \\
&= \sum_{i=0}^{8} {}_8C_i(-1)^{8-i}F_6((8-i)y) = DF_6(8y,-y) \\
&= 0
\end{aligned}$$

for all $y \in X\backslash\{0\}$. Hence we can say that $DF_6(x,y) = 0$ for all $x, y \in X$. Similarly we get $DF_7(x,y) = \cdots = DF_{12}(x,y) = 0$ for all $x, y \in X$. Therefore, we obtain that

$$D\hat{F}(x,y) = \sum_{k=6}^{12} DF_k(x,y) = 0$$

for all $x, y \in X$, i.e., the mapping $\hat{F}$ is a septic mapping. Now, together with the equation

$$\begin{aligned}
D\tilde{f}((1-n)x, nx) &= D\tilde{f}((1-n)x, nx) - D\hat{F}((1-n)x, nx) \\
&= \sum_{i=0}^{8} {}_8C_i(-1)^{8-i}\left(\tilde{f}((1-n)x + inx) - \hat{F}((1-n)x + inx)\right) \\
&= \tilde{f}((1-n)x) - \hat{F}((1-n)x) - 8\left(\tilde{f}(x) - \hat{F}(x)\right) \\
&\quad + \sum_{i=2}^{8} {}_8C_i(-1)^{8-i}\left(\tilde{f}((1-n)x + inx) - \hat{F}((1-n)x + inx)\right)
\end{aligned}$$

for all $x \in X$, it follows that

$$\begin{aligned}
&8\|\tilde{f}(x) - \hat{F}(x)\| \\
&\quad \le \lim_{n\to\infty}\left\|D\tilde{f}((1-n)x, nx)\right\| + \lim_{n\to\infty}\left\|\tilde{f}((1-n)x) - \hat{F}((1-n)x)\right\| \\
&\qquad + \sum_{i=2}^{8}\lim_{n\to\infty}\left\|{}_8C_i\left(\tilde{f}(((i-1)n+1)x) - \hat{F}(((i-1)n+1)x\right)\right\| \\
&\quad \le \lim_{n\to\infty}\left(|1-n|^p + n^p + \tilde{M}|1-n|^p + \tilde{M}\sum_{i=2}^{8}{}_8C_i\,|(i-1)n+1|^p\right)\theta\|x\|^p \\
&\quad = 0
\end{aligned}$$

for all $x \in X\backslash\{0\}$, by (3.6), (3.17) and the property $p < 0$, where

$$\tilde{M} := \frac{64K'}{2-2^p} + \frac{4K}{4-2^p} + \frac{84K'}{8-2^p} + \frac{5K}{16-2^p}$$

$$+\frac{21K'}{32-2^p} + \frac{K}{64-2^p} + \frac{K'}{128-2^p}.$$

Since $\tilde{f}(0) = 0 = F(0)$, we get $\tilde{f}(x) = F(x)$ for all $x \in X$. Therefore we conclude that,

$$Df(x, y) = D\tilde{f}(x, y) = DF(x, y) = 0$$

for all $x, y \in X$, as we desired.

□

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Sun-Sook Jin
Department of Mathematics Education
Gongju National University of Education
Gongju 32553, Republic of Korea
E-mail: `ssjin@gjue.ac.kr`

Yang-Hi Lee
E-mail: `yanghi22@naver.com`